Conditional-Value-at-Risk Estimation via Reduced-Order Models *


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This paper proposes and analyzes two reduced-order model (ROM) based approaches for the efficient and accurate evaluation of the Conditional-Value-at-Risk (CVaR) of quantities of interest (QoI) in engineering systems with uncertain parameters. CVaR is used to model objective or constraint functions in risk-averse engineering design and optimization applications under uncertainty. Evaluating the CVaR of the QoI requires sampling in the tail of the QoI distribution and typically requires many solutions of an expensive full-order model of the engineering system. Our ROM approaches substantially reduce this computational expense. Both ROM-based approaches use Monte Carlo (MC) sampling. The first approach replaces the computationally expensive full-order model by an inexpensive ROM. The resulting CVaR estimation error is proportional to the ROM error in the so-called risk region, a small region in the space of uncertain system inputs. The second approach uses a combination of full-order model and ROM evaluations via importance sampling, and is effective even if the ROM has large errors. In the importance sampling approach, ROM samples are used to estimate the risk region and to construct a biasing distribution. Few full-order model

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samples are then drawn from this biasing distribution. Asymptotically as the ROM error goes to zero, the importance sampling estimator reduces the variance by a factor \(1 - \beta \ll 1\), where \(\beta \in (0, 1)\) is the quantile level at which CVaR is computed. Numerical experiments on a system of semilinear convection-diffusion-reaction equations illustrate the performance of the approaches.

**Keywords:** Reduced-order models, Risk measures, Conditional-Value-at-Risk, Estimation, Sampling

1. Introduction

Designing reliable engineering systems requires taking into account the uncertainties associated with system parameters. In risk-averse optimization, so-called risk measures are applied to quantities of interest (QoI) to form the objective function and constraint functions. Originally used as a risk measure in financial applications, the Conditional-Value-at-Risk at level \(\beta\) (CVaR\(_\beta\)), \(\beta \in (0, 1)\), is now also more frequently used in engineering applications, see, e.g., the survey paper by Rockafellar and Royset [15]. Estimating CVaR\(_\beta\) (and many other risk measures) requires sampling in the tail of the distribution of the QoI and is computationally expensive. While the distribution of the uncertain system parameters is known, the resulting QoI is a random variable that depends on the solution of the system. Therefore, the distribution of the QoI is not explicitly known and naive sampling in the tail of this distribution to estimate CVaR\(_\beta\) requires many expensive system simulations. Each system simulation requires the expensive solution of a full-order model (FOM). In our application, the system is modeled by a system of partial differential equations (PDEs) and the FOM is a high-fidelity discretization of the PDEs.

This paper proposes and analyzes two reduced-order model (ROM) based approaches for the efficient and accurate estimation of CVaR\(_\beta\). In our first approach we replace the FOM by a ROM, and we analyze the resulting error in the CVaR\(_\beta\) estimate. This error analysis depends on so-called risk regions, small regions in parameter space that contain the parameters that lead to ‘risky’ QoI values. Specifically, we show that the error in the CVaR\(_\beta\) estimate due to ROMs is upper bounded by the ROM error in so-called \(\epsilon\)-risk regions associated with the ROMs. These \(\epsilon\)-risk regions are derived from ROM information, contain the risk region of the FOM, and as the ROM error goes to zero converges to the risk region of the FOM. This first approach relies on an error bound between the ROM and the FOM, and the error needs to be sufficiently small for good CVaR\(_\beta\) estimates.

Our second approach uses an importance sampling framework. Here, the ROM error bound is only used to obtain estimates of the risk region. The proposed importance sampling framework can be effective even if the ROM error is large. ROMs are used to construct a biasing distribution that biases samples towards the risk region. We then estimate CVaR\(_\beta\) from FOM samples drawn from this biasing distribution. The ROM approximation error only impacts the quality of the biasing distribution and therefore,
via its variance, the sampling error. The smaller this variance, the fewer FOM samples are needed to generate a CVaRβ estimate at a desired tolerance. We show that asymptotically as the ROM error goes to zero, the importance sampling estimator reduces the variance by a factor $1 - \beta \ll 1$, where $\beta \in (0, 1)$ is the quantile level at which CVaR is computed.

There are existing approaches for using ROMs in the estimation of CVaRβ and other risk measures. Variance reduction strategies (including importance sampling) for quantile (Value-at-Risk) estimation using reduced models are considered by Cannamela et al. [2]. Specifically, importance-sampling estimates are constructed using a biasing density from a parametrized family of densities, and the parameters are informed through a reduced model. In this work, we consider CVaRβ (superquantiles), which are based on VaRβ (quantiles), and we analyze the impact of the ROM error on the CVaRβ estimate. Harajli et al. [5] compute buffered failure probabilities (which are related to CVaR) via importance sampling. However, an analytic biasing density is chosen which depends on the specific test problem in [5]. We suggest a principled, general approach to obtain a biasing density via reduced-order models obtained from the original governing equations. Importance sampling via reduced models has been successfully applied in the context of computing failure probabilities by Li et al. [9] and Peherstorfer et al. [11, 12, 13]. However, these approaches do not directly translate to importance sampling for CVaRβ estimation, the focus of our work. The review [7, Sec. 2.1, 2.2] by Hong et al. states asymptotic properties of CVaRβ estimation via importance sampling given a biasing density, but does not address how one can compute a biasing density. Among other contributions, our paper proposes a construction of a biasing density using ROMs. Zou et al. [23] estimate CVaRβ by constructing a Voronoi tessellation of the parameter space and using localized reduced-basis surrogate models. While their ROM construction is different from what we use in our example, it could in principle be used as well, and their overall approach is comparable to our first approach. The main difference between the work [23] and our work is that the ROM error in the entire parameter region is needed in [23], whereas our error estimate depends only on the ROM error in the risk region. In addition, we introduce and analyze an importance sampling approach that uses ROMs to construct the biasing distributions. Proper orthogonal decomposition based ROMs have recently been used in [22] to minimize CVaRβ for an aircraft noise problem modeled by the Helmholtz equation. Their overall approach is comparable to our first approach, but they do not analyze the impact of ROMs on the CVaRβ estimation error. The design of a ultrahigh-speed hydrofoil by using CVaRβ optimization is considered by Royset et al. [18]. They propose to build surrogates of the CVaR of their QoI and model these surrogates as random variables “due to unknown error in the surrogate relative to the actual value” of the CVaR of their QoI. This randomness in the CVaR surrogate is then incorporated into the design process by applying CVaR again, but with a different quantile level to the surrogate. Ultimately, they use a surrogate for the quantity of interest that combines high-fidelity and low-fidelity QoI evaluations into a polynomial fit model. Our work does not require additional stochastic treatment of model error, and focuses on the efficient and accurate sampling of CVaR using ROMs of the QoI that satisfy the original governing equations.
This paper is structured as follows. Section 2 provides background material and notation for CVaR\(\beta\) computation. In Section 3 we derive error bounds for CVaR\(\beta\) estimation with ROMs and give results on confidence intervals for CVaR\(\beta\). Section 4 presents our results on CVaR\(\beta\) estimation with importance sampling. Our two ROM approaches are illustrated on a system of semilinear convection-diffusion-reaction PDEs in Section 5. A review and discussion is given in Section 6.

2. Problem formulation and background

This section specifies our problem set-up and reviews the results on CVaR\(\beta\) needed for our application of ROMs. Section 2.1 introduces the basic problem set-up and notation, followed by a brief introduction to CVaR\(\beta\) and some useful properties in Section 2.2. Sample estimates of VaR\(\beta\) and CVaR\(\beta\) together with a complete algorithm are given in Section 2.3.

2.1. The state equation and quantity of interest

We consider systems modeled by equations of the form

\[ F(y, \xi) = 0, \quad (1) \]

where \(\xi\) is a vector of random variables with values in \(\Xi \subset \mathbb{R}^M\) and with probability density function (p.d.f.) \(\rho\), and where \(y\) denotes the state of the system. Equation (1) is referred to as the state equation. Often the system is modeled by a system of partial differential equations (PDEs) in which case the state equation (1) is a high fidelity discretization of the PDEs. We assume that for every parameter \(\xi \in \Xi\) there exists a unique solution \(y = y(\xi) \in \mathbb{R}^N\) of (1). For discretized PDEs, \(N\) is typically large.

We are interested in a quantity of interest (QoI) \(s : \mathbb{R}^N \mapsto \mathbb{R}\) and we assume that

\[ \xi \mapsto s(y(\xi)) \]

is both in \(L^1_\rho(\Xi)\) and in \(L^2_\rho(\Xi)\). For notational convenience, we set the quantity of interest to be the random variable

\[ X = s(y(\cdot)). \quad (2) \]

Following the original setting of financial applications, it is helpful to think of high values of \(X\) as risky.

The expected value and variance of the random variable \(X\) are given by

\[ \mathbb{E}_\rho[X] = \int_\Xi X(\xi)\rho(\xi)d\xi \quad \text{and} \quad \mathbb{V}_\rho[X] = \mathbb{E}_\rho \left[ (X(\cdot) - \mathbb{E}_\rho[X])^2 \right] \]

respectively. The subscript \(\rho\) is used to indicate which density is used in the integration.
2.2. Conditional-Value-at-Risk – CVaR

The CVaR\(_\beta\) is based on the Value-at-Risk (VaR\(_\beta\)). For a given level \(\beta \in (0, 1)\) the VaR\(_\beta\)[\(X\)] is the \(\beta\)-quantile of the random variable \(X\),

\[
\text{VaR}_\beta[X] = \min_{t \in \mathbb{R}} \{\Pr[X \leq t] \geq \beta\}.
\]

Here

\[
\Pr[X \leq t] = \int_{\Xi} \mathbb{I}\{X(\xi) \leq t\} \rho(\xi) d\xi,
\]

the probability that \(X\) is less or equal than \(t\) and \(\mathbb{I}\) is the indicator function. Different equivalent definitions of CVaR\(_\beta\) exist. The following definition is due to Rockafellar and Uryasev \[16, 17\]. The CVaR\(_\beta\) at level \(\beta \in (0, 1)\) is

\[
\text{CVaR}_\beta[X] = \min_{t \in \mathbb{R}} \left\{ t + \frac{1}{1-\beta} \mathbb{E}_\rho[(X - \text{VaR}_\beta[X])_+] \right\},
\]

where \((\cdot)_+ = \max\{\cdot, 0\}\). The minimum of (4) is attained on the interval \([t^*, t^{**}]\) with

\[
t^* = \text{VaR}_\beta[X], \quad \text{and} \quad t^{**} = \sup\{t : \Pr[X \leq t] \leq \beta\}.
\]

Inserting \(t^* = \text{VaR}_\beta[X]\) into (4) one obtains

\[
\text{CVaR}_\beta[X] = \text{VaR}_\beta[X] + \frac{1}{1-\beta} \mathbb{E}_\rho \left[ (X - \text{VaR}_\beta[X])_+ \right].
\]

If the cumulative distribution function (c.d.f.) \(H_X(x) = \Pr[X \leq x]\) is continuous at \(x = \text{VaR}_\beta[X]\), then \(\Pr[X = \text{VaR}_\beta[X]] = 0\), and (5) can be simplified to

\[
\text{CVaR}_\beta[X] = \frac{1}{1-\beta} \mathbb{E}_\rho \left[ X \cdot \mathbb{I}\{X \geq \text{VaR}_\beta[X]\} \right].
\]

More generally, the coherent risk measure properties of CVaR\(_\beta\) and the biconjugate representation of coherent risk measures (see, e.g., \[19\], \[21\], Sec. 6.3, \[8\]) give the representation

\[
\text{CVaR}_\beta[X] = \sup_{\vartheta \in \mathcal{A}} \mathbb{E}_\rho \left[ \vartheta X \right],
\]

where

\[
\mathcal{A} = \{\vartheta \in L^\infty_{\rho}(\Xi) : 0 \leq \vartheta(\xi) \leq (1 - \beta)^{-1} \text{ a.e. } \Xi, \text{ and } \mathbb{E}_\rho [\vartheta] = 1\}\. \]

One can show \[21\] Example 6.16, that the supremum is attained at any \(\vartheta^*_X\) that satisfies

\[
\vartheta^*_X(\xi) = \begin{cases} 
0 & \text{if } X(\xi) < \text{VaR}_\beta[X], \\
\in [0, (1-\beta)^{-1}] & \text{if } X(\xi) = \text{VaR}_\beta[X], \\
(1-\beta)^{-1} & \text{if } X(\xi) < \text{VaR}_\beta[X],
\end{cases}
\]

for almost all \(\xi \in \Xi\), and \(\mathbb{E}_\rho [\vartheta^*_X] = 1\). If \(\Pr[X = \text{VaR}_\beta[X]] = 0\), \(\vartheta^*_X\) is unique and (8), (7) reduce to (6).
The representations (8), (7) and (6) show that \( \text{CVaR}_\beta[X] \) depends only on the values of \( X \) that lie in the upper tail of the c.d.f. Therefore, the values of the parameter vector \( \xi \) that correspond to these high values of \( X \) can be considered risky. This motivates the following definition.

