Review L8 - Mathematical basis of stability analysis

\[
\begin{align*}
\dot{x} &= f(x, y) & \text{system of two coupled differential equations} \\
\dot{y} &= g(x, y)
\end{align*}
\]

step 1

find nullclines and fixed point(s)

\[
\begin{align*}
\dot{x} &= 0 \Rightarrow f(x_o, y_o) = 0 \\
\dot{y} &= 0 \Rightarrow g(x_o, y_o) = 0
\end{align*}
\]

step 2

consider small deviation from fixed point

\[
\begin{align*}
\Delta x &= x - x_o \\
\Delta y &= y - y_o
\end{align*}
\]

step 3

consider small deviation from fixed point

\[
\begin{align*}
\Delta x &= x - x_o \\
\Delta y &= y - y_o
\end{align*}
\]

step 4

linearize around fixed point(s)

\[
\begin{align*}
\dot{x} &= \frac{\partial f}{\partial x}(x_o, y_o) \Delta x + \frac{\partial f}{\partial y}(x_o, y_o) \Delta y = a \Delta x + b \Delta y \\
\dot{y} &= \frac{\partial g}{\partial x}(x_o, y_o) \Delta x + \frac{\partial g}{\partial y}(x_o, y_o) \Delta y = c \Delta x + d \Delta y
\end{align*}
\]

step 5

determine matrix A

\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]

step 6

determine trace and determinant of A:

\[
\begin{align*}
\tau &= \text{trace}(A) = a + d \\
\Delta &= \det(A) = ad - bc
\end{align*}
\]

only if \( \tau < 0 \) and \( \Delta > 0 \), \((x_o, y_o)\) is a stable fixed point

!!! be careful: only valid for 2 dimensional systems !!!
Box 1

**The toggle model**

The behaviour of the toggle switch and the conditions for bistability can be understood using the following dimensionless model for the network:

\[
\frac{dU}{dt} = \frac{\alpha_1}{1 + U^2} - U \quad (1a)
\]

\[
\frac{dV}{dt} = \frac{\alpha_2}{1 + U^2} - V \quad (1b)
\]

where \(U\) is the concentration of repressor 1, \(V\) is the concentration of repressor 2, \(\alpha_1\) is the effective rate of synthesis of repressor 1, \(\alpha_2\) is the effective rate of synthesis of repressor 2, \(\gamma\) is the cooperativity of

How to obtain this stability diagram?

Two fundamental dynamical responses: Switches and Oscillators

'Switch'

oscillator

Two fundamental dynamical responses:

Switches & Oscillators

two stable fixed points

unstable fixed point
nullclines:
\[ u = \frac{\alpha_1}{1+v^8} \]
\[ v = \frac{\alpha_2}{1+u^4} \]

\[ \frac{du}{dt} = \frac{\alpha_1}{1+v^8} - u \]
\[ \frac{dv}{dt} = \frac{\alpha_2}{1+u^4} - v \]

A "toy" example
\[ \dot{x} = -x + ay + x^2 y \]
\[ \dot{y} = b - ay - x^2 y \]
model for glycolysis
nullclines:
\[ y = \frac{x}{a+x^2} \]
\[ y = \frac{b}{a+x^2} \]
fixed point: \( x^* = b \)
\[ \dot{y} = 0 \]
stable or unstable?

Oscillator (unstable fixed point)