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Generalised polynomial chaos expansion approaches to approximate stochastic model predictive control†

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This paper considers the model predictive control of dynamic systems subject to stochastic uncertainties due to parametric uncertainties and exogenous disturbance. The effects of uncertainties are quantified using generalised polynomial chaos expansions with an additive Gaussian random process as the exogenous disturbance. With Gaussian approximation of the resulting solution trajectory of a stochastic differential equation using generalised polynomial chaos expansion, convex finite-horizon model predictive control problems are solved that are amenable to online computation of a stochastically robust control policy over the time horizon. Using generalised polynomial chaos expansions combined with convex relaxation methods, the probabilistic constraints are replaced by convex deterministic constraints that approximate the probabilistic violations. This approach to chance-constrained model predictive control provides an explicit way to handle a stochastic system model in the presence of both model uncertainty and exogenous disturbances.

Keywords: stochastic model predictive control; generalised polynomial chaos expansions; chance constraints; Boole's inequality; convex relaxation; semidefinite programming; stochastic differential equations

1. Introduction

In recent years, stochastic programming formulations for model predictive control (MPC, aka receding horizon control) have been intensively studied in the context of many different areas of application including robot and vehicle path planning (Blackmore & Ono, 2009; Blackmore, Ono, Bektassov, & Williams, 2010; Blackmore, Ono, & Williams, 2011), network traffic control (Yan & Bitmead, 2005), chemical processes (Li, Wendt, & Wozny, 2000; Schwarm & Nikolaou, 1999; van Hessem & Bosgra, 2004) and economics (Couchman, Cannon, & Kouvaritakis, 2006; Herzog, Dondi, & Geering, 2007; Zhu, Li, & Wang, 2004). In such control problems, stochastic models are represented in terms of stochastic differential equations (SDEs) with the stochasticity resulting from exogenous disturbances, plant/model mismatches and sensor noise.

Robust MPC formulations can be categorised as being either deterministic or stochastic, based on the representation of the uncertainties. Deterministic robust MPC (e.g., see Bemporad & Morari, 1999; Campo & Morari, 1987; Wang, 2002 and references therein) analyses the stability and performance of systems against worst-case perturbations with the resulting optimisations being min-max problems that are computationally demanding to solve directly and so are typically replaced by approximate solutions that are more amenable to implementation. The worst-case perturbations may have a vanishingly small probability of occurring in practice, but any such information on probabilities is not taken into account in a deterministic formulation. Analysis or design based on worst-case uncertainties can be too conservative to be applied in practice, may result in an over-design of process equipment, or can result in infeasibility during real-time optimisation. From a practical point of view, it is rare that an engineer knows exactly what value for hard bounds to specify on the uncertainty (e.g. knows that the hard bound on uncertainty in a parameter should be exactly 10.6% instead of 11.3%), and a small perturbation in these bounds can mean the difference because a closed-loop system being robust to the uncertainties or being unstable.

Most parameter estimation algorithms generate models with probabilistic descriptions of the uncertainties. For such models, robustness characterisations are intrinsically stochastic and can be written in terms of a probability distribution or a level of confidence in estimates with probabilistic risk of incorrectness. Contrary to deterministic robust MPC, stochastic robust MPC incorporates such probabilistic uncertainties and probabilistic violations of constraints, and allows for specified levels of risk during operation. Commonly used probabilistic analysis approaches are

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Monte Carlo (MC) methods, in which many simulations are run with sampled random variables or random processes. The effects of uncertainty on the closed-loop system are quantified by simulating a large number of individual deterministic model realisations. While such MC approaches are applicable to most systems, the computational cost can be prohibitively expensive, especially in real-time optimal control algorithms such as MPC. Apart from simulation-based methods, convex relaxations and approximations for a receding horizon method of the constrained discrete-time stochastic control are considered in Cinquemani, Agarwal, Chatterjee, and Lygeros (2011), in which convexity of the resultant optimisation is carried out in the basis of robust optimisation (Ben-Tal & Nemirovski, 2002; Bertsimas, Brown, & Caramanis, 2011) that includes robust linear programmes and more generally robust convex programmes (see Ben-Tal, Ghaoui, & Nemirovski, 2009 for details of robust convex optimisation). However, such robust optimisation formulations of chance constraints are not applicable to the cases when the stochastic dynamical system has nonlinear parametric uncertainties, whereas this paper can manage the system model that is linear parameter-varying Gaussian, for which the system matrices have nonlinear dependence of random variables and there are additive Gaussian random processes corresponding to external disturbance and measurement noise.

The high computational cost of simulation-based methods has motivated the development of computationally efficient methods for uncertainty analysis that replace or accelerate MC methods (Ghanem & Spanos, 1991; Maitre & Kino, 2010; Xiu, 2010). The MPC formulation in this paper uses generalised polynomial chaos (gPC) expansions, which is a spectral method to approximate the solution of an SDE that has stochastic parametric uncertainties and exogenous disturbances. Polynomial chaos expansions were first introduced for turbulence modelling with the uncertainties being Gaussian random variables (Wiener, 1938), with later extensions considering other types of common probability distributions (Xiu & Karniadakis, 2002). Recently, many researchers have demonstrated the use of (generalised) polynomial chaos expansions as a computationally efficient alternative to MC approaches for the analysis and control of uncertain systems (Fisher & Bhattacharya, 2009; Hover, 2006, 2008; Kim, 2013; Kim & Braatz, 2012b; Nagy & Braatz, 2007, 2010). In Fisher (2008) and Fisher and Bhattacharya (2011), the gPC expansion is applied to formulate optimal trajectory generation problems in the presence of random uncertain parameters.

This article also presents several probabilistic collision conditions that are functions of the mean and covariance of the trajectory. We show that a gPC expansion that can provide an approximation of the solution of an SDE, in which both system parameters and exogenous disturbances are stochastic, converges in the mean-square sense as the number of terms in the expansion increases. The proposed approximation results in a convex optimisation for the control policy that does not use any sampling and is amenable to online computation.

This paper is organised as follows. Section 2 presents some mathematical background on gPC expansions. Section 3 states a stochastic MPC formulation with a chance constraint and Section 4 presents several probabilistic conditions in terms of chance constraints in which the conditions are functions of the mean and covariance of a random variable. The main results are presented in Section 5 where a gPC expansion is used in place of the exact solution of an SDE in the MPC formulation, which enables the control policy to be computed as the solution to a convex optimisation under the assumption that the computed approximate solution in the presence of stochastic parametric uncertainties and exogenous disturbances is Gaussian. Section 6 provides a numerical example to illustrate the proposed approach for stochastic MPC of dynamic systems subject to stochastic parametric uncertainty and exogenous disturbance. Section 7 presents heuristic convex semi-chance constrained MPC problems using gPC expansions and discusses several issues related to gPC expansion theory and associated computations. Section 8 concludes the paper.

Notation

The following notation is used throughout this article: $\langle \cdot, \cdot \rangle$ is the inner product in a Hilbert space $\mathcal{H}$; **Pr** is the probability; **E** or $\bar{(\cdot)}$ is the expectation or mean; **Var** or $\Sigma_{(\cdot)}$ is the variance or covariance; $\mathcal{N}(a, b)$ is the Gaussian distribution with the mean a and the variance b; $\mathcal{U}(S)$ is the uniform distribution with the support set $S \subset \mathbb{R}^n$; the symbol $\sim$ means "distributed as;" and erf$(\cdot)$ is the Gaussian error function. For a given sequence of vectors $x_i \in \mathbb{R}^{n_i}$, col(x_i) refers to $[x_1^{\mathrm{T}}, \ldots, x_k^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^n$ where $n = \sum_{i=1}^{k} n_i$. The matrix with diagonal blocks formed with matrices $A_1, \ldots, A_m$ and the other entries are all zeros is denoted by diag$(A_1, \ldots, A_m)$. $\mathbb{S}^n \subset \mathbb{R}^{n \times n}$ refers to the set of real symmetric matrices and its subsets $\mathbb{S}^n_+$ and $\mathbb{S}^n_{++}$ are used to denote the set of positive semidefinite and definite matrices, respectively. We also use $A \succeq 0$ for $A \in \mathbb{S}^n_+$ and $A \succ 0$ for $A \in \mathbb{S}^n_{++}$.

2. Theoretical background

2.1 Characteristics of gPC expansions

Polynomial approximations are commonly used when implementing functions on a computing system with the basic assumption being that a finite sum of polynomials can accurately approximate a function of interest. For polynomial approximations, orthogonal polynomials are often used, with their properties reviewed below.