**Definition 2.1.** The risk region corresponding to \( \text{CVaR}_\beta[X] \) is given by

\[
G_\beta[X] := \{ \xi \mid X(\xi) \geq \text{VaR}_\beta[X] \} \subset \Xi
\]

and the corresponding indicator function of the risk region \( G_\beta[X] \) is

\[
I_{G_\beta[X]}(\xi) := I\{X(\xi) \geq \text{VaR}_\beta[X]\}.
\]

Note that

\[
\Pr[G_\beta[X]] = \Pr[\{X > \text{VaR}_\beta[X]\}] + \Pr[\{X = \text{VaR}_\beta[X]\}]
= (1 - \beta) + \Pr[\{X = \text{VaR}_\beta[X]\}],
\]

by the definition (3) of \( \text{VaR}_\beta[X] \).

2.3. Sampling-based estimation of \( \text{VaR}_\beta \) and \( \text{CVaR}_\beta \)

In practice, \( \text{CVaR}_\beta[X] \) is estimated through sampling. The procedure for computing \( \text{VaR}_\beta[X] \) and \( \text{CVaR}_\beta[X] \) is described in Algorithm 1 below. The second term on the right hand side in (12) is nonzero for the case \( \sum_{j=1}^{k_{\beta-1}} p^{(j)} \neq 1 - \beta \) and is based on the idea of splitting the probability atom at \( \text{VaR}_\beta[X] \) (see [17]).

Asymptotic convergence properties of the estimators \( \hat{\text{VaR}}_\beta[X] \) and \( \hat{\text{CVaR}}_\beta[X] \) are given in the review [7, Sec. 2.1,2.2] by Hong et al. and we state some of these later in Lemma 4.3 when we discuss importance sampling.

3. \( \text{CVaR}_\beta \) estimation with reduced-order models

Recall that evaluating the QoI \( X(\xi) \) at a given parameter \( \xi \) requires solving an expensive FOM. To devise our computationally efficient framework, we assume the availability of an inexpensive ROM approximation of the parameter to QoI map, denoted as \( X_r(\xi) \).

For the purpose of this section the details of how \( X_r(\xi) \) is computed are not important. Later, in Section 3.2, we give an example of how \( X_r(\xi) \) is computed via state reduction for the convection-diffusion-reaction model problem.

Our first proposed approach, described in this section, approximates \( \text{CVaR}_\beta[X] \) by \( \text{CVaR}_\beta[X_r] \). Since ROM samples \( X_r(\xi) \) are relatively inexpensive to compute, the computation of \( \text{CVaR}_\beta[X_r] \) is relatively inexpensive. Section 3.1 presents estimates of the error between \( \text{CVaR}_\beta[X] \) and \( \text{CVaR}_\beta[X_r] \). Section 3.2 then states the algorithm for practical computation of the ROM errors for general nonlinear systems.

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1 Rockafellar and Uryasev [17] assume that samples are ordered \( X(\xi^{(1)}) < X(\xi^{(2)}) < \ldots < X(\xi^{(n)}) \) with corresponding probabilities \( p^{(1)}, \ldots, p^{(n)} \). Since \( \text{VaR}_\beta[X] \) and \( \text{CVaR}_\beta[X] \) depend only on (a few) samples with large values, we prefer the ordering (18).
Algorithm 1 Sampling-based estimation of VaR and CVaR.

Input: Parameter samples $\xi^{(1)}, \ldots, \xi^{(n)}$ with probabilities $p^{(1)}, \ldots, p^{(n)}$, risk level $\beta \in (0, 1)$, and random variable $X(\xi)$.

Output: Estimate $\hat{\text{VaR}}_{\beta}[X]$ and $\hat{\text{CVaR}}_{\beta}[X]$.

1: Evaluate $X$ at the parameter samples: $X(\xi^{(1)}), \ldots, X(\xi^{(n)})$.
2: Sort values of $X$ in descending order and relabel the samples so that
   \begin{equation}
   X(\xi^{(1)}) > X(\xi^{(2)}) > \ldots > X(\xi^{(n)}),
   \end{equation}
   and reorder the probabilities accordingly (so that $p^{(j)}$ corresponds to $\xi^{(j)}$).
3: Compute an index $k_{\beta}$ such that
   \begin{equation}
   \sum_{j=1}^{k_{\beta}-1} p^{(j)} \leq 1 - \beta < \sum_{j=1}^{k_{\beta}} p^{(j)}.
   \end{equation}
4: Set
   \begin{equation}
   \hat{\text{VaR}}_{\beta}[X] = X(\xi^{(k_{\beta})})
   \end{equation}
   and
   \begin{equation}
   \hat{\text{CVaR}}_{\beta}[X] = \frac{1}{1-\beta} \sum_{j=1}^{k_{\beta}-1} p^{(j)} X(\xi^{(j)}) + \frac{1}{1-\beta} \left(1 - \beta - \sum_{j=1}^{k_{\beta}-1} p^{(j)}\right) \hat{\text{VaR}}_{\beta}[X].
   \end{equation}

3.1. Error analysis for CVaR$_{\beta}$ estimation with ROM

For the purpose of the following derivation, assume the availability of a bound for the error between the original QoI $X$ and its approximation $X_r$,

\begin{equation}
|X(\xi) - X_r(\xi)| \leq \epsilon_r(\xi) \quad \text{for} \quad \xi \in \Xi.
\end{equation}

We will relax this assumption somewhat at the end of this section.

Define the maximum error in $\Xi$ as

\begin{equation}
\epsilon_r^{\max} := \max_{\xi \in \Xi} \epsilon_r(\xi),
\end{equation}

so that $|X(\xi) - X_r(\xi)| \leq \epsilon_r^{\max}$ for all $\xi \in \Xi$. Zou et al. [23] use this maximum error and the biconjugate representation of CVaR$_{\beta}$ to derive an error estimate for CVaR$_{\beta}$. First, using (7) gives

\begin{equation}
\text{CVaR}_{\beta}[X] \leq \sup_{\vartheta \in A} \mathbb{E}[\vartheta X] \leq \sup_{\vartheta \in A} \mathbb{E}[\vartheta X_r] + \sup_{\vartheta \in A} \mathbb{E} \left[ \vartheta |X - X_r| \right]
= \text{CVaR}_{\beta}[X_r] + \sup_{\vartheta \in A} \mathbb{E} \left[ \vartheta |X - X_r| \right] \leq \text{CVaR}_{\beta}[X_r] + \epsilon_r^{\max},
\end{equation}
where the final inequality follows from the fact that elements $\vartheta \in \mathcal{A}$ are probability densities. Similarly, $\text{CVaR}_\beta[X_r] \leq \text{CVaR}_\beta[X] + \epsilon_r^\text{max}$. Thus,

$$|\text{CVaR}_\beta[X] - \text{CVaR}_\beta[X_r]| \leq \sup_{\vartheta \in \mathcal{A}} \mathbb{E} \left[ |\vartheta|X - X_r \right] \leq \epsilon_r^\text{max}.$$  \hspace{1cm} (15)

Although $\text{CVaR}_\beta[X]$ and $\text{CVaR}_\beta[X_r]$ depend only on $X$ and $X_r$ in their respective risk regions, the error bound \textbf{(15)} depends on the maximum error in $\Xi$. In particular, if the distributions of $X$ and $X_r$ have identical tails but differ elsewhere in $\Xi$, the actual error $|\text{CVaR}_\beta[X] - \text{CVaR}_\beta[X_r]|$ will be zero, whereas the error bound in \textbf{(15)} will still be $\epsilon_r^\text{max}$. It turns out that only the error between $X$ and $X_r$ in the so-called $\epsilon$-risk region, defined next, is important.

\textbf{Definition 3.1.} The $\epsilon$-risk region corresponding to $\text{CVaR}_\beta[X]$ is given by

$$\mathcal{G}_\beta^\epsilon[X_r] := \{ \xi : X_r(\xi) + \epsilon_r(\xi) \geq \text{VaR}_\beta[X_r] - \epsilon_r \},$$ \hspace{1cm} (16)

and the corresponding indicator function is $\mathbb{1}_{\mathcal{G}_\beta^\epsilon[X_r]}(\xi)$.

\textbf{Lemma 3.2.} It holds that

$$\mathcal{G}_\beta[X] \subseteq \mathcal{G}_\beta^\epsilon[X_r] \quad \text{and} \quad \mathcal{G}_\beta[X_r] \subseteq \mathcal{G}_\beta^\epsilon[X_r]$$ \hspace{1cm} (17)

and

$$\epsilon_r^G,\text{low} := \max_{\xi \in \mathcal{G}_\beta[X_r]} \epsilon_r(\xi) \leq \max_{\xi \in \mathcal{G}_\beta[X] \cup \mathcal{G}_\beta[X_r]} \epsilon_r(\xi) \leq \max_{\xi \in \mathcal{G}_\beta^\epsilon[X_r]} \epsilon_r(\xi) =: \epsilon_r^G.$$ \hspace{1cm} (18)

\textbf{Proof.} Consider the random variables $X_r - \epsilon_r$ and $X_r + \epsilon_r$. Obviously,

$$X_r(\xi) - \epsilon_r(\xi) \leq X(\xi) \leq X_r(\xi) + \epsilon_r(\xi) \quad \forall \xi \in \Xi,$n

and, by monotonicity of $\text{VaR}_\beta$ (see, e.g., \textbf{14}),

$$\text{VaR}_\beta[X_r] - \epsilon_r \leq \text{VaR}_\beta[X] \leq \text{VaR}_\beta[X_r] + \epsilon_r.$$n

In particular, for $\xi \in \mathcal{G}_\beta^\epsilon[X]$ we have

$$\text{VaR}_\beta[X_r] - \epsilon_r \leq \text{VaR}_\beta[X] \leq X(\xi) \leq X_r(\xi) + \epsilon_r(\xi),$$

which implies that $\mathcal{G}_\beta[X] \subseteq \mathcal{G}_\beta^\epsilon[X_r]$. Similarly, for $\xi \in \mathcal{G}_\beta[X_r]$ we have

$$\text{VaR}_\beta[X_r] - \epsilon_r \leq \text{VaR}_\beta[X_r] \leq X_r(\xi) \leq X_r(\xi) + \epsilon_r(\xi),$$

i.e., that $\mathcal{G}_\beta[X_r] \subseteq \mathcal{G}_\beta^\epsilon[X_r]$. Inequality \textbf{(18)} is an immediate consequence of \textbf{(17)}. \hfill \Box

We can now make the opening statement of this section more precise. In particular, we do not need the error function $\epsilon_r(\xi)$ in all of $\Xi$ from \textbf{(13)}, but only the error $\epsilon_r^G$ in the $\epsilon$-risk region $\mathcal{G}_\beta^\epsilon[X_r]$ from \textbf{(18)}. 

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Theorem 3.3. The error between CVaR$_\beta$ of the full-order model $X$ and CVaR$_\beta$ of the reduced-order model $X_r$ is bounded as

$$|CVaR_\beta[X] - CVaR_\beta[X_r]| \leq \left(1 + \frac{\max \{Pr\{X = VaR_\beta[X]\}, Pr\{X_r = VaR_\beta[X_r]\}\}}{1 - \beta}\right) \varepsilon^G,$$

(19)

If $X$ and $X_r$ have c.d.f.'s that are continuous at $VaR_\beta[X]$ and at $VaR_\beta[X_r]$, respectively, then

$$|CVaR_\beta[X] - CVaR_\beta[X_r]| \leq \varepsilon^G.$$  

(20)

Proof. Let $\vartheta_X$ be given by (8) with $CVaR_\beta[X] = \sup_{\vartheta \in D} E[\vartheta X] = E[\vartheta X].$ Then using the definition of $\vartheta^*_X$, $E[\vartheta^*_X X_r] \leq \sup_{\vartheta \in D} E[\vartheta X_r] = CVaR_\beta[X_r]$, and (10) gives

$$CVaR_\beta[X] = E[\vartheta^*_X X] = E[\vartheta^*_X X_r] + E[\vartheta^*_X (X - X_r)]$$

$$\leq CVaR_\beta[X_r] + \frac{1}{1 - \beta} \int_{G_\beta[X]} |X - X_r| \rho(\xi) d\xi$$

$$\leq CVaR_\beta[X_r] + \frac{1 - \beta + Pr\{X_r = VaR_\beta[X]\}}{1 - \beta} \max_{\xi \in G_\beta[X_r]} \varepsilon_r(\xi).$$

Similarly,

$$CVaR_\beta[X_r] \leq CVaR_\beta[X] + \frac{1 - \beta + Pr\{X_r = VaR_\beta[X]\}}{1 - \beta} \max_{\xi \in G_\beta[X_r]} \varepsilon_r(\xi).$$

It follows that

$$|CVaR_\beta[X] - CVaR_\beta[X_r]|$$

$$\leq \left(1 + \frac{\max \{Pr\{X = VaR_\beta[X]\}, Pr\{X_r = VaR_\beta[X_r]\}\}}{1 - \beta}\right) \max_{\xi \in G_\beta[X] \cup G_\beta[X_r]} \varepsilon_r(\xi).$$

(21)

The first term on the right hand side of (21) is bounded by $1 + 1/(1 - \beta)$. If $X$ and $X_r$ have c.d.f.'s that are continuous at $VaR_\beta[X]$ and $VaR_\beta[X_r]$, respectively, then $Pr\{X = VaR_\beta[X]\} = Pr\{X_r = VaR_\beta[X_r]\} = 0$ and the first term becomes one. The second term on the right hand side of (21) is bounded by $\varepsilon^G$ (see (18)).

Remark 3.4. The bound (19) uses the crude bounds $Pr\{X = VaR_\beta[X]\} \leq 1$ and $Pr\{X_r = VaR_\beta[X_r]\} \leq 1$. This can be improved, e.g., as follows.

