Quantifying the potential benefits of constrained control for a large-scale system

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Abstract: It is of practical interest to identify which processes will benefit significantly from the use of constrained control algorithms such as model predictive control, and which will not. Explicit conditions are derived that identify whether a particular process may benefit from constraint handling. These conditions are also useful for understanding the interactions between design and control for a particular system, especially for actuator placement and selection. The conditions are computable for a large-scale system directly from its transfer function model, a simulation model (e.g. defined by a set of ordinary/partial-differential equations and algebraic conditions), or experimental input–output data. The formulation considers the effects of measurement noise, process disturbances, model uncertainties, plant directionality and the quantity of experimental data. The conditions are illustrated by application to a paper-machine model constructed from industrial data.

1 Introduction

Model predictive control began to be applied in the chemical-process industries in the late 1970s. In model predictive control [1–3] (and its many variants including GPC [4], DMC [5], and MMC [6]), the control objective is optimised on-line subject to the process constraints. The main focus is usually on manipulated variable constraints, although constraints on states or outputs also have been considered. For linear processes, the on-line optimisation is typically formulated as a linear or quadratic program to be solved at each sampling instance. These optimisation problems can be large. For example, a process with 50 manipulated variables and control horizon of 10 results in an optimisation problem with 500 variables and 1000 or more constraints.

Interestingly, control systems for many large-scale processes have been reported which provided adequate closed-loop performance with no or mild constraint-handling capabilities. For example, a control algorithm implemented on an industrial-scale adhesive coater at the Avery–Dennison company handled constraints with nearly optimal performance by just scaling back the unconstrained manipulated variable vector whenever the constraints were violated [7]. For a blown-film extruder, a controller designed to not manipulate uncontrollable plant directions did not need constraint handling to achieve the desired closed-loop performance [8]. It was suggested that explicit constraint handling may not be necessary for many sheet and film processes provided that the controller is designed to be robust to model uncertainties [9].

It is of practical interest to identify which processes will benefit significantly from the use of constrained control algorithms such as model predictive control, and which will not. This is especially important in the control of large-scale systems, where the computations associated with the implementation of constrained control techniques such as model predictive control may not be feasible without an expensive upgrade of the existing control hardware. This is also of interest for application to high-speed control systems such as controlling the idle speed in motor vehicles, where there is significant pressure to minimise control-hardware costs. In such applications, model predictive control should only be used if significant performance improvements can be achieved.

Explicit conditions are derived that identify whether a particular process may benefit from constraint handling. The conditions are computable for large-scale systems directly from its transfer-function model, a simulation model (e.g. defined by a set of ordinary/partial-differential equations and algebraic conditions), or experimental input–output data. The formulation allows the consideration of measurement noise, process disturbances, model uncertainties, plant directionality and the quantity of experimental data, all of which can be important in determining whether constraint handling is needed. The conditions are readily computable for large-scale systems. They do not require extensive time-domain simulations, or other computationally expensive testing. These conditions are also useful for understanding the interactions between design and control for a particular system, especially for actuator placement and selection.

2 Pseudo-singular value framework

Consider an $n \times n$ transfer function matrix $\boldsymbol{P}(s)$ that relates the manipulated variables u to the controlled variables y.

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The columns and rows of $\boldsymbol{P}(s)$ are scaled so that the manipulated variable constraints are described as having a norm bound less than 1 and the maximum allowed deviation in the controlled variables is described by a norm bound less than 1 [10]. Although for most practical purposes an ∞-norm would be appropriate, the more general Hölder p-norm is considered as it can have theoretical or practical advantages in certain situations.

All poles of $\boldsymbol{P}(s)$ are assumed to be in the open left-half plane, which is applicable to many chemical processes. The matrix $\boldsymbol{P}(s)$ is assumed to be square for simplicity in notation only; all definitions and results in the manuscript are generalised to nonsquare $\boldsymbol{P}(s)$ by augmenting the transfer-function matrix with either rows or columns of zeros. The real singular-value decomposition of the steady-state gain matrix is defined by

$$\boldsymbol{P}(0) = \hat{\boldsymbol{U}}\boldsymbol{\Sigma}\hat{\boldsymbol{V}}^T \tag{1}$$