2.1.1 Orthogonality

Consider a measure space $(\mathcal{X}, \mathcal{M}, \mu)$ where $\mathcal{X}$ is a nonempty set equipped with a σ-algebra $\mathcal{M}$ and a measure

μ. A set of orthogonal polynomials $\{\phi_n(x)\}$ for $x \in \mathcal{M}$ is defined by the orthonormality relation

$$\langle \phi_n, \phi_m \rangle \triangleq \int_{\mathcal{X}} \phi_n(x)\phi_m(x)d\mu(x) = \begin{cases} 1 \text{ if } n = m, \\ 0 \text{ otherwise.} \end{cases} \quad (1)$$

2.1.2 Recurrence relation

Any set of orthogonal polynomials $\{\phi_n(x)\}$ on the real line satisfies a three-term recurrence formula (Ball, 1999)

$$x\phi_n(x) = a_{n+1}\phi_{n+1}(x) + b_n\phi_n(x) + a_n\phi_{n-1}(x) \quad (2)$$

for $n = 0, 1, \ldots$ Along with $\phi_{-1}(x) = 0$, this formula holds consistently and ϕ_0 is always a constant.

2.1.3 Parameterisation of random inputs

For any analysis of a stochastic system, the random inputs must be specified and characterised appropriately.

2.1.3.1 Random variables. Consider the concatenated parameter vector $\theta : \Omega \to \Theta \subseteq \mathbb{R}^{n_\theta}$ that is a random vector defined on the events Ω, where the set Θ is assumed to be known. The true parameter θ^* that is a realisation of a random variable θ is assumed to be in the set, and the statistics of the random variable θ is known. For a given probability distribution of a random parameter of a system, the first step of analysis using a gPC expansion is to transform the parameters to a set of independent random normalised variables known as *standard random variables* (Isukapalli, 1999). Performing this step involves finding a diffeomorphism T: $\Xi \to \Theta$ such that $\theta = T(\zeta)$ for $\zeta \in \Xi$ and the state of a stochastic model x has equivalent representations $x(z, t; \theta(\omega)) = x(z, t; \zeta(\omega))$.

2.1.3.2 Random process and the KL expansion. The inputs of a system can also be random processes. The Karhunen-Loeve (KL) expansion has been used to represent the stochastic input quantities in stochastic systems and is compatible with spectral methods for system identification and analysis using gPC expansions; i.e. the KL expansion provides a natural way to parameterise the random process inputs so that such a parameterisation can be exploited in the spectral analysis to construct basis functions. The details of KL expansions are not presented here due to limited space; readers are referred to Ghanem and Spanos (1991), Maitre and Kino (2010), Xiu (2010).

2.2 Universal approximation and convergence of polynomial expansions

The Hermite polynomial chaos expansion has a universal approximation property for expanding second-order random processes in terms of orthogonal polynomials (Cameron & Martin, 1947), and second-order random processes are processes with finite variance, which applies to most physical processes (Xiu & Karniadakis, 2002).

Theorem 2.1 Cameron-Martin Theorem (Cameron & Martin, 1947)**:** *For any functional f in a Hilbert measure space $(\mathcal{X}, \mathcal{M}, \mu)$, there exist a set of polynomials $\{\phi_i\}$ and constants $\{a_i\}$ such that*

$$\lim_{N\to\infty} \int_{\mathcal{X}} (f(x) - \hat{f}_N(x))^2 d\mu(x) = 0, \quad (3)$$

where $\hat{f}_N(x) \triangleq \sum_{i=0}^{N} a_i\phi_i(x)$ and a_j is obtained from the Galerkin projection $a_i := \frac{\langle f,\phi_i\rangle}{\|\phi_i\|^2}$.

The proof of this result is not trivial (interested readers are referred to Cameron & Martin, 1947). The rate of convergence depends on the smoothness of the function f and the type of orthogonal polynomial basis functions $\{\phi_i\}$ used for approximation, and this subject has been heavily studied (e.g., see Newman & Raymon, 1969; Xiu, 2010). The limiting behavior of the approximation error $\|f - \hat{f}_N\|$ is $O(N^{-p})$ where p denotes the differentiability of the function $f : \mathcal{X} \to \mathcal{Y}$ and $O(\epsilon)$ (recall that $O(\epsilon) \to 0$ as $\epsilon \to 0$). For an analytic function f, the convergence rate is exponential, i.e. $\|f - \hat{f}_N\|$ is $O(e^{-\alpha N})$ for some constant $\alpha > 0$.

3. Problem statement

Consider a stochastic discrete-time linear parameter-varying system:

$$x_{t+1} = A(\delta)x_t + B_u(\delta)u_t + B_w(\delta)w_t, \quad (4)$$

where $\delta \in \Delta$ denotes the concatenation of the parametric uncertainties and $A : \Delta \to \mathbb{R}^{n\times n}$, $B_u : \Delta \to \mathbb{R}^{n\times m}$, and $B_w : \Delta \to \mathbb{R}^{n\times n_w}$ are uncertain system matrices. Assume that $w \in \mathbb{R}$ is a Gaussian white-noise process with known distribution, and the initial state x_0 and uncertainty δ are random variables with known probability density functions (pdfs). Under this stochasticity of parameters and disturbance, the solution trajectory of system (4) is a random process for which the main goal of analysis is to compute or approximate the statistical properties and the main goal of synthesis is to drive the random process x_t to have a desirable statistics.

In finite-horizon stochastic MPC, the goal is to determine a control policy $\mu_T \triangleq (u_0, \ldots, u_T)$ that solves the optimisation

$$\begin{aligned} &\min_{\mu_T} J(\bar{x}_0, \Sigma_{x_0}, \mu_T) \\ &\text{s.t. } x_{t+1} = A(\delta)x_t + B_u(\delta)u_t + B_w(\delta)w_t, \\ &y_t = Cx_t, \ \mathbf{Pr}[y_t \notin \mathcal{F}_y] \geq \beta, \\ &u_t \in \mathcal{U}, \ \text{for } t = 0, \ldots, T, \\ &w_t \sim f_w, \ \delta \sim f_\delta, \ x_0 \sim f_{x_0}, \end{aligned} \quad (5)$$

where $\mathcal{F}_y$ denotes the forbidden region for the output y_t and β is a lower bound of probabilistic collision avoidance.[1]

4. Feasibility of chance constraints: probabilistic collision checking

This section presents four different ways of formulation of chance constraints corresponding to probabilistic collision avoidance. In particular, for a motion-planning problem for a mobile system, it is necessary to impose constraints on the (controlled) state or output variables. Such constraints have the form of a vector inequality $\eta(x) \leq 0$, where $x \in \mathcal{X} \subset \mathbb{R}^n$ refers to the state variables and the function $\eta : \mathbb{R}^n \to \mathbb{R}^m$. Due to stochastic nature of the state variables, it is natural to introduce so-called *chance constraints* of the form $\Pr[\eta(x) \leq 0] \geq \alpha$ where $\alpha \in (0, 1)$ denotes a level of confidence. For a probabilistic collision avoidance problem, the formulation of chance constraints depends on the representation of obstacles and mobile agents that have stochastic uncertainty.

4.1 Obstacles as point masses in a large work space

The probability of collision to obstacles at time t and in the work space $W \subset \mathbb{R}^{n_s}$, $n_s \leq 3$, can be defined as (Lambert, Gruyer, & Pierre, 2008; Toit & Burdick, 2011)

$$P_t^c \triangleq \int_{x^v} \int_{x^a} I_c\left(x_t^v, x_t^a\right) dF_{va}\left(x_t^v, x_t^a\right), \tag{6}$$

where $F_{va}(\cdot, \cdot)$ is the joint cumulative distribution function (cdf), the indicator function for collision is defined by

$$I_c(x^v, x^a) \triangleq \begin{cases} 1, & \text{for } \mathcal{X}_v(x^v) \bigcap \mathcal{X}_a(x^a) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

$\mathcal{X}_v(x^v)$ and $\mathcal{X}_a(x^a)$ are the regions occupied by the vehicle and the obstacle whose global reference coordinates are x^v and x^a, respectively. Equipped with this definition of probabilistic collision, the chance constraint $\mathbf{Pr}[y_t \notin \mathcal{F}_y] \geq \beta$ in (5) can be rewritten as $P_t^c \leq 1 - \beta$. Consider the obstacles as point masses, which occurs when the volume of $\mathcal{X}_v(x^v)$ is much smaller than the work space W for all $x^v \in W$ and the volume of $\mathcal{X}_a(x^a)$ is 0 for all $x^a \in W$.

Lemma 4.1 (Lemma 1 in Toit and Burdick, 2011)**:** *For obstacles as point masses, suppose that $x^v \sim \mathcal{N}(\bar{x}^v, \Sigma_{x^v})$ and $x^a \sim \mathcal{N}(\bar{x}^a, \Sigma_{x^a})$ are independent Gaussian random variables. Then $P^c \leq 1 - \beta$ can be rewritten as the constraint on $(\bar{x}^v, \Sigma_{x^v}, \bar{x}^a, \Sigma_{x^a})$:*

$$(\bar{x}^v - \bar{x}^a)^T \Sigma_x^{-1} (\bar{x}^v - \bar{x}^a) \geq -2 \ln\left(\frac{1-\beta}{V_v}\sqrt{\det(2\pi\,\Sigma_x)}\right), \tag{7}$$

where $\Sigma_x = \Sigma_{x^v} + \Sigma_{x^a}$ and V_v is the volume of the vehicle.