The $VaR_\beta[X]$ is monotonically increasing in $\beta$. For any $\delta \in (0, \beta)$ such that $VaR_{\beta - \delta}[X] < VaR_{\beta}[X] < VaR_{\beta + \delta}[X]$ it holds that

$$Pr\{X = VaR_{\beta}[X]\} \leq Pr\{X > VaR_{\beta - \delta}[X]\} \cap \{X \leq VaR_{\beta + \delta}[X]\}$$

$$= Pr\{X > VaR_{\beta - \delta}[X]\} - Pr\{X > VaR_{\beta + \delta}[X]\}$$

$$= 1 - (\beta - \delta) - (1 - (\beta + \delta)) = 2\delta.$$
Note that if the c.d.f. of $X$ is continuous at $\text{VaR}_\beta[X]$, then $\text{VaR}_{\beta-\delta}[X] < \text{VaR}_\beta[X] < \text{VaR}_{\beta+\delta}[X]$ for any $\delta \in (0, \beta)$. The term $\Pr\{X_r = \text{VaR}_\beta[X_r]\}$ can be bounded the same way.

### 3.2. Practical estimation of errors

In practice the $\epsilon$-risk region $G_\epsilon^r[X_r]$, and the errors $\hat{\epsilon}_r^{\max}$, $\hat{\epsilon}_r^{G, \text{low}}$, and $\hat{\epsilon}_r^{G}$ are estimated using sampling. We evaluate the ROM and the full-order model at $n$ random samples and compute the maximum error $\hat{\epsilon}_r^{\max}$ first. With the ordered samples of $X_r$ we obtain $\text{CVaR}_\beta[X_r]$, and consequently the error $\hat{\epsilon}_r^{G, \text{low}}$ in the $\text{CVaR}_\beta[X_r]$ region. Next, the $\epsilon$-risk region $G_\epsilon^r[X_r]$ can be computed, which is needed to compute the error $\hat{\epsilon}_r^{G}$. The details are given in Algorithm 2 below.

**Algorithm 2** Sampling-based estimation of CVaR $\beta$ and its errors with ROM.

**Input:** Parameter samples $\xi^{(1)}, \ldots, \xi^{(n)}$ with probabilities $p^{(1)}, \ldots, p^{(n)}$, risk level $\beta$, random variable $X_r(\xi)$, and its error function $\epsilon_r(\xi)$.

**Output:** Estimate $\hat{\text{CVaR}}_\beta[X_r]$ and error estimates $\hat{\epsilon}_r^{\max}$, $\hat{\epsilon}_r^{G, \text{low}}$, and $\hat{\epsilon}_r^{G}$.

1. Evaluate $X_r$ and $\epsilon_r$ at the parameter samples: $X_r(\xi^{(1)}), \ldots, X_r(\xi^{(n)})$, $\epsilon_r(\xi^{(1)}), \ldots, \epsilon_r(\xi^{(n)})$.
2. Apply Steps 2, 3, and 4 of Algorithm 1 with $X$ replaced by $X_r$ to obtain the index $k_\beta$ and $\text{CVaR}_\beta[X_r]$.
3. Re-order $\epsilon_r(\xi^{(1)}), \ldots, \epsilon_r(\xi^{(n)})$ to match the order of $X_r(\xi^{(1)}), \ldots, X_r(\xi^{(n)})$ from the previous step.
4. Approximate $\hat{\epsilon}_r^{\max}$ and $\hat{\epsilon}_r^{G, \text{low}}$:
   \[
   \hat{\epsilon}_r^{\max} = \max_{1 \leq j \leq n} \epsilon_r(\xi^{(j)}), \quad \hat{\epsilon}_r^{G, \text{low}} = \max_{1 \leq j \leq k_\beta} \epsilon_r(\xi^{(j)}).
   \]
5. Estimate $\hat{\text{VaR}}_\beta[X_r - \epsilon_r]$ by applying Steps 2, 3 and 4 of Algorithm 1 with $X$ replaced by $X_r - \epsilon_r$.
6. Approximate the $\epsilon$-risk region $\{X_r : X_r(\xi^{(j)}) + \epsilon_r(\xi^{(j)}) \geq \text{VaR}_\beta[X_r - \epsilon_r]\}$ by a discrete set
   \[
   \hat{G}_\epsilon^r[X_r] := \{\xi^{(j)} : X_r(\xi^{(j)}) + \epsilon_r(\xi^{(j)}) \geq \text{VaR}_\beta[X_r - \epsilon_r]\}.
   \]
7. Compute
   \[
   \hat{\epsilon}_r^{G} = \max_{\xi^{(j)} \in \hat{G}_\epsilon^r[X_r]} \epsilon_r(\xi^{(j)}).
   \]

### 4. Importance sampling for estimation of CVaR $\beta$

In the previous section, we used a ROM to replace the FOM in the CVaR $\beta$ estimation and showed that the resulting error in CVaR $\beta$ is proportional to the ROM error in
the $\epsilon$-risk region. Thus, this approach works well if a ROM error estimate is available and the ROM error is sufficiently small. To relax these conditions, we now consider an importance sampling (IS) approach to compute $\text{VaR}_\beta$ and $\text{CVaR}_\beta$. The ROM is used to generate a so-called biasing density from which samples are drawn. Given this biasing density, few samples of the expensive FOM are used to estimate $\text{VaR}_\beta$ and $\text{CVaR}_\beta$. While our analysis of the proposed IS approach assumes availability of ROM error bounds to estimate the risk region, our IS approach can be used with fewer assumptions than for the approach in the previous section. Our analysis shows that the performance of our IS approach improves as the ROM error becomes smaller, but as the numerical results in Section 5 show, it is effective even with coarse ROMs.

We begin with a brief introduction of the IS framework in Section 4.1. Section 4.2 derives the optimal IS density for $\text{CVaR}_\beta$ estimation. This optimal IS density itself is impractical, since it relies on evaluations of the expensive FOM. Therefore, we propose an IS density in Section 4.3 that uses the previously introduced $\epsilon$-risk region and, hence, only uses inexpensive evaluations of the ROM. The details of our implementation of the proposed IS approach are provided in Section 4.4.

### 4.1. Importance sampling framework

Importance sampling (IS) estimators use samples from a biasing distribution (a distribution that is biased towards a specific event, e.g., the risk region) to estimate statistics of the quantity of interest. The estimator accounts for the increased occurrence of such events by including re-weighting to compensate within the sampling estimate. For a general introduction to importance sampling, see, e.g., Owen [10, Sec. 9]. Mathematically, IS amounts to changing the density and the following results are stated to allow such a change.

Recall that $\rho$ is the density of the random variable $\xi$. Define the support $\text{supp}(\rho) = \{\xi \in \Xi \mid \rho(\xi) > 0\}$. Let $\varphi$ be another density with $\text{supp}(\rho) \subseteq \text{supp}(\varphi)$. For any integrable function $g : \Xi \rightarrow \mathbb{R}$ we have

$$
\mathbb{E}_\varphi[g] = \int_\Xi g(\xi) \rho(\xi) \, d\xi = \int_\Xi g(\xi) w(\xi) \varphi(\xi) \, d\xi = \mathbb{E}_\varphi[gw],
$$

where $w := \rho/\varphi$ is the so-called likelihood ratio, or IS weight function. The subscript $\varphi$ in $\mathbb{E}_\varphi$ and $\mathbb{V}_\varphi$ indicates that the integrals in the definition of expectation and variance are computed with the density $\varphi$. To derive the IS method for $\text{CVaR}_\beta$ estimation, we make the following assumption throughout this section.

**Assumption 4.1.** The c.d.f. $H_X(x) = \text{Pr}[X \leq x]$ is continuous at $x = \text{VaR}_\beta[X]$.

Under this assumption, $\text{Pr}[X = \text{VaR}_\beta[X]] = 0$ and

$$
\text{CVaR}_\beta[X] = \frac{1}{1 - \beta} \int_\Xi X(\xi) \mathbb{I}_{G_\beta[X]}(\xi) \rho(\xi) \, d\xi,
$$

(22)
We note that the assumptions made in Hong et al. [7] to prove asymptotic properties of CVaRβ imply continuity of the c.d.f. $H_X(x)$ at $x = \text{VaR}_\beta[X]$. We emphasize that while this continuity condition is needed to construct the proposed biasing distribution, our IS procedure can be applied even if this assumption does not hold.

We perform a change of measure from the nominal density $\rho$ to the biasing density $\varphi$ in (22) and account for the change by re-weighting, to obtain

$$\text{CVaR}_\beta[X] = \frac{1}{1 - \beta} \int_\bar{\Xi} I_{G_\beta[X]}(\xi) X(\xi) \rho(\xi) d\xi$$

$$= \frac{1}{1 - \beta} \int_\bar{\Xi} I_{G_\beta[X]}(\xi) X(\xi) w(\xi) \varphi(\xi) d\xi,$$

where

$$w(\xi) := \frac{\rho(\xi)}{\varphi(\xi)}$$

is the weight and $\bar{\Xi}$ is the support of the new density $\varphi$ to be defined later. Recall that $\varphi$ does not need to be positive everywhere, it is sufficient (see, e.g., [10, Chapter 9]) that

$$\varphi(\xi) > 0 \quad \text{for} \quad \xi \in G_\beta[X].$$

Thus we make the following assumption throughout this section.

**Assumption 4.2.** The support $\bar{\Xi}$ of the biasing density $\varphi$ satisfies

$$G_\beta[X] \subset \bar{\Xi}.$$  

The IS estimates $\hat{\text{VaR}}^\text{IS}_\beta[X]$ and $\hat{\text{CVaR}}^\text{IS}_\beta[X]$ are again computed by Algorithm 1, but now we draw independent samples $\xi^{(1)}, \ldots, \xi^{(n)}$ from the biasing distribution $\varphi$, evaluate $X(\xi^{(j)})$, $j = 1, \ldots, n$, and define probabilities $p^{(j)} = w(\xi^{(j)})/n$, $j = 1, \ldots, n$. These are now the inputs into Algorithm 1 when IS is used.

To justify our choice of the biasing density $\varphi$ that we will introduce in Section 4.3 and to analyze the asymptotic properties of the resulting estimates $\text{VaR}^\text{IS}_\beta[X]$ and $\text{CVaR}^\text{IS}_\beta[X]$, we need the following result adapted from [7, Sec. 2.2]. Recall that $\mathbb{E}_\varphi[\cdot]$ and $\mathbb{V}_\varphi[\cdot]$ denote expected value and variance under the measure $\varphi(\xi) d\xi$. In the following result “⇒” denotes convergence in distribution and $\mathcal{N}(0, 1)$ stands for the standard normal distribution.

**Lemma 4.3.** If there exists a $\delta > 0$ and $C > 0$ such that $X$ has a positive and continuously differentiable density $h_X(x)$ for all $x \in (\text{VaR}_\beta[X] - \delta, \text{VaR}_\beta[X] + \delta)$ and the weight function satisfies $w(\xi) \leq C$ for all $\xi \in \{\xi \mid X(\xi) \in (\text{VaR}_\beta[X] - \delta, \infty)\}$, then $\text{VaR}^\text{IS}_\beta[X] \to \text{VaR}_\beta[X]$ w.p. 1 as $n \to \infty$ and

$$\sqrt{n} \left( \text{VaR}^\text{IS}_\beta[X] - \text{VaR}_\beta[X] \right) \Rightarrow \frac{\left( \mathbb{V}_\varphi[I_{G_\beta[X]}(\cdot) w(\cdot)] \right)^{1/2}}{h_X(\text{VaR}_\beta[X])} \mathcal{N}(0, 1).$$
If in addition $\mathbb{E}_\varphi[(X(\cdot) - \text{VaR}_\beta[X])^2_+] w^2(\cdot)] < \infty$, then $\widehat{\text{CVaR}}^{\text{IS}}[X] \to \text{CVaR}_\beta[X]$ w.p. 1 as $n \to \infty$ and
\[
\sqrt{n} \left( \widehat{\text{CVaR}}^{\text{IS}}[X] - \text{CVaR}_\beta[X] \right) \Rightarrow \frac{\mathbb{V}_\varphi((X(\cdot) - \text{VaR}_\beta[X])_+ w(\cdot))}{1 - \beta}^{1/2} \mathcal{N}(0, 1).
\] (25)

**Remark 4.4.** Asymptotic results for the standard Monte Carlo estimates, denoted by $\hat{\text{VaR}}^{\text{MC}}_\beta[X]$ and $\hat{\text{CVaR}}^{\text{MC}}_\beta[X]$ and computed by Algorithm 1 with independent samples $\xi^{(1)}, \ldots, \xi^{(n)}$ from the nominal distribution $\rho$ and equal probabilities $p^{(j)} = 1/n$, $j = 1, \ldots, n$, are a special case of Lemma 4.3 with $w \equiv 1$.

The goal of IS is to compute a biasing density $\varphi$ such that the variance of the estimator is small. We are not aware of an expression for the variance of $\widehat{\text{CVaR}}^{\text{IS}}_\beta[X]$ for fixed $n$. Therefore, we use the asymptotic result (25) and use
\[
\frac{\mathbb{V}_\varphi((X(\cdot) - \text{VaR}_\beta[X])_+ w(\cdot))}{n(1 - \beta)^2}
\] as the ‘variance’ of $\widehat{\text{CVaR}}^{\text{IS}}_\beta[X]$ for fixed $n$.