where $\hat{\boldsymbol{U}}$ and $\hat{\boldsymbol{V}}$ are constant real orthogonal matrices, and $\boldsymbol{\Sigma}$ is a constant real diagonal matrix whose diagonal elements are the singular values. The singular values are ordered and nonnegative, that is, $\Sigma_{11} \geq \Sigma_{22} \geq \cdots \geq \Sigma_{nn} \geq 0$. The singular values (and hence $\boldsymbol{\Sigma}$) are unique for a given matrix $\boldsymbol{P}(0)$, while the $\hat{\boldsymbol{U}}$ and $\hat{\boldsymbol{V}}$ matrices are nonunique. For example, multiplying the ith columns of $\hat{\boldsymbol{U}}$ and $\hat{\boldsymbol{V}}$ by -1 results in an additional pair of matrices $(\hat{\boldsymbol{U}}, \hat{\boldsymbol{V}})$ that satisfy (1). The singular-value decomposition of the transfer-function matrix, both at steady-state and as a function of frequency, has been applied to process control problems for more than ten years [11–15].

Now define the diagonal matrix $\boldsymbol{D}_U$ ($\boldsymbol{D}_V$) which has each element either $+1$ or -1 such that the topmost nonzero element in each column of $\hat{\boldsymbol{U}}\boldsymbol{D}_U$ ($\hat{\boldsymbol{V}}\boldsymbol{D}_V$) is positive. Then

$$\begin{aligned}\boldsymbol{P}(0) = \hat{\boldsymbol{U}}\boldsymbol{\Sigma}\hat{\boldsymbol{V}}^T &= (\hat{\boldsymbol{U}}\boldsymbol{D}_U)(\boldsymbol{D}_U\boldsymbol{\Sigma}\boldsymbol{D}_V)(\hat{\boldsymbol{V}}\boldsymbol{D}_V)^T \\ &= \boldsymbol{U}\boldsymbol{\Lambda}(0)\boldsymbol{V}^T\end{aligned} \tag{2}$$

where $\boldsymbol{\Lambda}(0) = \boldsymbol{D}_U\boldsymbol{\Sigma}\boldsymbol{D}_V$ is a constant real diagonal matrix whose diagonal elements are referred to as pseudosingular values [9]. The pseudosingular values can be of any sign (including zero). The unique matrix $\boldsymbol{\Lambda}(0)$ is the steady-state matrix for a transfer-function matrix $\boldsymbol{\Lambda}(s)$ defined by

$$\boldsymbol{\Lambda}(s) = \boldsymbol{U}^T\boldsymbol{P}(s)\boldsymbol{V} \tag{3}$$

which is referred to as the pseudosingular value matrix. While $\boldsymbol{\Lambda}(s)$ is diagonal at steady-state ($s = 0$), it is not necessarily diagonal for other values of s. Each off-diagonal element is a transfer function whose steady-state value is zero.

Pseudosingular values are closely related to singular values but are allowed to have sign. The term 'pseudosingular values' was first proposed in reference to a class of industrially relevant processes referred to as 'pseudoSVD processes' [16], which are defined as those processes which have $\boldsymbol{\Lambda}(s)$ diagonal for all values of s. These processes include paper machines, adhesive coaters, polymer-film extruders and certain classes of distribution networks, such as used in electric power systems and ship communication systems. The pseudoSVD-process structure was first most clearly defined in two papers by Hovd *et al.* [17, 18], where it was shown that controllers of the form $\boldsymbol{K}(s) = \boldsymbol{V}^T\boldsymbol{\Lambda}_K(s)\boldsymbol{U}$ provided optimal stability and performance robustness for pseudoSVD processes with a large number of uncertainty structures.

Later work proposed the above definition of pseudosingular values which applies to general plant transfer functions [9, 19].

Controllability can be assessed directly in terms of pseudosingular values. In particular, to control an output-disturbance direction with zero steady-state error, it is necessary to identify correctly the sign of the corresponding pseudosingular value at steady state. Statistically, the pseudosingular values whose steady-state signs are known with confidence define the controllable disturbance directions, while the pseudosingular values whose signs are not known with confidence define disturbance directions that cannot be controlled reliably. The pseudosingular values also have an interpretation in terms of signal-to-noise ratios [9, 19].

Multivariate statistics and Monte Carlo procedures have been developed for quantifying the accuracy of pseudosingular values from experimental input–output data [9, 19]. The accuracy estimates are dependent on the noise levels, the quantity of available experimental data and the plant directionality. Alternatively, if the pseudosingular values are applied in an early design phase (before the plant is built), then their accuracies can be estimated by applying one of the identification procedures to a simulation model which incorporates models for the process disturbances and measurement noise.