The constraint (7) is not convex in $(\bar{x}^v, \bar{x}^a)$ even for a fixed β, but is concave in $(\bar{x}^v, \bar{x}^a)$ due to positive definiteness of the inverse covariance matrix Σ_x^{-1}. However, a method of semidefinite programming (SDP) relaxation can be used to check its feasibility and solve related optimisations approximately.

Convex relaxation: Suppose that $\bar{x}^v$ is affine in the control input u, i.e. $\bar{x}^v = Mu + b$ with an appropriate matrix M and a vector b of compatible dimensions. Then the inequality (7) can be rewritten as

$$\begin{aligned} \begin{bmatrix} 1 \\ u \end{bmatrix}^{\mathrm{T}} \mathcal{Q} \begin{bmatrix} 1 \\ u \end{bmatrix} \geq \gamma \iff & \operatorname{Tr}\left(\mathcal{Q} \begin{bmatrix} 1 \\ u \end{bmatrix} \begin{bmatrix} 1 \\ u \end{bmatrix}^{\mathrm{T}}\right) \geq \gamma \\ \iff & \operatorname{Tr}(\mathcal{Q}U) \geq \gamma,\ U \succeq 0, \\ & U_{11} = 1,\ \operatorname{Rank}(U) = 1, \end{aligned} \tag{8}$$

where $Q \succeq 0$ and γ can be appropriately computed from (7). Suppose that a convex quadratic constraint of the form $u^{\mathrm{T}} Q_1 u + q_1^{\mathrm{T}} u + q_{10} \leq 0$ with $Q_1 \succeq 0$ is imposed on the control input. Minimising the probability of collision under that quadratic constraint can be represented as the optimisation

$$\begin{aligned} &\max_U \operatorname{Tr}(\mathcal{Q}U) \\ &\text{s.t. } \operatorname{Tr}(\mathcal{Q}_1 U) \leq 0,\ U \succeq 0,\ U_{11} = 1,\ \operatorname{Rank}(U) = 1, \end{aligned} \tag{9}$$

where the symmetric matrix $\mathcal{Q}_1$ satisfies the relation $\begin{bmatrix} 1 \\ u \end{bmatrix}^{\mathrm{T}} \mathcal{Q}_1 \begin{bmatrix} 1 \\ u \end{bmatrix} = u^{\mathrm{T}} Q_1 u + q_1^{\mathrm{T}} u + q_{10}$. It is well known that removing the rank constraint $\operatorname{Rank}(U) = 1$ in this particular optimisation does not change the optimum value, i.e, the corresponding SDP relaxation is exact (Nesterov, Wolkowicz, & Ye, 2000). Next, consider a similar problem with the box-type constraints $|u_i| \leq 1$ for $i = 1, \ldots n_u$ in the place of the quadratic constraint on u. Then minimisation of the probability of collision can be represented as the optimisation

$$\begin{aligned} &\max_U \operatorname{Tr}(\mathcal{Q}U) \\ &\text{s.t. } U_{ii} \leq 1,\ i = 2, \ldots, n_u + 1, \\ &\quad U \succeq 0,\ U_{11} = 1,\ \operatorname{Rank}(U) = 1. \end{aligned} \tag{10}$$

The associated primal SDP relaxation is the same as the optimisation (10) without the rank constraint, and its suboptimality is bounded by

$$\gamma^\star \leq \gamma_{\text{sdp}}^\star \leq \frac{\pi}{2}\gamma^\star \tag{11}$$

where $\gamma^\star$ refers to the optimal value of (10) and $\gamma_{\text{sdp}}^\star$ refers to the optimal value of the primal SDP relaxation (Nesterov, 1998; Nesterov et al., 2000).

4.2 Probabilistic safety regions

Instead of quantifying the probability of safety by $1 - P_c$, consider the dual definition of probability of safety:

$$P_t^s \triangleq \int_{x^v} \int_{x^s} I_s\left(x_t^v, x_t^s\right) dF_{vs}\left(x_t^v, x_t^s\right), \tag{12}$$

where x^s is a global reference coordinate that characterises a virtual safety region $\mathcal{X}_s(x^s)$ and the joint cdf F_{vs} and the indicator function I_s follow similar definitions as in the previous section. With this definition of probabilistic safety regions, the chance constraint $\mathbf{Pr}[y_t \notin \mathcal{F}_y] \geq \beta$ in (5) can be rewritten as $P_t^s \geq \beta$. Consider the obstacles as point masses, which is a case when the volume of $\mathcal{X}_v(x^v)$ is much smaller than the work space W for all $x^v \in W$ and the safety region $\mathcal{X}_s(x^s)$ defines a point or a sequence of points in W.

Lemma 4.2: *For point-mass obstacles, suppose that* $x^v \sim \mathcal{N}(\bar{x}^v, \Sigma_{x^v})$ *and* $x^s \sim \mathcal{N}(\bar{x}^s, \Sigma_{x^s})$ *are independent Gaussian random variables. Then* $P_s \geq \beta$ *can be rewritten as the constraint on* $(\bar{x}^v, \Sigma_{x^v}, \bar{x}^s, \Sigma_{x^s})$*:*

$$(\bar{x}^v - \bar{x}^s)^T \Sigma_x^{-1} (\bar{x}^v - \bar{x}^s) \leq -2 \ln\left(\frac{\beta}{V_v}\sqrt{\det(2\pi\,\Sigma_x)}\right), \tag{13}$$

where $\Sigma_x = \Sigma_{x^v} + \Sigma_{x^s}$ *and* V_v *is the volume of the vehicle.*

The constraint (13) is convex in $(\bar{x}^v, \bar{x}^s)$ for a fixed $\beta \in [0, 1]$.

4.3 Obstacles as linear constraints in a work space

Consider the concatenated system output y_t. The forbidden region for the system output can be defined as a union of N linear inequality constraints that is a nonconvex polyhedral set

$$\mathcal{F}_y \triangleq \bigcup_{i=1}^{N} \left\{y : h_i^{\mathrm{T}} y \geq b_i\right\}. \tag{14}$$

Assume that $y \sim \mathcal{N}(\bar{y}, \Sigma_y)$ and define $\eta_i \triangleq h_i^{\mathrm{T}} y$, which is a univariate Gausian random variable with mean $\bar{\eta}_i = h_i^{\mathrm{T}} \bar{y}$ and variance $\Sigma_{\eta_i} = h_i^{\mathrm{T}} \Sigma_y h_i$. The idea of risk allocation proposed by Ono and Williams (2008) and Blackmore and Ono (2009) can be used to derive a conservative convex condition for the constraint (14).

Lemma 4.3 (Lemmas 1, 2, and 3 in Blackmore and Ono, 2009)**:** *Consider a chance constraint* $\mathbf{Pr}[y \notin \mathcal{F}_y] \geq \beta$ *or the equivalent condition* $\mathbf{Pr}[y \in \mathcal{F}_y] \leq 1 - \beta$ *where* $\mathcal{F}_y$ *is defined in (14). Then the feasibility of the constraint*

$$\mathbf{Pr}[\eta_i \geq b_i] \leq \epsilon_i,\ \epsilon_i \in (0, 1),\ \textit{and}\ \sum_i \epsilon_i = 1 - \beta \tag{15}$$

implies the feasibility of the constraint $\mathbf{Pr}[y \notin \mathcal{F}_y] \geq \beta$*. Furthermore,* $\mathbf{Pr}[\eta_i \geq b_i] \leq \epsilon_i \iff \frac{1}{2}\left(1 - \mathrm{erf}\left(\frac{b_i - \bar{\eta}_i}{\sqrt{2\Sigma_{\eta_i}}}\right)\right) \leq \epsilon_i$*, and the constraint (15) is convex in* $(\bar{\eta}_i, \epsilon_i)$ *for* $\beta \geq 0.5$*.*

Alternatively, consider the forbidden region for the system output defined as an intersection of N linear inequality constraints that is a convex polyhedral set

$$\mathcal{F}'_y \triangleq \bigcap_{i=1}^{N} \left\{y : h_i^{\mathrm{T}} y \leq b_i\right\}. \tag{16}$$

In this case, the nonconvex chance constraint $\mathbf{Pr}[y \notin \mathcal{F}'_y] \geq \beta$ can be replaced by the relaxation

$$\mathbf{Pr}[\eta_i > b_i] \geq \epsilon_i,\ \epsilon_i \in (0, 1),\ \text{for } i = 1, \ldots, N, \tag{17}$$

where ϵ_i are appropriately defined functions of β.