### 4.2. Deriving the Optimal Biasing Distribution

Before we construct the proposed biasing density $\varphi$, we first compute the optimal biasing density, i.e., the biasing density that gives an estimator $\widehat{\text{CVaR}}^{\text{IS}}_\beta[X]$ with zero variance in equation (26). Although this optimal biasing density is usually impractical (it depends on the quantities to be estimated), it guides us in the construction of a computable biasing density.

**Theorem 4.5.** The biasing density resulting in zero variance in equation (26) is given by
\[
\varphi^* (\xi) = \frac{\mathcal{L} \mathcal{G}_\beta[X](\xi) (X(\xi) - \text{VaR}_\beta[X]) \rho(\xi)}{(1 - \beta) (\text{CVaR}_\beta[X] - \text{VaR}_\beta[X])}.
\] (27)
Proof. We begin by analyzing the variance term in (26), which gives
\begin{align*}
\mathbb{V}_\varphi \left[ (X(\cdot) - \text{VaR}_\beta[X])_+ w(\cdot) \right] \\
= \mathbb{E}_\varphi \left[ (X(\cdot) - \text{VaR}_\beta[X])^2_+ w^2(\cdot) \right] - \left( \mathbb{E}_\varphi \left[ (X(\cdot) - \text{VaR}_\beta[X])_+ w(\cdot) \right] \right)^2 \\
= \int_\mathbb{E} \left( \frac{(X(\xi) - \text{VaR}_\beta[X])_+ \rho(\xi)}{\varphi(\xi)} \right)^2 \varphi(\xi) d\xi - \left( \int_\mathbb{E} \frac{(X(\xi) - \text{VaR}_\beta[X])_+ \rho(\xi)}{\varphi(\xi)} \varphi(\xi) d\xi \right)^2 \\
= \int_\mathbb{E} \left( \mathbb{I}_{G^\beta[X]}(\xi) (X(\xi) - \text{VaR}_\beta[X]) \right)^2 \frac{\rho(\xi)}{\varphi(\xi)} \varphi(\xi) d\xi - \left( \int_\mathbb{E} \mathbb{I}_{G^\beta[X]}(\xi) (X(\xi) - \text{VaR}_\beta[X]) \rho(\xi) d\xi \right)^2 \\
= \int_\mathbb{E} \left( \mathbb{I}_{G^\beta[X]}(\xi) (X(\xi) - \text{VaR}_\beta[X]) \right)^2 \frac{\rho(\xi)}{\varphi(\xi)} \rho(\xi) d\xi - \left( \int_\mathbb{E} \mathbb{I}_{G^\beta[X]}(\xi) (X(\xi) - \text{VaR}_\beta[X]) \rho(\xi) d\xi \right)^2 \\
= \int_\mathbb{E} \left( \mathbb{I}_{G^\beta[X]}(\xi) (X(\xi) - \text{VaR}_\beta[X]) \right)^2 \frac{\rho(\xi)}{\varphi(\xi)} \rho(\xi) d\xi - (1 - \beta)^2 (\text{CVaR}_\beta[X] - \text{VaR}_\beta[X])^2 ,
\end{align*}
where the last identity follows from (22) and (10) with \( \Pr[X = \text{VaR}_\beta[X]] = 0 \). The change from \( \mathbb{E} \) to \( \mathbb{E}^c \) in the second to last inequality is justified, since for \( \xi \in \mathbb{E} \cap (\mathbb{E})^c \) we have \( \mathbb{I}_{G^\beta[X]}(\xi) = 0 \) (see (24)), while for \( \xi \in \mathbb{E} \cap (\mathbb{E})^c \) we have that \( \rho(\xi) = 0 \).

Now, define
\[ U(\xi) := \mathbb{I}_{G^\beta[X]}(\xi) (X(\xi) - \text{VaR}_\beta[X]) \rho(\xi) - (1 - \beta) (\text{CVaR}_\beta[X] - \text{VaR}_\beta[X]) \varphi(\xi). \]
Using the definition of \( U \) and the same arguments as above, we obtain
\begin{align*}
\int_\mathbb{E} \frac{(U(\xi))^2}{\varphi(\xi)} d\xi \\
= \int_\mathbb{E} \left( \mathbb{I}_{G^\beta[X]}(\xi) (X(\xi) - \text{VaR}_\beta[X]) \right)^2 \frac{\rho(\xi)}{\varphi(\xi)} d\xi \\
- 2(1 - \beta) (\text{CVaR}_\beta[X] - \text{VaR}_\beta[X]) \int_\mathbb{E} (X(\xi) - \text{VaR}_\beta[X])_+ \rho(\xi) d\xi \\
+ (1 - \beta)^2 (\text{CVaR}_\beta[X] - \text{VaR}_\beta[X])^2 \int_\mathbb{E} \varphi(\xi) d\xi \\
= \int_\mathbb{E} \left( \mathbb{I}_{G^\beta[X]}(\xi) (X(\xi) - \text{VaR}_\beta[X]) \right)^2 \frac{\rho(\xi)}{\varphi(\xi)} \rho(\xi) d\xi - (1 - \beta)^2 (\text{CVaR}_\beta[X] - \text{VaR}_\beta[X])^2 ,
\end{align*}
which shows that
\[ \mathbb{V}_\varphi [(X(\cdot) - \text{VaR}_\beta[X])_+ w(\cdot)] = \int_\mathbb{E} \frac{(U(\xi))^2}{\varphi(\xi)} d\xi. \]
The density \( \varphi = \varphi^* \) in (27) gives \( U \equiv 0 \).
The optimal density \( \varphi^* \) depends on quantities \( \text{CVaR}_\beta(X) \) and \( \text{VaR}_\beta(X) \) that we want to estimate, it requires knowledge of the true risk region \( G_\beta[X] \), and it requires evaluating \( X \). Thus it is impractical to use. However, it guides us in the construction of the biasing density, which is discussed next.

4.3. Using a ROM to generate the biasing density

The optimal biasing density (27) motivates the initial choice

\[
\varphi(\xi) = \frac{\mathbb{I}_{G_\beta[X]}(\xi) \rho(\xi)}{1 - \beta}.
\]

(Note that \( \Pr[G_\beta[X]] = 1 - \beta \) under our assumption that the c.d.f. \( H_X(x) = \Pr[X \leq x] \) is continuous at \( x = \text{VaR}_\beta[X] \).) This choice (28) is obtained from the optimal density (27) by dropping \( \text{CVaR}_\beta[X], \text{VaR}_\beta[X], \) and \( X \). However, (28) still depends in the risk region of the expensive \( X \). Therefore, we use a ROM and the \( \epsilon \)-risk region (16) to construct our biasing density

\[
\varphi(\xi) := \frac{\mathbb{I}_{G_{\beta}[X_r]}(\xi) \rho(\xi)}{\Pr[\mathbb{I}_{G_{\epsilon}[X_r]}]}, \tag{29}
\]

Since \( G_\beta[X] \subseteq G_{\beta}[X_r] \), the support \( \tilde{\Xi} = G_{\beta}[X_r] \) of this density satisfies (24). A crucial advantage of (29) over (27) or (28) is that its construction requires only inexpensive evaluations of the ROM.

With the density \( \varphi \) in (29) the weight function (23) is given by

\[
w(\xi) = \frac{\Pr[G_{\beta}[X_r]]}{\mathbb{I}_{G_{\beta}[X_r]}(\xi)}.
\]

(30)

Samples \( \xi \) from the biasing distribution \( \varphi \) in (29) satisfy \( \xi \in G_{\beta}[X_r] \). Therefore, \( w \) in (30) is well-defined. Furthermore, for these samples,

\[
1 - \beta \leq w(\xi) = \Pr[G_{\beta}[X_r]] \leq 1.
\]

Note, that the smaller the ROM error \( \epsilon_r \), the closer \( w(\xi) = \Pr[G_{\beta}[X_r]] \) to \( \Pr[G_{\beta}[X]] = 1 - \beta \).

The goal of IS is that the IS estimator has much lower variance than a standard Monte-Carlo estimator (obtained from (26) with \( w \equiv 1 \)). We show next, that our proposed density indeed reduces the variance in equation (26), and that the variance reduction is proportional to the size of the \( \epsilon \)-risk region.

**Theorem 4.6.** The variance (26) corresponding to the \( \widehat{\text{CVaR}}_\beta^{IS}[X] \) estimator with density (29) is reduced by a factor of at least \( \Pr[G_{\beta}[X_r]] \) compared to the standard Monte Carlo estimator \( \widehat{\text{CVaR}}_\beta^{MC}[X] \) (obtained from (26) with \( w \equiv 1 \)) with original density \( \rho \), i.e.,

\[
\frac{\mathbb{Var}_{\varphi} [\mathbb{I}_{G_{\beta}[X]}(\cdot) (X(\cdot) - \text{VaR}_\beta[X])] w(\cdot)}{\mathbb{Var}_{\rho} [\mathbb{I}_{G_{\beta}[X]}(\cdot) (X(\cdot) - \text{VaR}_\beta[X])] \leq \Pr[G_{\epsilon}[X_r]].
\]
Thus, dividing the variance term gives the stated result.

\[ \mathbb{V}_\phi \left[ I_{G_\beta[X]}(\cdot) (X(\cdot) - \text{VaR}_\beta[X]) w(\cdot) \right] \]
\[ = \mathbb{E}_\phi \left[ I_{G_\beta[X]}(\cdot) (X(\cdot) - \text{VaR}_\beta[X])^2 w(\cdot) \right] - \left( \mathbb{E}_\phi \left[ I_{G_\beta[X]}(\cdot) (X(\cdot) - \text{VaR}_\beta[X]) w(\cdot) \right] \right)^2 \]
\[ = \mathbb{E}_\rho \left[ I_{G_\beta[X]}(\cdot) (X(\cdot) - \text{VaR}_\beta[X])^2 w(\cdot) \right] - \left( \mathbb{E}_\rho \left[ I_{G_\beta[X]}(\cdot) (X(\cdot) - \text{VaR}_\beta[X]) \right] \right)^2 \]
\[ = \Pr[G_\beta^r[X_r]] \mathbb{E}_\rho \left[ I_{G_\beta[X]}(\cdot) (X(\cdot) - \text{VaR}_\beta[X]) \right] - \left( \mathbb{E}_\rho \left[ I_{G_\beta[X]}(\cdot) (X(\cdot) - \text{VaR}_\beta[X]) \right] \right)^2 \]
\[ \leq \Pr[G_\beta^r[X_r]] \mathbb{V}_\rho \left[ I_{G_\beta[X]}(\cdot) (X(\cdot) - \text{VaR}_\beta[X]) \right]. \]

Thus, dividing the variance term gives the stated result. \(\square\)

4.4. Implementation details of the IS approach

To execute our IS approach, we need to be able to sample from the distribution with density \( \rho \). Although \( \rho \) only involves an inexpensive ROM, sampling from the distribution with density \( \rho \) is still impossible to do exactly. There are several options including estimating the \( \epsilon \)-risk region and then using acceptance-rejection sampling, or estimating the \( \epsilon \)-risk region and then approximating \( \rho \) using a Gaussian mixture model or kernel density estimation. We have applied both acceptance-rejection sampling and a Gaussian mixture model and found that in our numerical example, where the original density \( \rho \) is constant, acceptance-rejection sampling performed better. Therefore, we describe acceptance-rejection sampling here. Our approach of approximating \( \rho \) using a Gaussian mixture model and corresponding numerical results are given in Appendix B.

First, we approximate the \( \epsilon \)-risk region \( G_\beta^r[X_r] := \{ \xi : X_r(\xi) + \epsilon_r(\xi) \geq \text{VaR}_\beta[X_r - \epsilon_r] \} \) by taking \( m \) samples of the ROM. Next, we employ the acceptance-rejection sampling strategy to generate samples from density \( \rho \).

The acceptance-rejection algorithm for continuous random variables generates samples from a desired distribution \( \phi \) given an easy to sample distribution \( \rho \) (see, e.g., [10], Section 4.7). The acceptance-rejection algorithm assumes the existence of \( C > 0 \) with \( \phi(\xi)/\rho(\xi) \leq C \) for all \( \xi \in \mathbb{X} \). In our case

\[ \frac{\phi(\xi)}{\rho(\xi)} = \frac{I_{G_\beta[X]}(\xi)}{\Pr[G_\beta^r[X_r]]}, \]

and the previous assumption is satisfied with \( C = \left( \Pr[G_\beta^r[X_r]] \right)^{-1} \). Then the standard acceptance-rejection algorithm generates a candidate sample \( \xi_c \) from \( \rho \) and a sample \( u \) from the uniform distribution \( U(0,1) \), and accepts the sample \( \xi_c \) if

\[ u \leq \frac{\phi(\xi_c)}{C\rho(\xi_c)}. \]
In our case,
\[
\frac{\varphi(\xi^c)}{C\rho(\xi^c)} = \mathbb{I}_{G^c_{\beta}[X_r]}(\xi^c).
\]

Thus, in order to accept the sample we just need to check that it belongs to \(G^c_{\beta}[X_r]\) which can be done by evaluating the ROM \(X_r\) and its error function \(\epsilon_r\) at \(\xi^c\).

The ROM-based acceptance-rejection method to generate the samples from a distribution \(\xi\) is given in Algorithm 3 on the next page. Finally, the proposed approach to compute CVaR\(\beta\) via IS is summarized in Algorithm 4 on the next page.