For assessing whether a process can benefit from constrained control, the pseudosingular-value decomposition of the plant model is first identified, with all the pseudosingular values classified as being controllable or uncontrollable. The uncontrollable pseudosingular values are statistically indistinguishable from zero. Without loss in generality it is assumed that the r controllable pseudosingular values appear first in the pseudosingular value matrix. If this assumption does not hold, then relax the condition that pseudosingular values are ordered by magnitude in the steady-state pseudosingular-value matrix $\boldsymbol{\Lambda}(0)$, and reorder the columns of the input and output rotation matrices in (2) so that the first r pseudosingular values are controllable.

3 Assessment based on steady-state models

The following mathematical relationships are needed in the presentation of the results. The steady-state true process-gain matrix $\boldsymbol{P}(0)$ can be written in terms of the pseudo SVD as

$$\boldsymbol{P}(0) = \sum_{i=1}^{n} \Lambda_{ii}(0)\boldsymbol{U}^i(\boldsymbol{V}^i)^T \tag{4}$$

where $\boldsymbol{U}^i$ and $\boldsymbol{V}^i$ are the ith columns of $\boldsymbol{U}$ and $\boldsymbol{V}$, respectively.

Since the columns of $\boldsymbol{V}$ form a complete orthonormal basis set, the steady-state value of the manipulated variable u can be written as

$$\boldsymbol{u} = \sum_{j=1}^{n} \alpha_j \boldsymbol{V}^j \tag{5}$$

where the real scalar $\alpha_j = (\boldsymbol{V}^j)^T\boldsymbol{u}$ quantifies the extent of manipulated variable movement $\boldsymbol{u}$ in the direction of $\boldsymbol{V}^j$. Similarly, the effect of the disturbances on the output (see Fig. 1) can be written as

$$\hat{\boldsymbol{d}} = \sum_{j=1}^{n} \beta_j \boldsymbol{U}^j \tag{6}$$

where the real scalar $\beta_j = (\boldsymbol{U}^j)^T\hat{\boldsymbol{d}}$ quantifies the extent of the steady-state disturbance in the direction of $\boldsymbol{U}^j$. A disturbance transfer function is incorporated into this framework by defining $\hat{\boldsymbol{d}} = \boldsymbol{P}_d\boldsymbol{d}$, where $\boldsymbol{P}_d$ is the disturbance transfer function and the columns of $\boldsymbol{P}_d$ are scaled so that the magnitude of the potential disturbances $\boldsymbol{d}$ is norm bounded by one [10]. A load disturbance $\boldsymbol{l}$ can be incorporated into this framework by defining $\hat{\boldsymbol{d}} = \boldsymbol{P}\boldsymbol{l}$.

Thus the steady-state controlled output $\boldsymbol{y}$ is given by

$$\boldsymbol{y} = \boldsymbol{P}(0)\boldsymbol{u} + \hat{\boldsymbol{d}} \tag{7}$$

$$\boldsymbol{y} = \sum_{i=1}^{n} \Lambda_{ii}(0)\boldsymbol{U}^i(\boldsymbol{V}^i)^T \sum_{j=1}^{n} \alpha_j \boldsymbol{V}^j + \sum_{j=1}^{n} \beta_j \boldsymbol{U}^j \tag{8}$$

$$\boldsymbol{y} = \sum_{j=1}^{n} \{\Lambda_{jj}(0)\alpha_j + \beta_j\}\boldsymbol{U}^j \tag{9}$$

While the focus of this manuscript is on disturbance suppression, the results also hold for reference tracking by replacing $\boldsymbol{y}$ with the error signal $\boldsymbol{e} = \boldsymbol{y} - \boldsymbol{r}$ and replacing $\hat{\boldsymbol{d}}$ with the negative of the setpoint signal $(-\boldsymbol{r})$.

3.1 Inputs for perfect control within the controllable subspace

Owing to limited experimental data, measurement noise and process disturbances that occur during identification, not all pseudosingular values of a large-scale-process model may be identified with sufficient accuracy to be used for control. Multivariate statistics show that the inherent identifiability of the pseudosingular values becomes even more difficult as the process dimension increases [20]. This implies that a significant number of pseudosingular values will be uncontrollable for large-scale processes, making it impossible to obtain zero steady-state error for all disturbance directions. This Section computes the magnitude of the manipulated variable vector needed to achieve zero steady-state error in the r controllable disturbance directions. The manipulated variable vector $\boldsymbol{u}$ that does this is defined by (5) with

$$\alpha_j = \begin{cases} \dfrac{-\beta_j}{\Lambda_{jj}(0)} & j = 1, \ldots, r \\ 0 & j = r+1, \ldots, n \end{cases} \tag{10}$$