Lemma 4.4: *With the condition* $\epsilon_i \geq 0.5$ *incorporated into the constraint (17), the combined constraint is convex in* $(\bar{\eta}_i, \epsilon_i)$*.*

Proof: $\mathbf{Pr}[\eta_i > b_i]$ is a concave function in $\bar{\eta}_i \geq b_i$, and for $\epsilon_i \geq 0.5$, the feasibility of the constraint (17) necessarily requires $\bar{\eta}_i \geq b_i$. Thus, if $(\bar{\eta}_i^1, \epsilon_i^1)$ and $(\bar{\eta}_i^2, \epsilon_i^2)$ are feasible solutions of the constraint (17) and $\epsilon_i^j \geq 0.5, j = 1, 2$, then $(\bar{\eta}_i^\lambda, \epsilon_i^\lambda)$ is also a feasible solution for all $\lambda \in [0, 1]$ where the superscript λ refers to the λ-convex combination of the feasible solutions with the superscripts 1 and 2. □

Imposing additional linear constraints on ϵ_i does not change the convexity of the combined constraint. For example, additional constraints $\epsilon_i \geq \ell(\beta)$ could be introduced in which $\ell := \beta \mapsto [0.5, 1)$ is a nondecreasing function. However, the most practically useful functional form for the ℓ is not obvious. One functional form that may be useful is $\ell(\beta) = \sqrt[N]{\beta}$, in which case $\beta \geq 0.5^N$ would satisfy the constraint $\epsilon_i \geq 0.5$.

5. Efficient approximation of feasibility of probabilistic constraints

The previous section presented methods for formulating chance constraints for probabilistic collision avoidance under stochastic uncertain circumstance and model uncertainty. Under the assumption that the state (or output) variables are jointly Gaussian random variables, the resultant chance constraints involve imposing constraints on the mean and covariance of the state. However, the state of the system model (4) is not Gaussian and even computations of its mean and covariance can necessitate sampling-based evaluation such as Monte Carlo simulation. This section presents and analyses methods for approximate uncertainty propagation in a stochastic dynamical system (4) that are based on generalised polynomial chaos expansions. The

methods provide numerically tractable computations of the mean and covariance of the controlled state variables for which closed-forms of the approximate mean and covariance can be obtained and the associated chance constraints can be efficiently evaluated.

5.1 Approximation of uncertainty propagation

Consider the concatenated parametric uncertainty $\theta := [x_0^{\mathrm{T}}, \delta^{\mathrm{T}}]^{\mathrm{T}}$. Suppose that there exists a diffeomorphism $T : \Xi \to \Theta = \Delta \times \mathcal{X}$ such that $\theta = T(\zeta)$ and $\zeta \in \Xi$ is a standard random variable. For application of the spectral method based on gPC expansions, assume that the solution of the SDE (4) has the form

$$x_t \approx \hat{x}_t \triangleq \sum_{i=0}^{p-1} \phi_i(\zeta) X_t^i \tag{18}$$

which is an approximation of the true solution x with p basis functions from the set $\{\phi_i\}$. Obtaining the approximate solution $\hat{x}$ involves determining the time-varying deterministic coefficients X_t^i. To do this, substitute the approximation $\hat{x}$ into x of the SDE (4) and solve for the X_t^i by intrusive or non-intrusive projections onto the probability space of the random variable ζ whose cdf is given by F_ζ. In particular, applying Galerkin projection (Maitre & Kino, 2010) results in another SDE

$$X_{t+1} = G_X X_t + G_u u_t + G_w w_t, \tag{19}$$

where $X_t := \mathrm{col}(X_t^i) \in \mathbb{R}^{np}$, the initial condition $X_0^i = \langle \phi_i(\zeta), x_0(\zeta) \rangle$, and the matrices $G_{(\cdot)}$ are computed from the inner product (1) defined on a measure space $(\Xi, \mathcal{M}, F_\zeta)$ for the Galerkin projection. The concatenated variables of interest over the time-horizon satisfy the equation

$$X_{0:T} = H_X X_0 + H_u u_{0:T} + H_w w_{0:T}, \tag{20}$$

where $X_{0:T} := \mathrm{col}(X_0, \ldots, X_T)$ and the matrices $H_{(\cdot)}$ have closed forms in terms of the matrices $G_{(\cdot)}$ in (19). From the assumption of Gaussian white noise w_t, the concatenated coefficients X_t is a Gaussian random process resulting in a Gaussian random variable $X_{0:=T}$ with mean and covariance

$$\begin{aligned} \bar{X}_{0:T} &= H_X X_0 + H_u u_{0:T} + H_w \bar{w}_{0:T}, \\ \Sigma_{X_{0:T}} &= H_X \Sigma_{X_0} H_X^{\mathrm{T}} + H_w \Sigma_{w_{0:T}} H_w^{\mathrm{T}}. \end{aligned} \tag{21}$$

The following proposition shows that the mean and covariance of the approximate solution $\hat{x}_t$ have closed-forms with respect to the mean and covariance of the coefficients of a generalised polynomial chaos expansion given in (21).

Proposition 5.1: *The concatenated approximation of the solution $\hat{x}_{0:T}$ satisfies*

$$\mathbf{E}[\hat{x}_{0:T}] = K_X X_0 + K_u u_{0:T} + K_w \bar{w}_{0:T}, \tag{22}$$

where the matrices $K_{(\cdot)}$ are functions of $G_{(\cdot)}$ and $H_{(\cdot)}$, and there exists an affine surjective map $\Omega : \mathbb{S}^{np(T+1)} \to \mathbb{S}^{n(T+1)}$ such that

$$\Sigma_{\hat{x}_{0:T}} = \Omega(\Sigma_{X_{0:T}}). \tag{23}$$

Proof: The proof is straightforward. Consider an approximate solution using a polynomial expansion (18). Due to independence of the random parameter θ and the random process w_t, its expectation is $\mathbf{E}[\hat{x}_t] = \mathbf{E}[\phi(\zeta)^{\mathrm{T}} \otimes \mathbf{I}_n]\mathbf{E}[X_t]$ where the first expectation is computed w.r.t. the random vector ζ and the second expectation is computed w.r.t. the random process w_t. Since the coefficient X_t is linear and has affine dependence on the control input u_t and the external disturbance w_t, the concatenated approximate state $\mathbf{E}[\hat{x}_{0:T}]$ is of the form given in (22). Similarly, the variance of the approximate state $\mathbf{E}[\hat{x}_t \hat{x}_t^{\mathrm{T}}]$ can be rewritten as $\mathbf{E}[(\phi(\zeta)^{\mathrm{T}} \otimes \mathbf{I}_n) X_t X_t^{\mathrm{T}} (\phi(\zeta)^{\mathrm{T}} \otimes \mathbf{I}_n)^{\mathrm{T}}]$ or equivalently, $\mathbf{E}[\underline{X}_t \phi(\zeta)\phi(\zeta)^{\mathrm{T}} \underline{X}_t^{\mathrm{T}}] = \mathbf{E}[\underline{X}_t \Phi \underline{X}_t^{\mathrm{T}}]$ where $\Phi \triangleq \mathbf{E}[\phi(\zeta)\phi(\zeta)^{\mathrm{T}}]$ and $\underline{X}_t \triangleq [X_t^0, \ldots, X_t^{p-1}] \in \mathbb{R}^{n \times p}$. From orthonormality of the basis functions, let $\Phi = \mathbf{I}_p$ without loss of generality. The (k, ℓ) element of the matrix $\mathbf{E}[\hat{x}_t \hat{x}_t^{\mathrm{T}}] = \mathbf{E}[\underline{X}_t \underline{X}_t^{\mathrm{T}}]$ is $\sum_{j=0}^{p-1} X_{t,k}^j X_{t,\ell}^j$ and $X_{t,k}^j X_{t,\ell}^j$ is an element of the matrix $\mathbf{E}[X_t X_t^{\mathrm{T}}]$. Therefore, $\mathbf{E}[\hat{x}_t \hat{x}_t^{\mathrm{T}}]$ is an affine function of $\mathbf{E}[X_t X_t^{\mathrm{T}}]$, which is equivalent to the concatenated covariance matrix $\Sigma_{\hat{x}_{0:T}}$ being an affine function of $\Sigma_{X_{0:T}}$. The corresponding mapping is a projection that is surjective. □

The random process X_t is Gaussian such that the mean and covariance given by (21) exactly characterise the probability distribution of X_t for all t, whereas the approximation $\hat{x}_t$ to the solution x_t is not necessarily Gaussian, due to the additional randomness of the parameters (x_0, δ). However, the mean and covariance of x_t can be approximated by the mean and covariance of $\hat{x}_t$ given by (22) and (23). More precisely, the next proposition shows the convergence of the approximation error in the mean-square sense.