**Algorithm 3** Acceptance-rejection sampling using surrogate model.

**Input:** Surrogate model QoI \(X_r\) with error function \(\epsilon_r\), risk level \(\beta \in (0,1)\), nominal distribution \(\rho\), # of samples \(m\) to estimate \(G^c_{\beta}[X_r]\), desired # of samples \(n\).

**Output:** \(n\) samples from (29) and an estimate of \(\text{Pr}[G^c_{\beta}[X_r]]\).

1. Sample \(m\) inputs \(\{\xi^{(1)}, \ldots, \xi^{(m)}\}\) from the nominal distribution \(\rho(\xi)\) with equal probabilities \(p^{(j)} \equiv 1/m\).
2. Compute ROM QoI values \(X_r(\xi^{(1)}), \ldots, X_r(\xi^{(m)})\).
3. Compute ROM error function values \(\epsilon_r(\xi^{(1)}), \ldots, \epsilon_r(\xi^{(m)})\).
4. Apply Steps 2, 3, and 4 of Algorithm 1 with \(X_r\) replaced by \(X_r - \epsilon_r\) to obtain \(\hat{\text{VaR}}_{\beta}[X_r - \epsilon_r]\).
5. Estimate \(\epsilon\)-risk region \(\hat{G}^c_{\beta}[X_r] := \{\xi^{(j)} : X_r(\xi^{(j)}) + \epsilon_r(\xi^{(j)}) \geq \hat{\text{VaR}}_{\beta}[X_r - \epsilon_r]\}\).
6. Estimate \(\text{Pr}[G^c_{\beta}[X_r]]\) as \(|\hat{G}^c_{\beta}[X_r]|/m\) where \(|\cdot|\) denotes cardinality of a set.
7. Set \(n_a = 0\) (counter of accepted samples), \(n_c = 0\) (counter of candidate samples).
8. while \(n_a \leq n\) do
9. Generate candidate sample \(\xi^c\) from \(\rho(\xi)\).
10. Compute ROM QoI value \(X_r(\xi^c)\).
11. Compute ROM error function value \(\epsilon_r(\xi^c)\).
12. if \(X_r(\xi^c) + \epsilon_r(\xi^c) \geq \hat{\text{VaR}}_{\beta}[X_r - \epsilon_r]\) then
13. Accept sample \(\xi^c\), set \(n_a = n_a + 1\).
14. end if
15. Set \(n_c = n_c + 1\).
16. end while

In the Algorithm 3, \(n_c\) is used to keep track of the number of candidate samples generated from \(\rho\). This number can be estimated as follows:

\[
n_c \approx n/\hat{\text{Pr}}[G^c_{\beta}[X_r]].
\]

Thus, the total cost of generating \(n\) biased samples is approximately

\[
m + n/\hat{\text{Pr}}[G^c_{\beta}[X_r]] \leq m + n/\hat{\text{Pr}}[G_{\beta}[X_r]] = m + \frac{n}{1-\beta} \tag{31}
\]

ROM evaluations.
**Algorithm 4** Estimating CVaR\(_\beta\) using importance sampling with ROM.

**Input:** FOM QoI \(X\), ROM QoI \(X_r\) with error function \(\epsilon_r\), risk level \(\beta \in (0, 1)\), nominal distribution \(\rho\), # of evaluations \(m\) of the ROM \(X_r\) and \(n\) or the FOM \(X\).

**Output:** Importance sampling estimate \(\hat{\text{CVaR}}_{\beta}[X]\).

1. Apply the acceptance-rejection Algorithm 3 to generate \(n\) samples from the biasing density (29) and obtain the estimate \(\hat{\text{Pr}}[G_{\beta}^\epsilon[X_r]]\) of \(\text{Pr}[G_{\beta}^\epsilon[X_r]]\).
2. Compute FOM outputs \(X(\xi^{(1)}), \ldots, X(\xi^{(n)})\).
3. Assign values of weight function \(w = \rho/\varphi\): \(w(\xi^{(1)}), \ldots, w(\xi^{(n)}) \equiv \hat{\text{Pr}}[G_{\beta}^\epsilon[X_r]]\).
4. Apply Algorithm 1 with \(X(\xi^{(j)}), p^{(j)} = w(\xi^{(j)})/n, j = 1, \ldots, n\), to obtain \(\hat{\text{CVaR}}_{\beta}^{\text{IS}}[X]\).

5. **Numerical results**

We present numerical results for our approach for estimating CVaR\(_\beta\) using ROMs. Section 5.1 introduces the PDE model, followed by a detailed description of its discretization and reduced-order modeling in Section 5.2. The numerical results are presented and discussed in Sections 5.3 and 5.4.

### 5.1. Convection-diffusion-reaction PDE model

We consider a simplified model of a premixed combustion flame at constant and uniform pressure, and follow the notation and set-up in [1, Sec. 3]. The model includes a one-step reaction of the species

\[
2H_2 + O_2 \rightarrow 2H_2O
\]

in the presence of an additional non-reactive species, nitrogen. The state is comprised of the components \(y = [T, Y_F, Y_O, Y_P]\), with the \(Y_i\) being the mass fractions of the species fuel (F, here \(H_2\)), oxidizer (O, here \(O_2\)), product (P, here \(H_2O\)), and \(T\) denoting the temperature. The physical combustor domain is 1.8 cm in length, and 0.9 cm in height. Thus \(\Omega = (0, 1.8) \times (0, 0.9)\). Dirichlet boundary conditions are specified on the left boundary \(\Gamma_D = \{0\} \times [0, 0.9]\) and homogeneous Neumann boundary conditions are specified in the top, bottom and right boundary \(\Gamma_N = \partial \Omega \setminus \Gamma_D\). The velocity field \(U\) is set to be constant in the \(x_1\) direction, and divergence free. The molecular diffusivity \(\kappa\) is modeled as constant, equal and uniform for all species. Constants in the nonlinear reaction term are random variables with values in \(\Xi\). For a given \(\xi \in \Xi\) the state equation is the system of diffusion-advection-reaction PDEs

\[
\begin{align*}
0 = \kappa \Delta y(x) - U(x) \nabla y(x) + \mathcal{N}(y(x), \xi) & \quad x \in \Omega, \quad (32a) \\
y(x)|_{\Gamma_D} = y_D(x) & \quad x \in \Gamma_D, \quad (32b) \\
\nabla y(x) \cdot n = 0 & \quad x \in \Gamma_N. \quad (32c)
\end{align*}
\]

The left boundary \(\Gamma_D\) is divided into three equal parts. The bottom and top third of the left boundary are held at \(T = 300\,\text{K}\) while the mass fractions are prescribed as
Table 1: Parameters for the PDE model (32)–(33) from [1].

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Physical meaning</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa$</td>
<td>molecular diffusivity</td>
<td>$2 \text{ cm}^2/\text{s}$</td>
</tr>
<tr>
<td>$U$</td>
<td>velocity</td>
<td>$50 \text{ cm}/\text{s}$</td>
</tr>
<tr>
<td>$W_{H_2}$</td>
<td>molecular weight</td>
<td>$2.016 \text{ g/mol}$</td>
</tr>
<tr>
<td>$W_{O_2}$</td>
<td>molecular weight</td>
<td>$31.9 \text{ g/mol}$</td>
</tr>
<tr>
<td>$W_{H_2O}$</td>
<td>molecular weight</td>
<td>$18 \text{ g/mol}$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>density of mixture</td>
<td>$1.39 \times 10^{-3} \text{ g/cm}^3$</td>
</tr>
<tr>
<td>$R$</td>
<td>univ. gas constant</td>
<td>$8.314 \text{ J mol/K}$</td>
</tr>
<tr>
<td>$Q$</td>
<td>heat of reaction</td>
<td>$9800 \text{ K}$</td>
</tr>
</tbody>
</table>

zero Dirichlet conditions. The middle third of the left boundary is the inflow boundary, where the incoming unburned mixture has temperature $T = 950 \text{ K}$ and mass fractions $Y_{H_2} = 0.0282, Y_{O_2} = 0.2259, Y_{H_2O} = 0$. The nonlinear reaction term $\mathcal{N}(y, \xi) = [\mathcal{N}_T, \mathcal{N}_F, \mathcal{N}_O, \mathcal{N}_P](y, \xi)$ is of Arrhenius type and modeled as

$$
\mathcal{N}_i(y, \xi) = -\nu_i \left( \frac{W_i}{\rho} \right) \left( \frac{\rho Y_F}{W_F} \right)^{\nu_F} \left( \frac{\rho Y_O}{W_O} \right)^{\nu_O} A \exp \left( -\frac{E}{RT} \right), \quad i = F, O, P \quad (33a)
$$

$$
\mathcal{N}_T(y, \xi) = Q \cdot \mathcal{N}_P(y, \xi). \quad (33b)
$$

The uncertain parameters of the model are considered to be

$$
\xi = [A, E],
$$

with values in the parameter domain

$$
\Xi = [A_{\text{min}}, A_{\text{max}}] \times [E_{\text{min}}, E_{\text{max}}] = [5.5 \times 10^{11}, 1.5 \times 10^{13}] \times [1.5 \times 10^3, 9.5 \times 10^3]. \quad (34)
$$

The random variable $\xi$ is uniformly distributed, i.e., $\rho$ is constant. The other parameters are defined in Table 1.

### 5.2. Discretization and reduced-order models

This section discusses the discretization of the PDE (32), which is our FOM, the quantity of interest, and the computation of ROMs.

#### 5.2.1. Full-order model

The PDE model is discretized using a finite difference approximation in two spatial dimensions, with a $72 \times 36$ grid, leading to $N = 10,804$ unknowns in the discretized model. Let $y$ be the vector with components corresponding to the approximations of the state $y$ at the grid points. The resulting nonlinear system becomes

$$
0 = Ky + \mathcal{N}(y; \xi), \quad (35)
$$
with boundary conditions as described above. Here, \( K \in \mathbb{R}^{N \times N} \) is the discretized representation of the derivative operators. The nonlinear system is solved with Newton’s method. Let \( T(\cdot) \in \mathbb{R}^{N/4} \) be the vector with components corresponding to the approximations of the temperature \( T(x, \cdot) \) at the grid points. Given the uncertainty in the input parameters, the quantity of interest is the random variable

\[
X : \Xi \mapsto \mathbb{R}, \quad X(\xi) = \exp \left( \frac{\|T(\xi)\|_\infty - 2000}{100} \right). \tag{36}
\]

The quantity of interest is non-dimensionalized, and is large when the maximum temperature is above 2000 K. For this example, the maximum temperature is between 1122 K and 2435 K, and therefore the quantity of interest is between \( 1.5 \times 10^{-4} \) and \( 7.8 \times 10^1 \). The maximum temperature \( \|T(\xi)\|_\infty \) and the QoI \( X(\xi) \) at parameters in \( \Xi \) are shown in Figure 1.

![Figure 1: Maximum temperature \( \|T(\xi)\|_\infty \) in K and the QoI \( X(\xi) \) for \( \xi = (A, E) \in \Xi \), computed using the FOM.](image)

The cumulative distribution function of the quantity of interest, together with \( \text{VaR}_{\beta}[X] \) and \( \text{CVaR}_{\beta}[X] \) for \( \beta = 0.95 \) are shown in Figure 2. These quantities are estimated with \( 10^4 \) samples of the FOM and included for illustration only. Here, \( \text{VaR}_{0.95}[X] = 43.92 \) and \( \text{CVaR}_{0.95}[X] = 53.94 \).

Figure 2 and \( \text{CVaR}_{\beta}[X] \) are generated from a large number of FOM samples for illustration only. The goal of this paper is to show that \( \text{CVaR}_{\beta}[X] \) can be estimated with far fewer FOM evaluations using ROMs. Next we describe how these ROMs are computed for this example.

### 5.2.2. Reduced-order Models

We use Proper Orthogonal Decomposition (POD) to compute reduced-order models of the form

\[
0 = K_r y_r + N_r(y_r; \xi),
\]
where the subscript $r$ is a label of the ROM. Here, $\mathbf{K}_r = V_r^T \mathbf{K} V_r$, $y \approx V_r y_r$ and we also project the boundary conditions. The projection matrix $V_r \in \mathbb{R}^{N \times N_r}$, $N_r \ll N$, is computed via singular value decomposition of snapshot data $\mathbf{Y} = [y(\xi^{(1)}), \ldots, y(\xi^{(S)})] \in \mathbb{R}^{N \times S}$, where $S = 100$ snapshots were generated from solutions of (35) at $10 \times 10$ equally-spaced values $A$ and $E$ in $\Xi$. Since the nonlinearity is of exponential type, we use the Discrete Empirical Interpolation Method (DEIM [3]) for an efficient evaluation of the nonlinear term. Four different surrogate models are built from $r = 1, 2, 3, 4$ POD basis vectors and the same number of DEIM selection points, respectively. A detailed description of the model reduction for this example is given in [1]. The surrogate models then define a new random variable for the (non-dimensional) quantity of interest, namely

$$X_r : \Xi \mapsto \mathbb{R}, \quad X_r(\xi) = \exp \left( \frac{||T(\xi)||_{\infty} - 2000}{100} \right),$$

where $T(\xi)$ is the first block of length $N/4$ in $V_r y_r$.

