I. A. P. Mathematics Lecture Series 1999

Homework due Friday, January 15, 1999

Lecture 5: *Lattice Inversion Problems with Applications in Solid State Physics*

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1. **Césaro Semigroups.**

   Let $a(n)$ be completely multiplicative. Show that $\varepsilon_m(\varepsilon_n(x)) = \varepsilon_{mn}(x)$ for each $\{\varepsilon_n(x)\}$:

   $$\varepsilon_n(x) \quad = \quad a(n)x, \quad x^{a(n)}, \quad \frac{x}{1 - x\log a(n)}, \quad \frac{xa(n)}{1 - x[a(n) - 1]}$$

   Extra Credit: Can you think of a Césaro semigroup not trivially contained in this list?

2. **Two-body inversion on the linear chain $\mathcal{Z}$**

   Solve for $f(x)$ and prove your answer:

   $$\sum_{k=0}^{\infty} (-1)^k f\left((2k+1)x\right) = \frac{1}{x^2}$$

   Hint: Express in the Césaro form $\sum_{n=1}^{\infty} a(n) f(nx) = F(x)$.

3. **Two-body inversion on the simple cubic lattice $\mathcal{Z}^3$**

   (a) Let $a(n)$ denote the number of points in $\mathcal{Z}^3$ at a distance $\sqrt{n}$ from the origin. Compute $a(n)$ for $n = 1, 2, \ldots 9$.

   (b) Compute the Dirichlet inverse $b(n) = a^{-1}(n)$ for $n = 1, 2, \ldots 9$.

   (c) Compute the first 8 nonzero terms in the Möbius expansion of $\phi(x)$:

   $$\sum_{\alpha \in \mathcal{Z}^3} \phi(d_\alpha x) = e^{-x^2}, \quad x > 0$$

   where the summation excludes the origin and $d_{(i,j,k)} = \sqrt{i^2 + j^2 + k^2}$.

4. **Three-body inversion on the square lattice $\mathcal{Z}^2$.** (Extra Credit)

   Derive formulae for $g(x)$ valid in the regions $x \geq 1$, $1/\sqrt{2} \leq x < 1$ and $1/2 \leq x < 1/\sqrt{2}$:

   $$\sum_{\alpha,\beta \in \mathcal{Z}^2} (1 + \cos\theta_{\alpha,\beta}) g(d_\alpha x) g(d_\beta x) = \begin{cases} (x-1)^4 & \text{if } 0 < x < 1 \\ 0 & \text{if } x \geq 1 \end{cases}$$

   where the summation excludes the origin, $d_{(i,j)} = \sqrt{i^2 + j^2}$ and $\theta_{\alpha,\beta}$ is the angle between the α and β subtended at the origin. Plot $g(x)$ for $x \geq 1/2$.