Proposition 5.2: *Consider the solution trajectory x_t of the system (4) and its approximation $\hat{x}_t$ using a gPC expansion given by (18) whose coefficients X_t solve (19). Assume that the random variables (x_0, δ) are independent of the random process w_t and X_t is a second-moment process.*[2] *Then $\|x_t - \hat{x}_t\| \to_{\mathrm{m.s.}} 0$ pointwisely in t as $p \to \infty$, where $\|\cdot\|$ can be any vector p-norm.*

Proof: An approximation $\hat{x}_t$ can be explicitly rewritten as $\sum_{i=0}^{p-1} \phi_i(\zeta(x_0, \delta)) X_t^i(w_{0:t-1})$. From Theorem 2.1, for any realisation of the random variable $w_{0:t-1} \in \mathcal{W}^t$, where $\mathcal{W}$ is the support of w_t and $\bar{\epsilon}$ is greater than zero, there exists

$\bar{p} \in \mathbb{N}$ such that $\int_\Xi \|x_t - \sum_{i=0}^{p-1} \phi_i(\zeta) X_t^i(w_{0:t-1})\|^2 d\mu_\zeta(\zeta) \le \bar{\epsilon}$ for all $p \ge \bar{p}$, where μ_ζ is the probability measure of ζ. The $\bar{\epsilon}$ is a function of $w_{0:t-1}$. Due to the linear dependence of x_t and X_t^i on $w_{0:t-1}$, which follows from (4) and (19), $\bar{\epsilon} = \epsilon w_{0:t-1}^{\mathrm{T}} w_{0:t-1}$ where $\epsilon > 0$ is an arbitrary constant that is independent of $w_{0:t-1}$. This implies that the mean-square approximation error is bounded above:

$$\int_{\mathcal{W}^t} \int_\Xi \left\| x_t - \sum_{i=0}^{p-1} \phi_i(\zeta) X_t^i(w_{0:t-1}) \right\|^2 d\mu_\zeta(\zeta) d\mu_w(w_{0:t-1}) \le \epsilon \int_{\mathcal{W}^t} w_{0:t-1}^{\mathrm{T}} w_{0:t-1} d\mu_w(w_{0:t-1}) \le \epsilon M,$$

where μ_w is the corresponding probability measure of the random variable $w_{0:t-1}$ and $M < \infty$ whose boundedness follows from the second-moment assumption of the random process w_t. Since $\epsilon > 0$ is arbitrary, the convergence is ensured. □

Furthermore, if the system matrices are analytic functions of the random variables (x_0, δ) then the convergence rate of the approximation error $\|x_t - \hat{x}_t\|$ to 0 in mean-square is exponential, which follows from the solution trajectory x_t being an analytic function of (x_0, δ) under those assumptions.

5.2 Gaussian approximation and convexifications of chance-constrained MPC: information theoretic justification

For a Gaussian random process y_t (or x_t), Section 4 shows that the chance constraint corresponding to probabilistic collision avoidance $\mathbf{Pr}[y_t \notin \mathcal{F}_y] \ge \beta$ can be rewritten as conditions in terms of its mean $\bar{y}_t$ and covariance Σ_{y_t}. In particular, conditions (13), (15) and (17) are jointly convex in y_t (or x_t) and the other decision variables (ϵ_i), under some mild assumptions.

However, the solution x_t of the system dynamics (4) and its spectral approximation $\hat{x}_t$ given in (18) are not generally Gaussian random processes that make the optimisation (5) difficult to solve in the sense that the chance constraint does not have a closed-form expression and its feasibility is hard to check. To avoid the use of any sampling or simulation-based methods to evaluate the feasibility of the chance constraint $\mathbf{Pr}[y_t \notin \mathcal{F}_y] \ge \beta$, the approximate solution $\hat{x}_t$ is substituted in the place of x_t and Gaussian fitting of the random variables under consideration is applied. More specifically, assume that $\hat{x}_t \sim \mathcal{N}(\bar{\hat{x}}_t, \Sigma_{\hat{x}_t})$, for which there are closed-form expressions given by (22) and (23).[3] A theoretical justification of this assumption $\hat{x}_t \sim \mathcal{N}(\bar{\hat{x}}_t, \Sigma_{\hat{x}_t})$ can be made from the principle of maximum entropy (Cover & Thomas, 2006, Chap. 12). Maximum entropy can be used to determine or approximate a probability distribution that incorporates only known information. If only the first and second moments of $\hat{x}_t$ are used to approximate its probability distribution then the maximum entropy distribution has the form $\mathcal{N}(\bar{\hat{x}}_t, \Sigma_{\hat{x}_t})$, i.e. a Gaussian distribution. Furthermore, since $\hat{x}_t$ converges to x_t in the mean-square sense as the number of basis functions increases, the approximate probability distribution $\mathcal{N}(\bar{\hat{x}}_t, \Sigma_{\hat{x}_t})$ can be made arbitrarily close to the probability distribution of x_t that maximises entropy subject to the constraints corresponding to the first and second moments.

Proposition 5.3: *Consider the solution trajectory x_t of the system (4) and its approximation $\hat{x}_t$ using a gPC expansion given by (18) whose coefficients X_t solve (19). Assume that the random variables (x_0, δ) are independent of the random process w_t and X_t is a second-moment process (for notation convenience, the subscript t is dropped from here on). Suppose that a probability density $f^\star$ solves the optimisation*

$$\begin{aligned} \max_f \; & -\int_S f(x) \log f(x) dx \\ \text{s.t. } & f(x) \ge 0, \; \int_S f(x) dx = 1, \\ & \int_S f(x) x^i dx = M_i, \; i = 1, 2, \end{aligned} \tag{24}$$

where S denotes the support for the random variable x, and M_1 and M_2 correspond to the given first and second moments, respectively. Then an approximate Gaussian distribution $\hat{f}_2 \triangleq \mathcal{N}(\bar{\hat{x}}, \Sigma_{\hat{x}})$ obtained from the solution of (19) converges to $f^\star$ as $p \to \infty$ in the $\mathcal{L}_1$-norm sense, i.e.,

$$\lim_{p \to \infty} \int_S |f^\star(x) - \hat{f}_2(x)| dx = 0.$$

Proof: Due to limited space, consider the scalar case (the extension to the multivariable case is straightforward). From the principle of maximum entropy, a unique $f^\star$ has the form of $e^{\lambda_0 + \lambda_1 x + \lambda_2 x^2}$ that corresponds to a Gaussian distribution. Similarly, $\hat{f}_2$ is a unique maximum entropy distribution that solves the optimisation (24) with given approximate moments $\hat{M}_i$, $i = 1, 2$, in the place of M_i and can be rewritten as $e^{\hat{\lambda}_0 + \hat{\lambda}_1 x + \hat{\lambda}_2 x^2}$ for some constants $\hat{\lambda}_j$, $j = 0, 1, 2$. Since convergence in mean-square implies convergence in distribution and $\hat{M}_i$ can be arbitrarily close to M_i as $p \to \infty$ from Proposition 5.2, this implies that $\lim_{p \to \infty} \max_j |\lambda_j - \hat{\lambda}_j| = 0$. Therefore, for any arbitrary constant $\epsilon > 0$, there exists $\bar{p} \in \mathbb{N}$ such that $\min\{e^\epsilon, e^{-\epsilon}\} \le \hat{f}_2(x)/f^\star(x) \le \max\{e^\epsilon, e^{-\epsilon}\}$ uniformly in $x \in S$ for all $p \ge \bar{p}$. This implies that $\hat{f}_2$ converges to $f^\star$ in the $\mathcal{L}_1$-norm sense as $p \to \infty$. □

Remark 1: The above Gaussian approximation is a suboptimal way to estimate the probability distribution of x_t, which produces convex chance constraints that are more computationally tractable by ignoring the extra information

in the higher-order moments of $\hat{x}_t$. This method of approximation has the same characteristics as the extended Kalman filter (EKF) and unscented Kalman filter (UKF) that are widely used in practical applications although there are no theoretical guarantees that those estimation methods will always work well or even converge.

Using the Gaussian approximation, the design problem reduces to finding a control policy μ_T (or $u_{0:T}$) that solves the optimisation

$$\begin{aligned}
&\min_{\mu_T} J(\bar{x}_0, \Sigma_{x_0}, \mu_T)\\
&\text{s.t. } \bar{\hat{x}}_{0:T} = K_X X_0 + K_u u_{0:T} + K_w \bar{w}_{0:T},\\
&\quad \Sigma_{\hat{x}_{0:T}} = \Omega(\Sigma_{X_{0:T}}),\\
&\quad \hat{x}_{0:T} \sim \mathcal{N}(\bar{\hat{x}}_{0:T}, \Sigma_{\hat{x}_{0:T}}),\ y_{0:T} = (\oplus_{i=0}^{T} C)\hat{x}_{0:T},\\
&\quad (\bar{y}_{0:T}, \Sigma_{y_{0:T}}) \in \mathcal{F}(\beta) \text{ or } (\bar{y}_{0:T}, \Sigma_{y_{0:T}}, \epsilon) \in \mathcal{F}(\beta),\\
&\quad u_{0:T} \in \mathcal{U}^{T+1},
\end{aligned} \tag{25}$$

where the matrices $K_{(\cdot)}$ and $\Sigma_{X_{0:T}}$, and the injection map Ω are precomputed, $\oplus_{i=0}^{T} C \triangleq \text{diag}(C, \ldots, C)$, and the constraints $\mathcal{F}(\beta)$ can be one of the sets:

$$\left\{(y, \Sigma_y) : \text{Equation(13)},\ \bar{x}^v = y,\ \Sigma_{x^v} = \Sigma_y\right\}; \tag{26}$$

$$\left\{(y, \Sigma_y, \epsilon) : \text{Equation(15)},\ \bar{\eta} = y,\ \Sigma_\eta = \Sigma_y\right\}; \tag{27}$$

$$\left\{(y, \Sigma_y, \epsilon) : \text{Equation(17)},\ \bar{\eta} = y,\ \Sigma_\eta = \Sigma_y,\ \epsilon \geq 0.5\right\}, \tag{28}$$

where $\epsilon = \text{col}(\epsilon_i)$, $\beta \geq 0.5$ is required for the second feasible solution set to be convex in (y, ϵ), and the first and the third sets are convex in y and (y, ϵ), respectively. With the standard performance specification that the objective function J is convex quadratic in μ_T and the set $\mathcal{U}$ is a convex polytope, the optimisation (25) is a convex quadratically constrained quadratic program (QCQP) when $\mathcal{F}(\beta)$ is given by (26) and a convex nonlinear program when $\mathcal{F}(\beta)$ is given by (27) or (28).