where $\beta_j = (\boldsymbol{U}^j)^T \boldsymbol{P}_d \boldsymbol{d}$. This manipulated variable vector $\boldsymbol{u}$ does not make any manipulations in plant-input directions associated with uncontrollable disturbance directions. Inserting into (9), the resulting output vector is

$$\boldsymbol{y} = \sum_{j=r+1}^{n} \beta_j \boldsymbol{U}^j \tag{11}$$

The norm of $\boldsymbol{y}$ should be computed and compared with one to determine whether the desired output performance can be achieved. No constraints on the manipulated variable has been considered yet, so the norm of $\boldsymbol{y}$ represents the best output performance achievable by any control algorithm, whether constrained, unconstrained, linear, and/or nonlinear. If the norm of $\boldsymbol{y}$ is greater than one, then the engineer needs to redesign the process (e.g. purchasing higher-quality sensors so that more directions are controllable when estimating model parameters from input–output data, or incorporating surge tanks to suppress the effect of disturbances on the output), weaken the performance specifications, and/or collect more experimental data so as to provide a more accurate model.

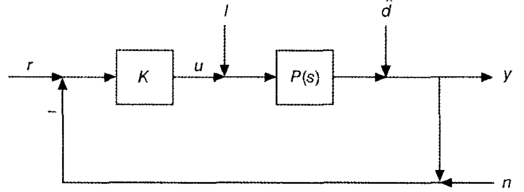

Fig. 1 *Standard feedback control system*

The manipulated variable is $\boldsymbol{u}$, the process output is y, the setpoint is $\boldsymbol{r}$, the measurement noise is $\boldsymbol{n}$ and the disturbances are $\hat{\boldsymbol{d}}$ and $\boldsymbol{l}$. The process transfer function is $\boldsymbol{P}(s)$. The controller $\boldsymbol{K}$ can be linear or nonlinear (e.g. a model predictive controller)

Recall that the plant transfer function was scaled so that the allowable manipulated variables are described by having the norm of $\boldsymbol{u}$ less than one. Inserting (10) into (5) gives

Condition 1:

$$\|\boldsymbol{u}\|_p = \left\| \sum_{j=1}^{r} \frac{(\boldsymbol{U}^j)^T \boldsymbol{P}_d \boldsymbol{d}}{\Lambda_{jj}(0)} \boldsymbol{V}^j \right\|_p < 1 \tag{12}$$

If condition 1 is satisfied, then there exists a control action that can achieve zero steady-state error in the controllable disturbance directions for the given disturbance, such that the constraint on the actuator moves is inactive. Hence a constrained controller such as model predictive control will not provide any improvement in the closed-loop steady-state performance over unconstrained control.

If condition 1 is not satisfied, then a violation of the actuator constraints is required for any controller to achieve zero steady-state error in the controllable disturbance directions for the given disturbance. In this case, a control law that drives the manipulated variable vector to the boundary of the actuator constraint set is optimal at steady-state. This implies that a control law that takes the actuator constraints into account, such as a model predictive control law, may achieve better performance than an unconstrained control law.

The norm of the vector of manipulated variables $\boldsymbol{u}$ in (12) depends on how much of the disturbance occurs in controllable disturbance directions, and on the magnitude of the controllable pseudosingular values. Controlling disturbance directions associated with small pseudosingular values requires large manipulated variable moves. As will be seen later in this Section, condition 1 generalises conditions written in terms of the minimum singular value of the plant matrix [12, 21], which only considered the 2-norm, and did not take into account the fact that some plant directions can be uncontrollable owing to model uncertainty.

Whereas condition 1 was derived for a particular disturbance, condition 2 gives the appropriate condition for determining whether constrained control should be used, based on a set of norm-bounded disturbances.

Condition 2:

$$\max_{\|\boldsymbol{d}\|_p \le 1} \|\boldsymbol{u}\|_p = \left\| \sum_{j=1}^{r} \frac{1}{\Lambda_{jj}(0)} \boldsymbol{V}^j (\boldsymbol{U}^j)^T \boldsymbol{P}_d \right\|_{ip} < 1 \tag{13}$$

where $\|\cdot\|_{ip}$ is the induced Hölder p-norm. This is proved by rearranging the expression in (12) and applying the definition of the induced matrix norm. If condition 2 is satisfied, then there exists a control action that can achieve zero steady-state error in the controllable disturbance directions for any norm-bounded disturbance $\boldsymbol{d}$, such that the constraint on the actuator moves is inactive. In this case, no improvement in the closed-loop steady-state performance is possible by using a control action that is on the boundary of the actuator constraint region. In other words, when condition 2 is satisfied, an unconstrained control law is optimal at steady-state, for each norm-bounded disturbance $\boldsymbol{d}$.