Estimates of the errors $\epsilon_r(\xi)$ in the QoI for the ROMs with $r = 1, 2, 4$ are shown in Figure 3 (note the different error-bar magnitudes). The error plot for ROM $X_3$ is somewhat similar to that of ROM $X_4$ and is omitted because of space limitations. To generate these plots, we compute $\epsilon_r(\xi)$ exactly for parameter values $\xi$ on the $10 \times 10$ parameter grid used to generate the ROMs, and then interpolate between these values. Figure 3 shows that the maximum errors vary significantly for these four ROMs. In particular, the error in ROM 1 is huge relative to the size of the QoI $X$, see Figures 1 and 3a. This is due to the structure of the QoI (36). In our case, the ROM maximum temperature exceeds the FOM maximum temperature $||T(\xi)||_{\infty}$ in the bottom right corner of the parameter domain, where $||T(\xi)||_{\infty} > 2000$ K. This difference is magnified by the exponential function. For example, if the FOM $||T(\xi)||_{\infty} = 2400$ K and the ROM approximation exceeds this value by only 10%, then $X_r(\xi) = e^{6.4} \approx 601.9$ exceeds $X(\xi) = e^{4} \approx 54.6$ by a factor of 11.
5.3. Estimating CVaR $\beta$ by sampling from ROMs

Following our algorithm and error analysis in Section 3, we use the ROMs $X_r$ with $r = 1, 2, 3, 4$ basis functions to estimate $\text{CVaR}_\beta[X_r]$ with $\beta = 0.95$. Section 5.3.1 presents the computed risk regions for the FOM and the ROM. Section 5.3.2 computes further ROM errors that occur in the bound for the $\text{CVaR}_\beta$ approximation. Section 5.3.3 then presents the results for $\text{CVaR}_\beta$ estimation.

5.3.1. Risk regions of FOM and ROM

Estimates of the FOM risk region, of the ROM risk regions and of the $\epsilon$-risk regions are shown in Figure 4. These regions are computed using five independent batches of $10^4$ samples of the parameters $A$ and $E$ – one batch for the FOM and one batch for each ROM. In every case the ROM risk region is in the bottom right part of the domain $\Xi$. However, the sizes of the ROM risk regions vary. A sufficiently accurate ROM always gives a good approximation of the true risk region, as is the case with the finest ROM $r = 4$. As expected from Lemma 3.2, the estimated $\epsilon$-risk regions $\tilde{G}^{\epsilon}[X_r]$ contain the estimated FOM risk region $\tilde{G}^{\epsilon}[X]$ and shrink as the ROM becomes more accurate. Note that the large number of samples ($n = 10^4$) of the FOM is used only for illustration of the FOM risk region and the error estimates derived in this paper. However, these FOM samples are not used in our ROM or ROM-IS approach to estimate $\text{CVaR}_\beta$.

5.3.2. ROM errors in risk-regions

Table 2 shows the estimate of the error $\tilde{\epsilon}_{\max}$ between the FOM and the ROMs in all of $\Xi$ (see (14)), the error $\tilde{\epsilon}_{G,\text{low}}$ between the FOM and the ROM in the ROM risk region $\tilde{G}_\beta[X_r]$ (see (18)), and the error $\tilde{\epsilon}_G$ between the FOM and the ROM in the $\epsilon$-risk region $\tilde{G}^{\epsilon}_\beta[X_r]$ (see (18)). To compute these estimates we use the linear interpolant of the error function $\epsilon_r(\xi)$ displayed in Figure 3, evaluate it at the same $10^4$ random parameter values that were used to obtain the risk regions, and compute errors as in Algorithm 2. In this example, the maximum errors occur in the risk region (compare columns two and four...
Figure 4: Top row: Estimated FOM risk region $\hat{\mathbb{G}}_{\beta} [X]$. Middle row: Estimated ROM risk regions $\hat{\mathbb{G}}_{\beta} [X_r]$, $r = 1, 2, 3, 4$. Bottom row: Estimated $\epsilon$-risk regions $\hat{\mathbb{G}}_{\beta}^\epsilon [X_r]$ of ROMs $r = 1, 2, 3, 4$. In all plots $\beta = 0.95$; samples in the risk regions are shown as yellow stars, all other samples as blue circles.
of Table 2).

Table 2: Estimates of errors between the FOM and the ROMs. The maximum error over the entire parameter domain is $\hat{\epsilon}_{r, \text{max}}$, the maximum error in the ROM risk region $\tilde{G}_\beta[X_r]$ is given by $\hat{\epsilon}_{r, \text{low}}$ and the maximum error in the $\epsilon$-risk region $\tilde{G}_\beta^\epsilon[X_r]$ is $\hat{\epsilon}_{r, \text{G, low}}$.

<table>
<thead>
<tr>
<th>ROM, r</th>
<th>$\hat{\epsilon}_{r, \text{max}}$</th>
<th>$\hat{\epsilon}_{r, \text{low}}$</th>
<th>$\hat{\epsilon}_{r, \text{G}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>776.00</td>
<td>776.00</td>
<td>776.00</td>
</tr>
<tr>
<td>2</td>
<td>24.47</td>
<td>24.47</td>
<td>24.47</td>
</tr>
<tr>
<td>3</td>
<td>9.04</td>
<td>8.72</td>
<td>9.04</td>
</tr>
<tr>
<td>4</td>
<td>0.96</td>
<td>0.96</td>
<td>0.96</td>
</tr>
</tbody>
</table>

5.3.3. CVaR$_\beta$ estimates with ROM and FOM

Table 3 shows the CVaR$_\beta$ estimates with $\beta = 0.95$ for the FOM and the ROMs. These estimates were obtained using Algorithm 1 with the $n = 10^4$ samples shown in Figure 4a (for FOM) and in Figures 4b–4e (for ROMs). We denote these CVaR$_\beta$ estimates by CVaR$_\beta^{\text{MC}}$ to distinguish them from the estimates obtained with IS in the next section. The third column shows the radius of the 95% confidence interval of the respective estimate. We give more details on this computation below. The radius of the 95% confidence interval is a measure of the error between the MC estimate $\hat{\text{CVaR}}_{\beta}^{\text{MC}}$ and the true CVaR$_\beta$ for the given FOM QoI or ROM QoI. The last two columns of Table 3 compare $\hat{\text{CVaR}}_{\beta}^{\text{MC}}[X_r]$ with $\text{CVaR}_{\beta}^{\text{MC}}[X]$.

Table 3: Estimates of CVaR$_\beta$ with $\beta = 0.95$ for FOM and ROMs $r = 1, 2, 3, 4$ (estimated with $10^4$ MC samples and Algorithm 1). Absolute and relative errors are computed with respect to the FOM estimate $\text{CVaR}_{\beta}^{\text{MC}}[X] = 53.94$.

<table>
<thead>
<tr>
<th>CVaR$_\beta^{\text{MC}}$</th>
<th>CI radius</th>
<th>Abs error</th>
<th>Rel error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>FOM</td>
<td>53.94</td>
<td>1.09</td>
<td>—</td>
</tr>
<tr>
<td>ROM 1</td>
<td>361.40</td>
<td>17.89</td>
<td>307.47</td>
</tr>
<tr>
<td>ROM 2</td>
<td>44.80</td>
<td>0.46</td>
<td>9.14</td>
</tr>
<tr>
<td>ROM 3</td>
<td>49.91</td>
<td>0.92</td>
<td>4.02</td>
</tr>
<tr>
<td>ROM 4</td>
<td>53.87</td>
<td>1.04</td>
<td>0.07</td>
</tr>
</tbody>
</table>

Tables 2 and 3 show that

$$\left| \hat{\text{CVaR}}_{\beta}^{\text{MC}}[X] - \text{CVaR}_{\beta}^{\text{MC}}[X_r] \right| \leq \hat{\epsilon}_{r, \text{G}}$$
in this example. In general this error depends on the bound (20) (Figure 2 indicates that the c.d.f.’s of $X$ and $X_r$ are continuous) as well as on the MC sampling error.

Note that the $\hat{\text{CVaR}}_{\beta}^{\text{MC}}[X]$ estimate, computed with $10^3$ expensive FOM samples can be approximated with high accuracy using ROMs that only require 100 FOM evaluations to generate the ROM. In particular the estimate $\hat{\text{CVaR}}_{\beta}^{\text{MC}}[X_3]$ obtained with the second best ROM, has an absolute error 4.02 that is close to the CI width 1.09 of the FOM MC estimator. The estimate $\hat{\text{CVaR}}_{\beta}^{\text{MC}}[X_4]$ obtained with the best ROM, happens to have an absolute error that is even below the CI width 1.09 of the FOM MC estimator. Thus in this example, a good ROM can speed up (measured in FOM evaluations) by a factor 100.

The confidence interval is derived from the asymptotic results in [7, Sec. 2.1, 2.2] (see also Lemma 4.3). Specifically, the $100(1 - \alpha)\%$ confidence interval (CI) for $\text{CVaR}_\beta[X]$ is

$$
\left[ \hat{\text{CVaR}}_{\beta}^{\text{MC}}[X] - z_\alpha \frac{\hat{\kappa}_\beta}{\sqrt{n}}, \hat{\text{CVaR}}_{\beta}^{\text{MC}}[X] + z_\alpha \frac{\hat{\kappa}_\beta}{\sqrt{n}} \right],
$$

(37)

where $z_\alpha = \Phi^{-1}(1 - \alpha/2)$ with $\Phi$ being the c.d.f. of the standard normal variable, and $
\hat{\kappa}_\beta = \psi_\beta/(1 - \beta)$ with

$$
(\hat{\psi}_\beta)^2 = \frac{1}{n} \sum_{j=1}^{n} \mathbb{I}_{\hat{G}_\beta[X]}(\xi(j)) \left( X(\xi(j)) - \hat{\text{VaR}}_{\beta}^{\text{MC}}[X] \right)^2 w(\xi(j))^2

- \left( \frac{1}{n} \sum_{j=1}^{n} \mathbb{I}_{\hat{G}_\beta[X]}(\xi(j)) \left( X(\xi(j)) - \hat{\text{VaR}}_{\beta}^{\text{MC}}[X] \right) w(\xi(j)) \right)^2,
$$

(38)

where

$$
\mathbb{I}_{\hat{G}_\beta[X]}(\xi) = \begin{cases} 1, & \text{if } X(\xi) \geq \hat{\text{VaR}}_{\beta}^{\text{MC}}[X], \\ 0, & \text{else}, \end{cases}
$$

and $w(\xi) \equiv 1$. The confidence intervals for $\text{CVaR}_\beta[X_r]$ of ROMs $r = 1, 2, 3, 4$ are constructed in a similar manner. The third column of Table 3 reports the radius (half the width) of the corresponding confidence interval with $z_\alpha = 1.96$, which gives a 95% confidence interval.

Note that $(\hat{\psi}_\beta)^2$ is an estimate for the variance $\mathbb{V}_\rho \left[ (X(\cdot) - \text{VaR}_\beta[X])_+ \right]$ which appears in Lemma 4.3 and [7, Eqn. (8)]. We do not have a proof of consistency of $\hat{\psi}_\beta$, which is beyond the scope of this paper, but refer the reader to the discussion of a similar result regarding the consistency of an estimator of variance for $\text{VaR}_\beta$ which can be found in [4].

### 5.4. Estimating CVaR$_\beta$ via ROM-informed importance sampling

Next we use importance sampling to estimate $\text{CVaR}_\beta[X]$. Section 5.4.1 shows the computed estimates. By using IS, we reduce the variance of the estimator compared to a

---

2We include $w$ here because [38] will be used in the next section with non-constant IS weight function.
standard MC estimator. We discuss variance reduction for this example in Section 5.4.2 and show its alignment with the theoretical results.

5.4.1. Importance sampling estimates of CVaR

We generate IS samples from the biasing densities as in Section 4.4. Since we consider multiple ROMs, we label $\varphi_r$ as the biasing density corresponding to ROM $X_r$. The estimates $\hat{\text{CVaR}}^\text{IS}_\beta[X]$ with densities $\varphi_r$, $r = 1, 2, 3, 4$, are reported in Table 4. The estimates are obtained using Algorithm 4 with $m = 10^4$ ROM evaluations to explore the risk regions. We use $n_c = n/\Pr[G_\beta[X_r]] \leq n/0.05 = 2 \cdot 10^3$ ROM evaluations in the acceptance-rejection step to get $n = 100$ samples (see (31)). Only $n = 100$ FOM evaluations are used to construct the IS estimates. The reference value is $\text{CVaR}^\text{ref}_\beta := \hat{\text{CVaR}}^\text{MC}_\beta[X] = 53.94$ from $10^4$ samples as reported in Table 3. We define the 95% confidence intervals as in (37) with $\hat{\text{CVaR}}^\text{MC}_\beta[X]$ substituted by $\hat{\text{CVaR}}^\text{IS}_\beta[X]$, $w$ given by the IS weight, and $z_\alpha = 1.96$. The reported results are averaged over $K = 100$ independent trials. The presented CVaR$\beta$ estimates are the average values over these trials, similarly the radii of the confidence intervals, and the absolute and relative errors are the average values of respective quantities computed for each independent trial. We refer to average absolute error as mean absolute error (MAE) and average relative error as mean relative error (MRE):

$$
\text{MAE} = \frac{1}{K} \sum_{k=1}^{K} \left| \hat{\text{CVaR}}^\text{IS}_\beta^{(k)}[X] - \text{CVaR}^\text{ref}_\beta[X] \right|, \quad \text{MRE} = \frac{\text{MAE}}{\text{CVaR}^\text{ref}_\beta[X]} \times 100\%,
$$

where $\hat{\text{CVaR}}^\text{IS}_\beta^{(k)}[X]$ is the estimate obtained on $k$-th trial and $K = 100$ is the number of trials. As can be seen from Table 4, the absolute errors of the averaged estimates are smaller than the presented average absolute errors. Additionally, we compute the mean-squared error (MSE) of each estimate as follows:

$$
\text{MSE} = \sum_{k=1}^{K} \left( \hat{\text{CVaR}}^\text{IS}_\beta^{(k)}[X] - \text{CVaR}^\text{ref}_\beta[X] \right)^2.
$$

Table 3 showed that coarse ROMs lead to CVaR$\beta$ estimates with larger errors when substituted for the FOM (as is expected from our theory). However, Table 4 shows that ROMs can be used to build effective biasing densities. For example, the absolute errors for $\hat{\text{CVaR}}^\text{MC}_\beta[X_r]$ with ROMs 1–3 in Table 3 are substantially larger than for the importance sampling estimate $\hat{\text{CVaR}}^\text{IS}_\beta[x]$ that used ROMs 1–3 to compute biasing densities, see Table 4.