6. A demonstration example

This section compares the accuracy of the proposed gPC-based MPC formulations for a numerical example. Consider the parametric uncertain linear time-invariant system

$$x_{t+1} = \begin{bmatrix} 0.9 + \rho_1\delta_1 & 0.1 \\ 0.1 & 0.85 \end{bmatrix} x_t + \begin{bmatrix} 0.25 - \rho_2\delta_2 \\ 0.75 + \rho_2\delta_2 \end{bmatrix} u_t + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} w_t$$

with initial condition $x_0 = [20,\ 10]^{\mathrm{T}}$, $\rho_1 = 0.001$ and $\rho_2 = 0.05$ are weights on normalised standard random variables $\delta_1 \sim \mathcal{N}(0, 1)$ and $\delta_2 \sim \mathcal{N}(0, 1)$, respectively, and the exogenous process noise $w_t \sim \mathcal{N}(0, 0.001)$ is assumed to have autocorrelation $\mathbf{E}[w_t w_s] = 0$ for all $t \neq s$. This example

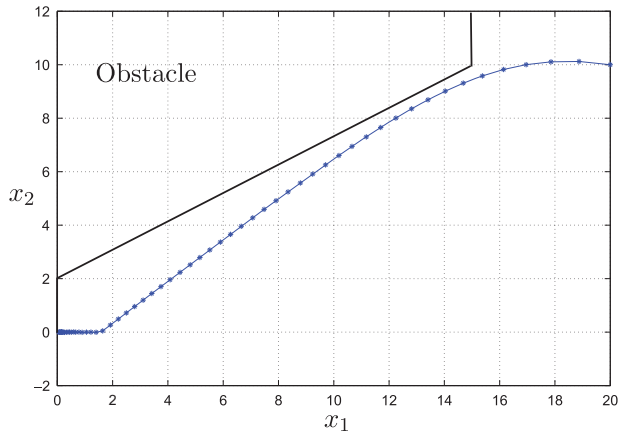

Figure 1. A controlled trajectory produced by the proposed stochastic MPC formulation. For description of the forbidden regions at each step of receding horizon control design to avoid obstacle collisions, the linear inequality constraints (14) are used. To do this, only partial linear constraints are imposed at each computation of receding horizon control input, whereas the full description of the obstacle in this example is indeed an intersection of linear constraints.

computes the control inputs by solving the optimisation (31), for which the covariance constraints are imposed based on the 99% level of confidence for collision avoidance, and comparing the probability of collisions obtained from different methods presented in the paper. Consider $Q_t = \text{diag}\{100, 100\}$ and $R_t = 1$ for all t, a prediction horizon of $T = 4$, and input constraint $u_t \in [-0.5, 0.5]$. The constraints are constructed from the obstacle shown in Figure 1. The resultant controlled system trajectory generated by a system model with fixed parameters $\delta = [0.01, 0.05]$ and randomly chosen exogenous disturbances w_t is shown in Figure 1, which avoids the obstacle as desired. Figure 2 shows Monte Carlo simulations with 5000 samples of (δ, w), which indicates that the stochastic MPC algorithm was effective in avoiding the obstacle while allowing the closed-loop trajectory to become rather close to the obstacle so as to optimise the closed-loop performance objective. Figure 3 compares the computed probabilities of collision using the methods presented in this paper with the probabilities quantified by the Monte Carlo simulations. At each time the probability of collision obtained by the gPC expansion is very close to the value computed using either Monte Carlo applied to the original system or Monte Carlo applied to the convex relaxation. The approximate probabilities of collision follow nearly identical trends to the true probabilities while enabling the optimal control problem at each time instance of MPC to be computed from a convex program that can be solved in polynomial-time.[4]

To further assess the accuracy of the gPC expansion, let $\Sigma_x^{\text{mc}}(t)$ and $\Sigma_x^{\text{pce}}(t)$ be the computed covariance of the controlled system trajectory obtained from Monte Carlo

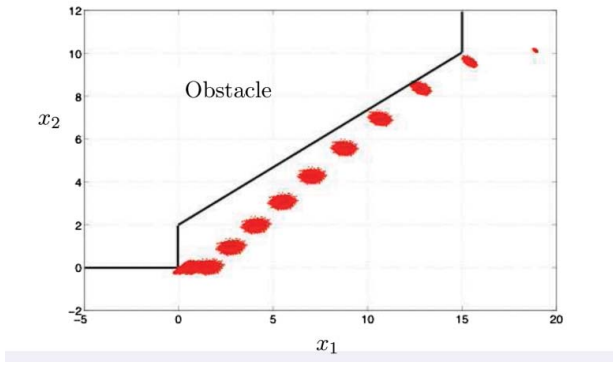

Figure 2. Monte Carlo simulations. The red dots correspond to simulated states at each 4*th* sampling instance for 5000 samples.

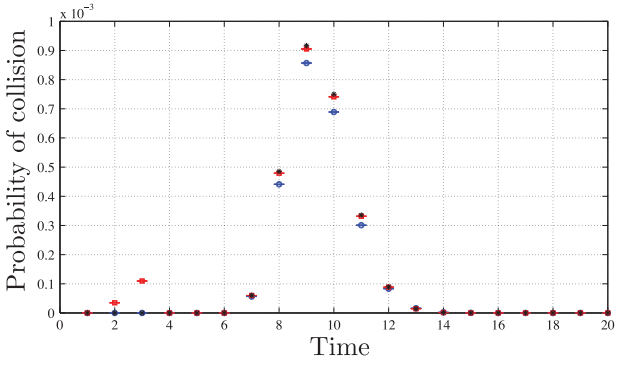

Figure 3. A comparison of the computed probability of collision during the first 20 [sec] of simulation for the true system and the approximation using a gPC expansion, estimated using Monte Carlo (MC) simulations where the error bars were obtained from 1000 Monte Carlo simulations with different sets of 5000 samples. The blue circle refers to the MC simulation result and the red box refers to the collision probability that is obtained from the MC simulations with the convex relaxation (15). For both computations, the corresponding error bars were generated at the 95% confidence level. The widths of the computed confidence intervals were smaller than 10^{-7}, which is negligible compared to the collision probability. The black star refers to the collision probability obtained from the presented gPC method that incorporates the convex relaxation (15).

simulations with 5000 samples of (δ, w) and the polynomial chaos expansion with a specified order of Hermite polynomials, respectively. Table 1 compares the worst-case deviation of $\max_{0\le t\le 60} \|\Sigma_x^{\mathrm{mc}}(t) - \Sigma_x^{\mathrm{pce}}(t)\|_F$, where $\|\cdot\|_F$ denotes the Frobenius norm, for different degrees of Hermite polynomials. The approximation error of the covariance matrix is small and, as expected from the theoretical analysis, the error in the state covariance matrix decreases as the number of terms in the polynomial expansion increases.

Table 1 Covariance approximation errors for different degrees of Hermite polynomial expansions.

Degree of Hermite polynomials	$\max_{0\le t\le 60} \|\Sigma_x^{\mathrm{mc}}(t) - \Sigma_x^{\mathrm{pce}}(t)\|_F$
1^{st}	$\approx 1.0001 \times 10^{-3}$
2^{nd}	$\approx 6.0878 \times 10^{-4}$
3^{rd}	$\approx 5.3627 \times 10^{-4}$
4^{th}	$\approx 2.8365 \times 10^{-4}$

7. Discussions and further remarks

7.1 Approximate solution using spectral methods with the KL expansions

This paper considers two sources of uncertainties: (1) parametric uncertainty and (2) exogenous disturbance. Uncertainty propagations induced by parametric uncertainty are approximated by using a gPC expansion and additive exogenous disturbances affect the coefficients of the resultant gPC expansion. Another possible approach to the same problem is to use a KL expansion to approximate the random process w_t, i.e. replace the random process w_t by its principal component approximation with random variables and solve a larger dimension deterministic ordinary differential equation (ODE) to approximate the true solution x_t. This approach requires higher online computational expense as the dimension of a deterministic ODE increases, even though the system data for such an ODE can be precomputed.