If condition 2 is not satisfied, then a violation of the actuator constraints is required for any controller to achieve zero steady-state error in the controllable disturbance directions, for some of the disturbances $\boldsymbol{d}$. This implies that a control law that takes the actuator constraints into account, such as a model predictive-control law, may achieve better performance than an unconstrained control law.

Conditions 1 and 2 are generalisations of conditions presented in Section 6.9.1 of [10] to the case where not all disturbance directions are controllable, and to more general norms. It is straightforward to generalise the above results so that different norms are used on the disturbances $\boldsymbol{d}$ and manipulated variables $\boldsymbol{u}$. It may be useful, for example, to use an ∞-norm to represent the set of allowable manipulated variable moves $\boldsymbol{u}$ and a 2-norm to represent the set of norm-bounded disturbances $\boldsymbol{d}$.

To gain more insight and to make the connections to previous work clearer, consider the case of the 2-norm, where the norm of the uncontrollable part of the plant output $\boldsymbol{y}$ is given by

$$\|\boldsymbol{y}\|_2^2 = \left\| \sum_{j=r+1}^{n} \beta_j \boldsymbol{U}^j \right\|_2^2 = \beta_{r+1}^2 + \cdots + \beta_n^2 \tag{14}$$

The magnitude of the output error is monotonically decreasing in r—the larger the number of controllable disturbance directions, the smaller the output error can be made. If all the disturbance directions are controllable, then zero steady-state error is possible using a controller with integral action.

In the example of the 2-norm, the norm of the manipulated variable vector that gives zero output error in the controllable disturbance directions is given by

$$\|\boldsymbol{u}\|_2^2 = \left\| \sum_{j=1}^{r} \frac{-\beta_j}{\Lambda_{jj}(0)} \boldsymbol{V}^j \right\|_2^2 = \frac{\beta_1^2}{\Lambda_{11}^2(0)} + \cdots + \frac{\beta_r^2}{\Lambda_{rr}^2(0)} \tag{15}$$

The magnitude of the control action is monotonically increasing in r—the larger the number of controllable disturbance directions, the more control action needed to reduce the output error to zero in the controllable disturbance directions. Assume for convenience that the controllable pseudosingular values are ordered in terms of largest to smallest magnitude. For a disturbance whose 2-norm is bounded by one, the maximum magnitude of the manipulated variable vector $\boldsymbol{u}$ occurs for $\beta_r = 1$ with all other $\beta_j = 0$:

$$\|\boldsymbol{u}\|_2 = \frac{1}{|\Lambda_{rr}(0)|} \tag{16}$$

If all the disturbance directions were controllable, then this is equal to

$$\|\boldsymbol{u}\|_2 = \frac{1}{\underline{\sigma}(\boldsymbol{P}(0))} \tag{17}$$

which was reported in earlier work [12, 21].

While the results for the 2-norm are particularly intuitive, other norms such as the ∞-norm are more useful for application to large-scale systems.

3.2 Inputs for acceptable control

Rather than requiring zero steady-state error in all controllable disturbance directions, an alternative approach is to compute the minimum control effort $\boldsymbol{u}$ necessary to obtain an acceptable output error ($\boldsymbol{y}$ norm bounded by 1). This can be formulated as

$$\min_{\|\boldsymbol{P}\boldsymbol{u}+\boldsymbol{P}_d\boldsymbol{d}\|_p \le 1} \|\boldsymbol{u}\|_p < 1 \tag{18}$$

Inserting (5) and (9), and taking into account that manipulated variable moves should only occur in controllable directions ($\alpha_j = 0$ for $j = r+1, \ldots, n$), gives condition 3:

Condition 3:

$$\min_{\left\| \sum_{j=1}^{r} \Lambda_{jj}(0)\alpha_j \boldsymbol{U}^j + \boldsymbol{P}_d\boldsymbol{d} \right\|_p \le 1} \left\| \sum_{j=1}^{r} \alpha_j \boldsymbol{V}^j \right\|_p < 1 \tag{19}$$

If condition 3 is satisfied, than an unconstrained control law can provide satisfactory closed-loop performance. If condition 3 is not satisfied, a control law that takes the actuator constraints into account, such as a model predictive-control law, may achieve better performance than an unconstrained control law.

The left-hand side is a convex optimisation over α_j that can be solved using standard techniques such as interior-point algorithms. In the example of a 2-norm, the convex optimisation can be solved much more efficiently using Lagrange multipliers [22]. The use of the 1-norm or ∞-norm results in a linear program. It is worth mentioning that (19) is automatically satisfied if $\|\boldsymbol{P}_d\boldsymbol{d}\|_p \le 1$, since in this case the left-hand side of (19) is equal to zero.