5.4.2. Variance reduction via IS

Table 5 reports the estimated reduction in variance of IS estimators of CVaR$\beta$ compared to MC estimators, as well as the computed upper bound. The variances of $\hat{\text{CVaR}}^\text{MC}_\beta[X]$
Table 4: Estimates $\widehat{\text{CVaR}}_{\beta}^{\text{IS}}[X]$ with $\beta = 0.95$ obtained using Algorithm 4 with reduced-order models $X_r$, $r = 1, 2, 3, 4$. Here $m = 10^4$ ROM samples are used to identify the $\epsilon$-risk region, and the IS estimator uses $n = 100$ samples. Absolute and relative errors are computed with respect to the FOM estimate from Table 3. All values are averaged over $K = 100$ trials.

<table>
<thead>
<tr>
<th>IS 1</th>
<th>Av $\widehat{\text{CVaR}}_{\beta}^{\text{IS}}[X]$</th>
<th>Av CI radius</th>
<th>MAE (39)</th>
<th>MRE (%) (39)</th>
<th>MSE (40)</th>
</tr>
</thead>
<tbody>
<tr>
<td>IS 1</td>
<td>54.02</td>
<td>5.19</td>
<td>1.99</td>
<td>3.70</td>
<td>6.44</td>
</tr>
<tr>
<td>IS 2</td>
<td>54.39</td>
<td>4.26</td>
<td>1.59</td>
<td>2.96</td>
<td>4.22</td>
</tr>
<tr>
<td>IS 3</td>
<td>53.74</td>
<td>2.89</td>
<td>1.20</td>
<td>2.23</td>
<td>2.17</td>
</tr>
<tr>
<td>IS 4</td>
<td>53.94</td>
<td>1.61</td>
<td>0.66</td>
<td>1.22</td>
<td>0.65</td>
</tr>
</tbody>
</table>

Table 5: Estimated variance reduction (4.6) computed with 100 samples for IS densities $r = 1, 2, 3, 4$. The right-most column reports estimated probability of $\overline{G}_\beta[X_r]$.  

<table>
<thead>
<tr>
<th>IS 1</th>
<th>$\overline{V}<em>\phi[\text{CVaR}</em>{\beta}^{\text{IS}}[X]]/\overline{V}<em>\rho[\text{CVaR}</em>{\beta}^{\text{MC}}[X]]$</th>
<th>$\overline{\text{Pr}}[G_{\beta}[X_r]]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>IS 1</td>
<td>0.2258</td>
<td>0.2463</td>
</tr>
<tr>
<td>IS 2</td>
<td>0.1519</td>
<td>0.1771</td>
</tr>
<tr>
<td>IS 3</td>
<td>0.0691</td>
<td>0.0967</td>
</tr>
<tr>
<td>IS 4</td>
<td>0.0214</td>
<td>0.0519</td>
</tr>
</tbody>
</table>
and $\text{CVaR}_{\beta}^{\text{IS}}[X]$ are estimated using 100 samples from $\rho$ and $\varphi$ respectively and are averaged over 100 trials. The second column of Table 5 shows the resulting ratios. The relative sizes of the $\epsilon$-risk regions, $\Pr[G_{\beta}[X_r]]$, are estimated using the ROM samples as described in Algorithm 3 and are reported in the third column of Table 5. Theorem 4.6 shows that the density (29) leads to a reduction of the variance from equation (26) by a factor $\Pr[G_{\beta}[X_r]]$. The observed variance reduction in Table 5 is in agreement with the theoretical estimate from Theorem 4.6. Observe, that as the ROMs become more accurate, $\epsilon_r \to 0$, and $\hat{\Pr}[G_{\beta}[X_r]] \to 1 - \beta = 0.05$. Moreover, note that using ROM 4 results in a variance reduction by a factor $1/0.0214 \approx 47$, which shows the strength of the IS approach.

5.5. Benefits of using ROMs–Computational Budget Comparisons

Since the ROM construction requires FOM evaluations, the question is, would we be better off using MC or IS with only the FOM? This sections shows that in general the answer is no, it is better to use some FOM evaluations to construct a ROM and then use inexpensive ROM evaluations.

Computing the $\hat{\text{CVaR}}_{\beta}^{\text{MC}}[X_r]$ estimates in Table 3 required an initial 100 FOM samples to train the ROMs; but afterwards no additional FOM evaluations were needed. In comparison, computing the estimate $\hat{\text{CVaR}}_{\beta}^{\text{IS}}$ also required 100 FOM samples to train the ROMs, and then another $n = 100$ FOM samples to sample from the biasing density. (In practice one would not average over many runs, so we do not count the cost of those repetitions.) We next compare the ROM and ROM-IS estimators to an estimator that uses 200 FOM samples to directly estimate $\text{CVaR}_{\beta}$.

Table 6 reports the $\text{CVaR}_{\beta}$ estimate obtained using only FOM evaluations. The MC estimate with 200 samples (MC 200 in Table 5) is at best as good as IS 1 from Table 4 with respect to MSE, MAE, etc. Therefore, the proposed strategy that includes ROMs to get a biasing distribution is more accurate and computationally efficient.

Table 6: Estimates $\hat{\text{CVaR}}_{\beta}^{\text{MC}}[X]$ with $\beta = 0.95$ obtained using $n = 100$ (MC 100) and $n = 200$ (MC 200) samples. The estimate $\hat{\text{CVaR}}_{\beta}^{\text{IS}}[X]$ (IS FOM) also uses $n = 200$ samples, namely 100 samples to build the biasing distribution with the FOM by fitting a Gaussian Mixture Model (see the supplement (B)) and then 100 IS samples. All values are averaged over $K = 100$ trials.

<table>
<thead>
<tr>
<th></th>
<th>Av $\hat{\text{CVaR}}_{\beta}[X]$</th>
<th>Av CI radius</th>
<th>MAE (39)</th>
<th>MRE (%) (39)</th>
<th>MSE (40)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MC 100</td>
<td>52.78</td>
<td>10.45</td>
<td>4.14</td>
<td>7.68</td>
<td>27.11</td>
</tr>
<tr>
<td>MC 200</td>
<td>53.39</td>
<td>7.52</td>
<td>3.66</td>
<td>6.78</td>
<td>20.31</td>
</tr>
<tr>
<td>IS FOM</td>
<td>57.29</td>
<td>0.02</td>
<td>28.12</td>
<td>52.13</td>
<td>988.57</td>
</tr>
</tbody>
</table>

For comparison Table 6 also reports the result of IS with FOM evaluations only. Acceptance-rejection sampling using only FOM evaluations is too expensive: given an
estimate of the risk region, we would need $n_c \approx n / (1 - \beta) = 2 \cdot 10^3$ FOM evaluations to generate $n = 100$ samples. Therefore, we use IS with the Gaussian mixture model. Specifically, we use 100 MC FOM samples—with only five samples falling into risk region—and fit a biasing density using a Gaussian mixture model (see Appendix B). We then sample 100 parameters from the obtained density and report the resulting estimate as IS FOM in Table 6. This estimate has the largest error of all, mostly due to the small number of samples used to fit the mixture model. Therefore, it appears beneficial to invest the 100 FOM samples to build a ROM and follow the importance sampling strategy described above.

Figure 5 presents several of the 100 trials averaged in Tables 4 and 6 in more detail. It supports the previous observation that investing 100 FOM samples to build a ROM and then generate many inexpensive ROM samples to build a biasing density is beneficial since it substantially reduces the variation of the resulting estimate, given an overall budget of FOM evaluations.

Figure 5: Estimates $\hat{\text{CVaR}}_{\beta}^{\text{IS}}(X)$ for $k = 1, \ldots, 100$ with 95% confidence intervals. Red dotted line corresponds to $\text{CVaR}^{\text{ref}}_{\beta}[X]$. 

(a) MC 200.
(b) IS 2.
(c) IS 3.
(d) IS 4.
6. Conclusions

We have presented two methods to estimate CVaR$_\beta$ with the help of reduced-order models, together with analyses of their performances. One method directly works with the ROM, and the second approach uses the IS framework to reduce the number of high-fidelity samples needed for CVaR$_\beta$ estimation.

For the first approach in Section 3, we showed that the CVaR$_\beta$ estimation error when using a ROM instead of the high-fidelity model is proportional to the ROM error in the $\epsilon$-risk region of the ROM. Since the $\epsilon$-risk region is small relative to the entire parameter region, this can improve the CVaR$_\beta$ error estimate. Moreover, it could be used to compute ROMs more efficiently.

For the second approach, we derived the optimal biasing distribution for the IS framework, and used it to derive a biasing distribution that is computed from using only ROM information. We proved that the variance resulting from the proposed biasing distribution is reduced at least by a factor equal to the probability of the risk-region. This factor is small and is asymptotically (as the ROM error goes to zero) equal to $1 - \beta \ll 1$.

Both approaches were applied to CVaR$_\beta$ estimation of a quantity of interest related to heat release modeled by a system of diffusion-advection-reaction PDEs derived from a simple combustion model. For this example, our ROM CVaR$_\beta$ estimation error from Section 3 reduced the computational cost (measured in FOM evaluations) by a factor $10^2$. The importance sampling framework led to substantially better CVaR$_\beta$ estimates when coarser ROMs are used compared to simply replacing FOM samples by ROM samples, but at the expense of using $10^2$ additional FOM samples. With the same budget of 200 total FOM evaluations, the MC estimates based on the FOM is at best as good as the IS estimate computed with the coarsest ROM 1; more accurate ROMs improve the estimation and especially lead to smaller confidence intervals. With the most accurate ROM 4, our importance sampling framework reduced the variance of the CVaR$_\beta$ estimator by a factor of about 47 compared to the standard MC estimator. Overall, our numerical results showed that it appears beneficial to invest FOM samples to train a ROM—which is then used to compute a biasing distribution—than to sample from the FOM directly.

The results in this paper point to future work. Currently, we generate ROMs once from a given number of high-fidelity model evaluations. Since we only need the ROM error to be small in the small $\epsilon$-risk region one could alternate ROM improvement and $\epsilon$-risk region estimation to adaptively generate ROMs and overall use fewer high-fidelity model evaluations.

This supplement contains additional numerical results. In particular, Section A summarizes the performance of our approach on a different QoI for which CVaR$_\beta$ is closer to $\text{VaR}_\beta$, and Section B reports results obtained when the IS biasing density (29) is approximated using a Gaussian mixture model.
A. Another quantity of interest

The quantity of interest can have a significant effect on the computation of VaR\textsubscript{\beta} and CVaR\textsubscript{\beta} and in turn, on the efficiency of our proposed method. In addition to the QoI (36) we also considered the quantity of interest

\[ X : \Xi \mapsto \mathbb{R}, \quad X(\xi) = \|T(\xi)\|_\infty. \]  

(41)

The values of the QoI (41) are in K. The CVaR\textsubscript{\beta} estimation results using our ROM techniques are similar to those reported for the QoI (36) in the previous sections. Therefore, we do not report detailed results, but only point out some interesting differences compared to the QoI (36).

Figure 6 shows the c.d.f. with the values of VaR\textsubscript{\beta}[X] and CVaR\textsubscript{\beta}[X] for \beta = 0.95 for the QoI (41). Note, that the tail of the c.d.f. is much shorter and, consequently, that VaR\textsubscript{\beta}[X] and CVaR\textsubscript{\beta}[X] are much closer. Thus, any sample in the short tail will give \(X(\xi) \approx \text{VaR}_\beta[X] \approx \text{CVaR}_\beta[X]\) and therefore the estimation of CVaR\textsubscript{\beta}[X] is easier in this case than it was for the QoI (36). While we still observed gains by using our ROM approaches to estimate CVaR\textsubscript{\beta}[X], the benefit of using ROMs is less than it was for the QoI (36)—simply because estimating CVaR\textsubscript{\beta}[X] of the QoI (41) is easier.