7.2 Heuristic convexification methods for chance constraints with stochastic parametric uncertainty

Here the convexification methods are illustrated for the prototypical stochastic MPC problem

$$\begin{aligned} \min_{u_{0:T-1}} \ & \mathbf{E}\left[\sum_{t=1}^{T} x_t^{\mathrm{T}} Q_t x_t + u_{t-1}^{\mathrm{T}} R_{t-1} u_{t-1}\right] \\ \text{s.t.} \ & x_{t+1} = A(\delta)x_t + B(\delta)u_t, \\ & \Pr_{\Delta}[H_t x_t \ge b_t] \le \epsilon_t, \\ & u_{\min,t} \le u_t \le u_{\max,t}, \end{aligned} \tag{29}$$

with the stochastic uncertainties δ (the incorporation of the external noise perturbation is straightforward as described in the theoretical sections of this paper but not included in this example to shorten the presentation). The constraints are defined over the time interval $[1, T]$ for x_t and $[0, T-1]$ for u_t, and this time interval consideration is omitted here for notational convenience and will always be clear from the context. The optimisation (29) is further simplified by replacing the chance constraint $\Pr_{\Delta}[H_t x_t \ge b_t] \le \epsilon_t$ by $H_t \mathbf{E}[x_t] \le b_t - \beta_t$ where $\beta_t > 0$ is an additional decision

variable. By doing this, optimisation (29), in which the dynamic constraint is an SDE, reduces to the deterministic optimisation

$$\begin{aligned}\min_{u_{0:T-1},\,\beta_{1:T}} \quad & \sum_{t=1}^{T}\left(X_t^{\mathrm{T}}\bar{Q}_t X_t + u_{t-1}^{\mathrm{T}} R_{t-1} u_{t-1}\right) - \gamma\sum_{t=1}^{T}\ell(\beta_t)\\ \text{s.t.}\quad & X_{t+1} = FX_t + Gu_t,\\ & c_0 H_t X_t^0 + \beta_t \le b_t,\ \beta_t > 0,\\ & u_{\min,t} \le u_t \le u_{\max,t},\end{aligned} \tag{30}$$

where x_t in the constraint of optimisation (29) is approximated by $\hat{x}_t$ in (18), $\bar{Q}_t \triangleq \mathbf{E}\left[(\phi(\zeta)\otimes\mathbf{I}_n)^{\mathrm{T}} Q_t (\phi(\zeta)\otimes\mathbf{I}_n)\right]$, $c_0 \triangleq \langle 1, \phi_0(\zeta)\rangle$, $\phi(\zeta)\triangleq \mathrm{col}(\phi_i)$, $\gamma > 0$ is a user-defined weight in the optimisation that corresponds to the maximisation of the feasibility of the chance constraint $\Pr_\Delta[H_t x_t \ge b_t] \le \epsilon_t$, and $\ell(\beta_t)$ is an incentive for decision variables to maximise the feasibility of the chance constraint $\Pr_\Delta[H_t x_t \ge b_t] \le \epsilon_t$; a typical choice can be $\sum_{i=1}^{m}\beta_{t,i}$ or $\max_i \beta_{t,i}$ that is linear in β_t, where $\beta_{t,i}$ is the ith entry of β_t.[5]The constrained optimisation (30) is a convex QP that can be solved efficiently.[6]

For a different formulation of constraints, consider constraints on the deviation of the solution trajectory from the expectation:

$$\begin{aligned}\min_{u_{0:T-1}} \quad & \sum_{t=1}^{T} X_t^{\mathrm{T}}\bar{Q}_t X_t + u_{t-1}^{\mathrm{T}} R_{t-1} u_{t-1}\\ \text{s.t.}\quad & X_{t+1} = FX_t + Gu_t,\\ & c_0 H_t X_t^0 \le b_t,\\ & (\bar{X}_t^i)^{\mathrm{T}} W \bar{X}_t^i - (c_0 X_{t,i}^0)^2 \le \sigma_{t,i}^2,\\ & u_{\min,t} \le u_t \le u_{\max,t},\end{aligned} \tag{31}$$

where $\bar{Q}_t$ and c_0 are the same as defined before, $W\triangleq \mathrm{diag}(\|\phi_i\|^2)$, and $\bar{X}_t^i$ denote the concatenation of coefficients of the polynomial expansion for the ith state. The constant vector c_0 and matrix W can be assumed to be normalised to be $\mathbf{1}_n$ and $\mathbf{I}_n$ without loss of generality. The constrained optimisation (31) is a convex QCQP that can be solved efficiently. More precisely, it is not hard to see that optimisation (31) can be rewritten as

$$\min_u \mathcal{Q}_0(u) \ \text{ s.t. } \mathcal{Q}_i(u) \le 0,\ i = 1, \ldots, m_q, \tag{32}$$

where $u \triangleq u_{0:T-1}$ and $\mathcal{Q}_i$ are convex quadratic forms for all $i = 0, 1, \ldots, m_q$. From Megretski and Treil (1993),[7]if the constraints $\mathcal{Q}_i \le 0$ are regular, i.e. satisfy a constraint qualification such as Slater's condition (Boyd & Vandenberghe, 2004, Section 5.2.3), then the static optimisation (32) has the same optimum as the optimisation

$$\max_{\lambda\ge 0}\min_u \mathcal{Q}_0(u) + \sum_i^{m_q}\lambda_i \mathcal{Q}_i(u) \tag{33}$$

for which fixing arbitrarily large $\lambda_i > 0$ results in the same optimal solution u^* as obtained from solving the constrained optimisation (32).

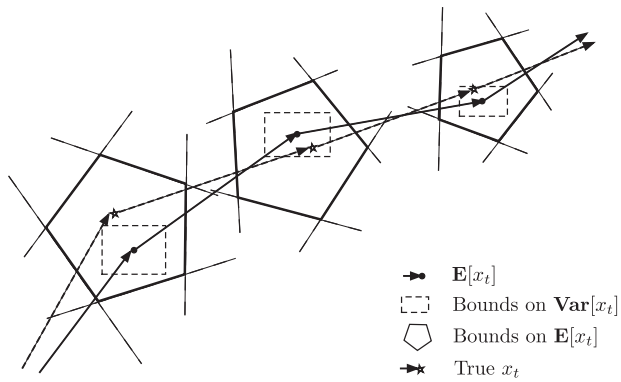

Figure 4. A schematic cartoon of constrained state trajectories with semi-chance constraints.

The conceptual picture of a constrained trajectory in Figure 4 shows how the constraints in (31) can be used to impose desired bounds on the controlled trajectory.

7.3 *The use of concentration-of-measure inequalities for probabilistic validation certificates of joint chance constraints*

This section shows that the Boole inequality can be incorporated into some well-known concentration-of-measure inequalities to provide probabilistic validation certificates for joint chance constraints. Consider the constraint $H^{\mathrm{T}}X \le b$, or equivalently $h_i^{\mathrm{T}}X \le b_i$ for $i = 1, \ldots, m$ where X is a random vector and h_i denotes the ith column of the matrix H. The associated probabilistic constraint can be written as $\Pr[H^{\mathrm{T}}X > b] = \Pr\left[\cup_{i=1}^{m}\{h_i^{\mathrm{T}}X > b_i\}\right]$. The Boole inequality gives an upper bound on this probabilistic violation of constraints:

$$\Pr\left[\bigcup_{i=1}^{m}\left\{h_i^{\mathrm{T}}X > b_i\right\}\right] \le \sum_{i=1}^{m}\Pr\left[h_i^{\mathrm{T}}X > b_i\right]. \tag{34}$$

Suppose that $b_i > 0$ for all $i = 1, \ldots, m$ without loss of generality. Some concentration-of-measure inequalities can be used for upper bounds on the right-hand side of (34) (Boucheron, Bousquet, & Lugosi, 2004):

- The Chernoff's bound: $\Pr\left[h_i^{\mathrm{T}}X > b_i\right] \le \frac{\mathbf{E}[e^{s_i h_i^{\mathrm{T}}X}]}{e^{s_i b_i}}$ where $s_i > 0$ for all $i = 1, \ldots, m$.

- The generalised Markov inequality:
  $\Pr\left[h_i^{\mathrm{T}} X > b_i\right] \leq \frac{\mathbf{E}[\phi_i(h_i^{\mathrm{T}} X)]}{\phi_i(b_i)}$ where $\phi_i : \mathbb{R} \to \mathbb{R}_+$ for all $i = 1, \ldots, m$.
- The Chebyshev inequality:
  $\Pr\left[|h_i^{\mathrm{T}} X - \mathbf{E}[h_i^{\mathrm{T}} X]| > t_i\right] \leq \frac{\mathrm{Var}(h_i^{\mathrm{T}} X)}{t_i^2} = \frac{h_i^{\mathrm{T}} \mathrm{Var}(X) h_i}{t_i^2}$.

Here we use the Chebyshev inequality.

Proposition 7.1: *If the random vector X satisfies the constraints on its expectation and variance*

$$h_i^{\mathrm{T}} X \leq b_i,\ h_i^{\mathrm{T}} \mathrm{Var}(X) h_i \leq t_i^2 \epsilon_i, \quad i = 1, \ldots, m \tag{35}$$

then the inequality $H^T X \leq b + t$ is satisfied with at least probability $1 - \sum_{i=1}^m \epsilon_i$, i.e., $\Pr\left[H^T X \leq b + t\right] \geq 1 - \sum_{i=1}^m \epsilon_i$.