An outer maximisation over norm-bounded disturbances $\boldsymbol{d}$ defines condition 4:

Condition 4:

$$\max_{\|\boldsymbol{d}\|_p \le 1} \min_{\left\| \sum_{j=1}^{r} \Lambda_{jj}(0)\alpha_j \boldsymbol{U}^j + \boldsymbol{P}_d\boldsymbol{d} \right\|_p \le 1} \left\| \sum_{j=1}^{r} \alpha_j \boldsymbol{V}^j \right\|_p < 1 \tag{20}$$

Condition 4 can be computed using global optimisation algorithms [23], at least for medium-scale processes. If $\|\boldsymbol{P}_d\|_{ip} \le 1$, then condition 4 is automatically satisfied, since in this case the left-hand side of (19) is equal to zero.

If condition 4 is satisfied, than an unconstrained control law can provide satisfactory closed-loop performance for all norm-bounded disturbances $\boldsymbol{d}$.

4 Assessment based on frequency-domain models

As before, assume that $\boldsymbol{P}(s)$ and $\boldsymbol{P}_d(s)$ are stable transfer functions. The controlled output $\boldsymbol{y}$ is given by

$$\boldsymbol{y}(s) = \boldsymbol{P}(s)\boldsymbol{u}(s) + \boldsymbol{P}_d(s)\boldsymbol{d}(s) \tag{21}$$

where the manipulated variable $\boldsymbol{u}$, disturbance $\boldsymbol{d}$, and output $\boldsymbol{y}$ are written in terms of deviation variables.

The single-input/single-output case will be used to illustrate the basic approach before deriving the multivariable case.

4.1 Single-input/single-output (SISO) case

The input that provides zero output error is

$$\boldsymbol{u}(s) = -\boldsymbol{P}^{-1}(s)\boldsymbol{P}_d(s)\boldsymbol{d}(s) \tag{22}$$

If the plant $\boldsymbol{P}(s)$ is nonminimum phase (has right-half-plane zeros or time delays), then this control action will not internally stabilise the closed-loop system. In this case, replace $\boldsymbol{P}^{-1}(s)$ by a physically realisable stable inverse $\tilde{\boldsymbol{P}}^{\dagger}(s)$, such as the inverse of the minimum phase part of $\boldsymbol{P}(s)$, or an optimal inverse (e.g. as in theorem 4.1-1 of [24]).

$$\boldsymbol{u}(s) = -\tilde{\boldsymbol{P}}^{\dagger}(s)\boldsymbol{P}_d(s)\boldsymbol{d}(s) \tag{23}$$

Also, limitations on the closed-loop performance due to model uncertainty can be addressed by multiplying the realisable inverse by a filter [24].

To derive a frequency-domain condition, assume that the disturbance is a sine function, i.e. $\boldsymbol{d}(t) = \sin(\omega t)$. Then the manipulated variable $\boldsymbol{u}$ is

$$\boldsymbol{u}(s) = -\frac{\omega}{s^2+\omega^2}\tilde{\boldsymbol{P}}^{\dagger}(s)\boldsymbol{P}_d(s) \tag{24}$$

After the initial transients die away, only terms associated with sines and cosines are persistent [25, 26]. The manipulated variable as $t \to \infty$ is

$$\boldsymbol{u}(t) = -|\tilde{\boldsymbol{P}}^{\dagger}(j\omega)\boldsymbol{P}_d(j\omega)|\sin(\omega t + \phi) \tag{25}$$

where $\phi = \arg\{\tilde{\boldsymbol{P}}^{\dagger}(j\omega)\boldsymbol{P}_d(j\omega)\}$. Thus

$$\limsup_{t\to\infty} |\boldsymbol{u}(t)| = |\tilde{\boldsymbol{P}}^{\dagger}(j\omega)\boldsymbol{P}_d(j\omega)| \tag{26}$$

which forms the basis for the next condition.

Condition 5a:

$$|\tilde{\boldsymbol{P}}^{\dagger}(j\omega)\boldsymbol{P}_d(j\omega)| < 1 \tag{27}$$

If condition $5a$ is satisfied at a particular frequency ω, then there exists a control action that at long times can achieve the output error for a sinusoidal disturbance at that frequency, such that the constraint on the actuator moves is inactive. In this case, no improvement in the performance is possible by using a control action that is on the boundary of the actuator constraint region. If condition $5a$ is not satisfied, then a control law that takes the actuator constraints into account, such as a model predictive-control law, may achieve better performance than an unconstrained control law.