Estimates of the errors \(\epsilon_r(\xi)\) in the QoI (41) for the ROMs with \(r = 1, 2, 3, 4\) are shown in Figure 7 (note the different error-bar magnitudes). For the QoI (41) the estimates of the FOM risk region, of the ROM risk regions and of the \(\epsilon\)-risk regions happen to be virtually identical to those shown in Figure 4. Comparing the ROM error plots in Figure 7 with the location of the risk regions in Figures 4b–4e shows that in this case, the ROM error \(\epsilon_r^\beta\) in the risk region is significantly smaller than the global ROM error \(\epsilon_{\text{max}}^\beta\). This demonstrates the benefit of our error estimate in Theorem 3.3, where the accuracy of the \(\hat{\text{CVaR}}_\beta[X_r]\) estimate was upper bounded by the ROM error in the \(\epsilon\)-risk region. For this example, this error is much lower than the error in the overall domain, which was used previously in (15) to bound the \(\text{CVaR}_\beta[X_r]\) error.
(a) Error of ROM 1, $\epsilon_1(\xi)$.
(b) Error of ROM 2, $\epsilon_2(\xi)$.
(c) Error of ROM 3, $\epsilon_3(\xi)$.
(d) Error of ROM 4, $\epsilon_4(\xi)$.

Figure 7: Absolute errors between the QoI $X$ [41] and the ROMs $X_r$ for $r = 1, 2, 3, 4$. Note the different magnitudes of the error bars.
B. Importance sampling densities via a Gaussian mixture model

As mentioned in Section 4.4, we can approximate the biasing density (29) using a Gaussian mixture model (GMM). In this section we describe our GMM approach and numerical results obtained with it.

Again we start by approximating the $\epsilon$-risk region $G_{\beta}^\epsilon \left[ X_r \right] := \{ \xi : X_r(\xi) + \epsilon_r(\xi) \geq \text{VaR}_\beta [X_r - \epsilon_r] \}$ by taking $m$ samples of the reduced-order model. Define

$$J^\epsilon = \{ j \in \{1, \ldots, m\} : \xi^{(j)} \in \hat{G}_{\beta}^\epsilon \left[ X_r \right] \}.$$

Next, we fit a GMM to the parameters in the estimated $\epsilon$-risk region. Consider the p.d.f.

$$\psi(\xi, \mu, \Sigma) = \frac{1}{(2\pi)^{M/2} |\Sigma|^{1/2}} \exp \left( -\frac{1}{2} (\xi - \mu)^T \Sigma^{-1} (\xi - \mu) \right)$$

of the $M$-dimensional multivariate normal distribution with mean $\mu \in \mathbb{R}^M$ and covariance $\Sigma \in \mathbb{R}^{M \times M}$. The p.d.f. of a GMM is the weighted sum of $k$ normal distributions

$$\psi_{\text{mix}}(\xi; \mu_1, \ldots, \mu_k; \Sigma_1, \ldots, \Sigma_k; \alpha) = \sum_{i=1}^k \alpha_i \psi(\xi; \mu_i, \Sigma_i),$$

with $\mu_1, \ldots, \mu_k \in \mathbb{R}^M$, $\Sigma_1, \ldots, \Sigma_k \in \mathbb{R}^{M \times M}$, and mixture weights $\alpha$ such that $\sum_{i=1}^k \alpha_i = 1$. The samples $\xi^{(j)} \in \hat{G}_{\beta}^\epsilon \left[ X_r \right]$, $j = 1, \ldots, |J^\epsilon|$, in the estimated $\epsilon$-risk region are used to fit the parameters of the GMM by solving the following maximum likelihood problem

$$\max_{\mu_1, \ldots, \mu_k; \Sigma_1, \ldots, \Sigma_k} \prod_{j \in J^\epsilon} \psi_{\text{mix}}(\xi^{(j)}; \mu_1, \ldots, \mu_k; \Sigma_1, \ldots, \Sigma_k; \alpha)$$

s.t. $\sum_{i=1}^k \alpha_i = 1$. (42a)

Problem (42) is well-analyzed in machine learning literature and there exist efficient algorithms to solve it (see, e.g., [6]). One of the ways to solve (42) is to employ the expectation-maximization (EM) algorithm. In MATLAB it is implemented by calling `gmdistribution.fit`. Algorithm 5 below summarizes the computational procedure for generating the biasing distribution from a reduced-order model and its error function $\epsilon_r(\xi)$. At the last step, when calling `gmdistribution.fit`, one of the provided options is a priori chosen number $k$ of Gaussian mixture components. In our numerical experiments we use

$$k = \min \left\{ \lceil |J^\epsilon|/2 \rceil, 5 \right\}.$$

Gaussian mixture components. In summary, we approximate the density (29) by

$$\varphi(\xi) = \psi_{\text{mix}}(\xi; \mu_1, \ldots, \mu_k; \Sigma_1, \ldots, \Sigma_k; \alpha),$$

which is the density used in the computational experiments in this paper. The ROM-based method to generate the biasing density is given in Algorithm 5. Finally, the proposed approach to compute CVaR$_\beta$ via importance sampling is summarized in Algorithm 6.
Algorithm 5 Generating biasing distribution using surrogate model.

**Input:** Surrogate model QoI $X_r$ with error function $\epsilon_r$, risk level $\beta \in (0, \beta)$, nominal distribution $\rho$, # of samples $m$.

**Output:** Biasing distribution $\varphi(\xi)$.

1. Sample $m$ inputs $\{\xi(1), \ldots, \xi(m)\}$ from the nominal distribution $\rho(\xi)$ with equal probabilities $p^{(j)} \equiv 1/m$.
2. Compute ROM QoI values $X_r(\xi(1)), \ldots, X_r(\xi(m))$.
3. Compute ROM error function values $\epsilon_r(\xi(1)), \ldots, \epsilon_r(\xi(m))$.
4. Apply Steps 2, 3, and 4 of Algorithm 1 with $X$ replaced by $X_r - \epsilon_r$ to obtain $\widehat{\text{VaR}}_{\beta}^r[X_r - \epsilon_r]$.
5. Estimate $\epsilon$-risk region $G_{\beta}^r[X_r] := \{\xi(j) : X_r(\xi(j)) + \epsilon_r(\xi(j)) \geq \widehat{\text{VaR}}_{\beta}^r[X_r - \epsilon_r]\}$.
6. Fit a mixture model distribution $\varphi$ to the risk region $G_{\beta}^r[X_r]$, e.g., for MATLAB users: gmdistribution.fit($G_{\beta}^r[X_r]$, options).

Algorithm 6 Estimating $\text{CVaR}_\beta^r$ using importance sampling with ROM.

**Input:** FOM QoI $X$, ROM QoI $X_r$ with error function $\epsilon_r$, risk level $\beta \in (0, 1)$, nominal distribution $\rho$, # of evaluations $m$ of the ROM $X_r$ and $n$ or the FOM $X$.

**Output:** Importance sampling estimate $\text{CVaR}_\beta^IS[X]$.

1. Generate the GMM biasing distribution $\varphi(\xi)$ using Algorithm 5 with $m$ samples from the nominal distribution $\rho$ of the ROM $X_r$.
2. Sample $n$ inputs $\{\xi(1), \ldots, \xi(n)\}$ from distribution $\varphi$.
3. Compute FOM outputs $X(\xi(1)), \ldots, X(\xi(n))$.
4. Compute values $w(\xi(1)), \ldots, w(\xi(n))$ of weight function $w = \rho/\varphi$.
5. Apply Algorithm 1 with $X(\xi(j))$, $p^{(\xi)} = w(\xi(j))/n$, $j = 1, \ldots, n$, to obtain $\text{CVaR}_\beta^IS[X]$.

### B.1. Numerical implementation and results

Due to the large variations of parameters, see equation (34), we scale the $A$ parameter by $10^{-10}$ before fitting the biasing distributions. Once samples are drawn, we scale the $A$ parameter back again.

The estimates $\text{CVaR}_\beta^IS[X]$ with densities $\varphi_r$, $r = 1, 2, 3, 4$, are reported in Table 7 below. The estimates are obtained using Algorithm 6 with $m = 10^4$ samples (ROM evaluations) to explore the risk regions and only $n = 100$ samples (FOM evaluations) to construct the estimates. The reference value is $\text{CVaR}_\beta^{ISf} := \text{CVaR}_\beta^{MC}[X] = 53.94$ from $10^4$ samples as reported in Table 3. We define the 95% confidence intervals as in (37) with $\text{CVaR}_\beta^{MC}[X]$ substituted by $\text{CVaR}_\beta^IS[X]$, $w$ given by the IS weight, and $z_\alpha = 1.96$. The reported results are averaged over $K = 100$ independent trials. The presented $\text{CVaR}_\beta$ estimates are the average values over these trials, similarly the radii of the confidence intervals, and the absolute and relative errors are the average values of respective quantities computed for each independent trial.
Table 7: Estimates $\hat{\text{CVaR}}^\text{IS} \beta [X] \text{ with } \beta = 0.95$ obtained using Algorithm \text{\textit{4}} with reduced-order models $X_r, r = 1, 2, 3, 4$. Here $m = 10^4$ ROM samples are used to generate the biasing density, and the IS estimator uses $n = 10^2$ samples. Absolute and relative errors are computed with respect to the FOM estimate from Table 3. All values are averaged over $K = 100$ trials.

<table>
<thead>
<tr>
<th>IS</th>
<th>$\text{Av } \hat{\text{CVaR}}^\text{IS} \beta [X]$</th>
<th>$\text{Av CI radius}$</th>
<th>MAE (39)</th>
<th>MRE (%) (39)</th>
<th>MSE (40)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>53.63</td>
<td>6.86</td>
<td>3.28</td>
<td>6.08</td>
<td>15.64</td>
</tr>
<tr>
<td>2</td>
<td>54.31</td>
<td>6.17</td>
<td>2.80</td>
<td>5.19</td>
<td>12.41</td>
</tr>
<tr>
<td>3</td>
<td>54.39</td>
<td>3.97</td>
<td>1.61</td>
<td>2.98</td>
<td>3.94</td>
</tr>
<tr>
<td>4</td>
<td>54.03</td>
<td>2.54</td>
<td>1.02</td>
<td>1.89</td>
<td>1.71</td>
</tr>
</tbody>
</table>

Table 8 below reports the estimated reduction in variance of IS estimators of CVaR $\beta$ compared to MC estimators. The variances of $\text{CVaR}^\text{MC} \beta [X]$ and $\text{CVaR}^\text{IS} \beta [X]$ are estimated using $10^2$ samples from $\rho$ and $\varphi$ respectively and are averaged over 100 trials. The second column of Table 8 shows the resulting ratios. In particular using ROM 4 results in a variance reduction by a factor $1/0.0545 \approx 18$. The relative sizes of the $\epsilon$-risk regions, $\text{Pr}[G^\epsilon \beta [X_r]]$, are estimated using the ROM samples that were used in the construction of the biasing densities and are reported in the third column of Table 8. The observed variance reduction is larger than the estimate of $\text{Pr}[G^\epsilon \beta [X_r]]$ for the more accurate ROMs $r = 3$ and $r = 4$. The results did not change fundamentally when we increased the sample size to $10^4$. Observe, that as the ROMs become more accurate, $\epsilon_r \to 0$, and $\text{Pr}[G^\epsilon \beta [X_r]] \to 1 - \beta = 0.05$.

Table 8: Estimated variance reduction (4.6) computed with $10^2$ samples for IS densities $r = 1, 2, 3, 4$. The right-most column reports estimated probability of $\hat{G}^\epsilon \beta [X_r]$.

<table>
<thead>
<tr>
<th>IS</th>
<th>$\hat{\text{V}}<em>{\hat{\varphi}^\beta} [\text{CVaR}^\text{IS} \beta [X]]/\hat{\text{V}}</em>{\hat{\rho}^\beta} [\text{CVaR}^\text{MC} \beta [X]]$</th>
<th>$\text{Pr}[G^\epsilon \beta [X_r]]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.4119</td>
<td>0.2383</td>
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<tr>
<td>2</td>
<td>0.3276</td>
<td>0.1791</td>
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<tr>
<td>3</td>
<td>0.1331</td>
<td>0.0964</td>
</tr>
<tr>
<td>4</td>
<td>0.0545</td>
<td>0.0528</td>
</tr>
</tbody>
</table>

We suspect that the error between the ‘ideal’ biasing density (29) and the actually used biasing density from Algorithm \text{\textit{5}} is the reason for the discrepancy between Theorem 4.6 and Table 8. Figure 8 below shows the computed biasing densities $\varphi_r$ computed by Algorithm 5 and the $\epsilon$-risk regions $G^\epsilon \beta [X_r]$ computed with ROMs $r = 2, 3, 4$. The yellow stars indicating the risk region are plotted at height $\hat{\text{Pr}}[G^\epsilon \beta [X_r]]$ and approximate the ‘ideal’ biasing density (29). The computed biasing densities show variations, which are
bigger for coarser ROMs.

Figure 8: The biasing densities $\varphi_2$ to $\varphi_4$ computed by Algorithm 5 and the $\epsilon$-risk regions $G_{\beta}[X_r]$ (yellow stars). The yellow stars indicating the risk region are plotted at height $\hat{\text{Pr}}[G_{\beta}[X_r]]$ and approximate the ‘ideal’ biasing density (29).

An important note with regards to Figure 8 is that the densities plotted here are the ones that were fitted to the re-scaled parameters $A$ and $E$. As noted at the beginning of this subsection, due to the large difference in the magnitudes of $A$ and $E$, we scale the $A$ parameter by $10^{-10}$ before fitting the biasing distributions. The magnitude of the $z$-axis in Figure 8 reflects this change of scale.
References


[12] B. Peherstorfer, B. Kramer, and K. Willcox, Combining multiple surrogate models to accelerate failure probability estimation with expensive high-fidelity models,