Figure 5 illustrates the outer polytopic certificate (colored in red) for the associated chance constraint $\Pr[H^{\mathrm{T}} X > b] \leq \epsilon$ with $\sum_{i=1}^m \epsilon_i \leq \epsilon$. The polytope colored in blue corresponds to the constraints on the expectation of the trajectories. Such certificates can be computed from the results in Prop. 7.1.

From (21) and the results in Prop. 5.1, gPC expansions can provide closed forms for the expectation and variance of the controlled predicted state and output trajectories. This implies that any chance constraints of polyhedral inequalities can be certificated by deterministic polyhedral inequalities that are obtained from incorporating gPC expansions into the Boole inequality and the Chebyshev inequality of the form presented in Prop. 7.1.

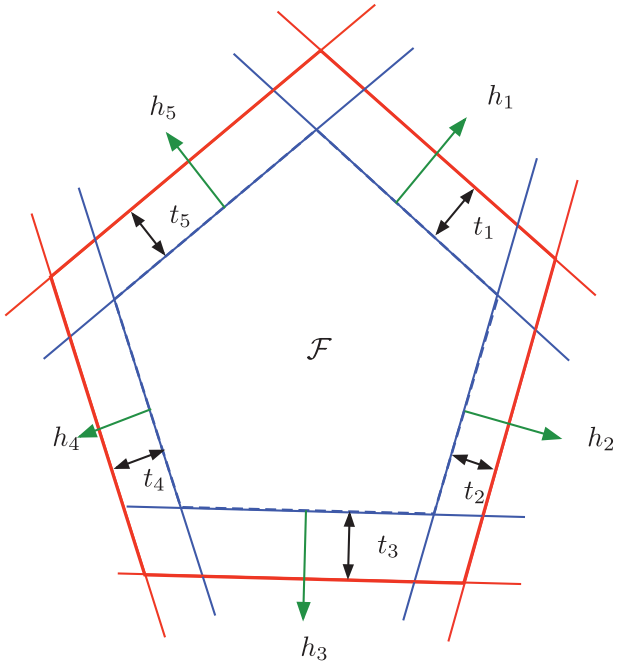

Figure 5. A cartoon of the outer bound that can be obtained from the Boole inequality and concentration-of-measure inequalities.

7.4 *Computation of first-time excursion probabilities*

An important problem in reliability analysis of stochastic dynamical systems is to compute the cumulative probability that the system output variables exceed a given threshold for a given time interval $[0, T]$ (Au & Beck, 2001; Greytak & Hover, 2011):

$$P_{\mathrm{f}}(T) \triangleq \Pr\left[\bigcup_{i=1}^m \{\exists t \in [0, T] : y_i(t) > b_i(t)\}\right], \tag{36}$$

where $b_i(t)$ denote some specified time-varying threshold levels. The probability (36) is called the *first-time excursion probability*. It is straightforward to show that computation of $P_{\mathrm{f}}(T)$ is a problem of checking the feasibility of chance constraints on the (controlled) system output and the aforementioned polynomial chaos expansion methods of control input design for stochastic MPC problems can be directly applied.

In particular, similar to the probability of violation in (34), the Boole inequality can be used to provide an upper bound on the probability of first-time excursion (36):

$$P_{\mathrm{f}}(T) \leq \sum_{i=1}^m \Pr\left[\{\exists t \in [0, T] : y_i(t) > b_i(t)\}\right] \tag{37}$$

for which the probability $\Pr\left[\{\exists t \in [0, T] : y_i(t) > b_i(t)\}\right]$ can be approximated by using the methods presented in Sections 5, 7.2, or 7.3.

7.5 *Affine feedback control policy*

The aforementioned methods for computation of a suboptimal control policy use the new measurements to compute a control action as well as to initialise the state-transition constraint in the optimisations at each step of prediction. In the presence of model/plant mismatch and external disturbances, the predicted control trajectory at time k can significantly deviate from the true controlled trajectory and the variance of the trajectory can increase such that the optimisation is feasible only for a short-time horizon, which is undesirable in terms of closed-loop stability. This situation can be avoided by incorporating a feedback control in each step of solving the optimisation, as has been done in many deterministic robust MPC formulations (e.g. see Kothare, Balakrishnan, & Morari, 1996). For example, the affine control law $u_t = K_t z_t + v_t$ can be inserted into the optimisation, where z_t is an estimated state or measured output. For a precomputed K_t (or a stationary control gain K), the resultant problem is exactly same as the open-loop

feedback control in which v_t is the only decision variable in each step of optimisation. If K_t is considered as an additional decision variable in each step of optimisation, then the resulting optimisation will retain the same degree of convexification as for u_t considered before.

7.6 Time-varying uncertain parameters

Consider the system dynamics given in (4) where the uncertain parameter vector $\delta \in \Delta$ (or $\theta = (\delta, x_0) \in \Theta$, if the initial condition is considered to be uncertain) is assumed to be an unknown constant vector. In the aforementioned MPC formulations, the uncertain parameters were assumed to be fixed only in the prediction step. The approaches can also be applied to slowly time-varying uncertain parameters, i.e. when the prediction horizon multiplied by the sampling interval is less than the time interval of significant parameter variation. A more accurate study of time-varying uncertain parameters can be performed by considering a large dimensional space of uncertain parameters. In particular, for the time-varying uncertain parameter vector $\delta_t \in \Delta$, consider the stacked vector $\delta_{0:T-1} \triangleq [\delta_0', \ldots, \delta_{T-1}']' \in \Delta^T$ where the superscript $'$ denotes the transpose and T denotes the prediction horizon. Then the approximation based on a polynomial chaos expansion is represented in terms of the stacked uncertain parameter vector $\delta_{0:\,T-1}$. This approach requires more basis functions for the corresponding spectral representation, but the time-dependent coefficients corresponding to the uncertain parameters in future can be set to zeros, which reduces the computation of projections to determine the coefficients of the gPC expansion.

8. Conclusions and future work

This paper considers a new approach for stochastic MPC problems in the presence of both parametric model uncertainty and exogenous stochastic disturbances. To approximate the solution of a stochastic differential equation and solve the corresponding stochastic MPC problem, a spectral method known as generalised polynomial chaos expansion is applied and constraints corresponding to the probability of safety/collision are imposed on the approximately predicted controlled trajectories, based on the model of a stochastic differential equation. The first and second moments of the approximate solution were exploited to estimate the probability distribution of the true solution. Under these technical assumptions, the chance constraints were replaced by convex constraints for the mean and covariance of the trajectory that are analytically computed from the gPC expansion. It was also shown that concentration-of-measure inequalities combined with the Boole inequality can provide conservative probabilistic certificates for chance constraints of polyhedral inequalities, for which applications of the gPC expansions are straightforward. Further studies to follow are to apply the presented methods to more complicated case studies and compare the heuristic convexification methods discussed in Section 7 to convex nonlinear programmes presented in Section 4, to study the trade-off between complexity and accuracy.

Notes

1. $\mathcal{F}_y$ and β can be time-varying, where the forbidden region might correspond to moving objectives and time-varying β can be used to assign different risk of collision in different time sequences in the predicted motions.
2. Consider the time interval $[0, T]$ in which X_t is a second-moment process.
3. The computation of deterministic constant matrices $K_{(\cdot)}$ and $\Sigma_{X_{0:T}}$ (or $\Sigma_{\hat{x}_{0:T}}$) can be performed off-line.
4. In particular, the computational complexity using a standard interior-point method (Boyd & Vandenberghe, 2004) is at most $O(\ell M^4 \log M)$ where $M = npT$ (n is the dimension of the state variables, p is the number of basis functions for a gPC expansion, T is the length of prediction horizon), and ℓ denotes the number of probabilistic polyhedral constraints. The average computation time at each sampling instance was $\approx$0.36 CPU seconds for $n = 2$, $p = 3$, and $T = 5$. This computation time includes the computation of the optimisation data, i.e., the time for computing matrices associated with the objective function and constraints, as well as the time for solving the resulting constrained optimisation. Optimisation is performed by the CVX toolbox (Grant, Boyd, & Ye, 2009) on a MacBook Pro laptop (2.53 GHz Intel Core 2 Duo, 4GB DDR3).
5. To be a convex program, $\|\beta_t\|_2$ cannot be used, since it results in a concave function term in the objective function in a minimisation problem.
6. By *efficiency*, it is meant that there is a numerical algorithm whose convergence is guaranteed and that provides an infeasibility certificate. Convex programs are such cases.
7. They applied a method of relaxation called the *S-procedure* (Yakubovich, 1971). Our case is a special case in which all the quadratic forms are convex, whereas Megretski and Treil (1993) considered more general cases where some of quadratic forms might be nonconvex. They proposed a sufficient condition for the set of quadratic form constraints to be lossless, i.e., the resultant relaxation obtained from the S-procedure gives an exact optimum.

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