Condition $5a$ should be tested for those frequencies where disturbance suppression is desired, namely for frequencies less than the desired closed loop bandwidth. Condition $5a$ is a generalisation of equation (5.57) of [10] to handle broader classes of physically realisable inverses.

4.2 Multi-input/multi-output (MIMO): fully controllable case

Here the SISO results are extended to the MIMO case. First assume that the plant matrix $\boldsymbol{P}(s)$ is invertible and the rank of the controllable subspace is n. The conditions for the cases in which the rank of the controllable subspace is less than n will be derived later.

The input that provides zero output error is

$$\boldsymbol{u}(s) = -\boldsymbol{P}^{-1}(s)\boldsymbol{P}_d(s)\boldsymbol{d}(s) \tag{28}$$

If the plant transfer function is nonminimum phase (e.g. has right-half-plane transmission zeros), then $\boldsymbol{P}^{-1}$ is replaced by a physically realisable stable inverse $\tilde{\boldsymbol{P}}^{\dagger}$ (e.g. chapter 12 of [24]):

$$\boldsymbol{u}(s) = -\tilde{\boldsymbol{P}}^{\dagger}(s)\boldsymbol{P}_d(s)\boldsymbol{d}(s) \tag{29}$$

where, as in the SISO case, the physically realisable stable inverse can be multiplied by a filter to take model uncertainty into account. For MIMO processes, the constraints are checked for each input channel separately. Then the results for SISO processes can be used to derive the conditions for MIMO processes. Let a particular sinusoidal disturbance be given by

$$\boldsymbol{d}(t) = \sin(\omega t)\tilde{\boldsymbol{d}} \tag{30}$$

where $\tilde{\boldsymbol{d}}$ is a constant vector with dimension m. For constraints on the ith manipulated variable u_i to be inactive at long times, the following condition has to be satisfied:

$$\limsup_{t\to\infty} |u_i(t)| = |[\tilde{\boldsymbol{P}}^{\dagger}(j\omega)]_i\boldsymbol{P}_d(j\omega)\tilde{d}| \tag{31}$$

where $[\tilde{\boldsymbol{P}}^{\dagger}(s)]_i$ denotes ith row of the matrix $\tilde{\boldsymbol{P}}^{\dagger}(s)$. Testing whether the left-hand side is less than one gives the following condition:

Condition 5b:

$$|[\tilde{\boldsymbol{P}}^{\dagger}(j\omega)]_i\boldsymbol{P}_d(j\omega)\tilde{d}| < 1 \tag{32}$$

For any disturbance direction of interest and each manipulated variable, condition $5b$ is interpreted in the same manner as for condition $5a$.

In practice, it is useful to check whether constrained control is needed for any disturbance from a norm-bounded set of disturbances:

$$\|\tilde{\boldsymbol{d}}\|_p \leq 1 \tag{33}$$

The appropriate result can be derived using a property of the dual norm [27].

$$\sup_{\|\tilde{\boldsymbol{d}}\|_p\leq 1}\ \limsup_{t\to\infty} |u_i(t)| = \|[\tilde{\boldsymbol{P}}^{\dagger}(j\omega)]_i\boldsymbol{P}_d(j\omega)\|_q \tag{34}$$

where $1/p + 1/q = 1$. This results in a condition appropriate for a set of norm-bounded disturbances:

Condition 6a:

$$\|[\tilde{\boldsymbol{P}}^{\dagger}(j\omega)]_i\boldsymbol{P}_d(j\omega)\|_q < 1 \tag{35}$$

Condition $6a$ is interpreted in the same manner as for condition $5b$, except that condition $6a$ applies to a set of norm-bounded disturbances. Constrained control is recommended for processes in which condition $6a$ is violated for some i.

The above analysis takes into account nonminimum phase behaviour, general norms on the disturbances (in condition $6a$), and when a filter is used, dynamic limitations posed by model uncertainty. When the plant transfer function is restricted to be minimum phase and semiproper, condition $5b$ is equivalent to a condition on page 229 of [10].

When $\boldsymbol{P}_d$ is not a vector, then the frequency-domain conditions in this section do not have counterparts to the results in section 6.9.1 of [10]. For example, condition (6.38) of [10] does not apply here, because it does not take into account that different elements of $\tilde{\boldsymbol{P}}^{\dagger}(s)\boldsymbol{P}_d(s)$ can have different phase angles at any particular frequency, so that the 'lim sup' for any particular manipulated variable $u_i(t)$ cannot be computed directly from the magnitude of each element of $\tilde{\boldsymbol{P}}^{\dagger}(j\omega)\boldsymbol{P}_d(j\omega)$.

4.3 Multi-input/multi-output (MIMO): general case

For the general case, it is assumed that the control moves are only made in the r manipulated variable directions that are controllable. Hence

$$\boldsymbol{u}(s) = \sum_{j=1}^{r} \alpha_j(s)\boldsymbol{V}^j = \underbrace{[\boldsymbol{V}^1 \cdots \boldsymbol{V}^r]}_{\tilde{V}} \underbrace{\begin{bmatrix} \alpha_1(s) \\ \vdots \\ \alpha_r(s) \end{bmatrix}}_{\underline{\alpha}(s)} \tag{36}$$

Inserting into (21) gives

$$\boldsymbol{y}(s) = \boldsymbol{P}(s)\tilde{\boldsymbol{V}}\underline{\alpha}(s) + \boldsymbol{P}_d(s)\boldsymbol{d}(s) \tag{37}$$

It is assumed that the controller only attempts to provide zero error in disturbance directions that are controllable. The vector $\underline{\boldsymbol{\alpha}}$ that does this satisfies

$$\underbrace{\begin{bmatrix} (U^1)^T \\ \vdots \\ (U^r)^T \end{bmatrix}}_{\tilde{U}^T} \boldsymbol{y}(s) = \tilde{\boldsymbol{U}}^T\boldsymbol{P}(s)\tilde{\boldsymbol{V}}\underline{\alpha}(s) + \tilde{\boldsymbol{U}}^T\boldsymbol{P}_d(s)\boldsymbol{d}(s) = 0 \tag{38}$$

Inserting $\boldsymbol{P}(s)$ from (3) and rearranging gives

$$\tilde{\boldsymbol{U}}^T\boldsymbol{y}(s) = \tilde{\boldsymbol{\Lambda}}(s)\underline{\alpha}(s) + \tilde{\boldsymbol{U}}^T\boldsymbol{P}_d(s)\boldsymbol{d}(s) = 0 \tag{39}$$

where $\tilde{\boldsymbol{\Lambda}}(s)$ is the first r rows and r columns of $\boldsymbol{\Lambda}(s)$. The manipulated variables that provide zero output error for a sinusoidal disturbance (30) are given by

$$\underline{\alpha}(s) = -\frac{\omega}{s^2+\omega^2}\tilde{\boldsymbol{\Lambda}}^{-1}(s)\tilde{\boldsymbol{U}}^T\boldsymbol{P}_d(s)\tilde{\boldsymbol{d}} \tag{40}$$

If $\tilde{\boldsymbol{\Lambda}}^{-1}(s)$ is nonminimum phase, then replace it with a physically realizable stable inverse [24]

$$\underline{\alpha}(s) = -\frac{\omega}{s^2+\omega^2}\tilde{\boldsymbol{\Lambda}}^{\dagger}(s)\tilde{\boldsymbol{U}}^T\boldsymbol{P}_d(s)\tilde{\boldsymbol{d}} \tag{41}$$

Written in terms of the original manipulated variables $\boldsymbol{u}$, this is

$$\boldsymbol{u}(s) = -\tilde{\boldsymbol{V}}\underline{\alpha}(s) = -\frac{\omega}{s^2+\omega^2}\tilde{\boldsymbol{V}}\tilde{\boldsymbol{\Lambda}}^{\dagger}(s)\tilde{\boldsymbol{U}}^T\boldsymbol{P}_d(s)\tilde{\boldsymbol{d}} \tag{42}$$

For constraints on the ith manipulated variable u_i to be inactive at long times, the following condition has to be satisfied:

$$\limsup_{t\to\infty} |u_i(t)| = |[\tilde{\boldsymbol{V}}]_i\tilde{\boldsymbol{\Lambda}}^{\dagger}(j\omega)\tilde{\boldsymbol{U}}^T\boldsymbol{P}_d(j\omega)\tilde{\boldsymbol{d}}| \tag{43}$$

Testing whether the left-hand side is less than one gives the following condition:

Condition 5c:

$$|[\tilde{\boldsymbol{V}}]_i\tilde{\boldsymbol{\Lambda}}^{\dagger}(j\omega)\tilde{\boldsymbol{U}}^T\boldsymbol{P}_d(j\omega)\tilde{\boldsymbol{d}}| < 1 \tag{44}$$

Condition $5c$ is interpreted in the same manner as for condition $5b$.

It is also useful to check whether constrained control is needed for any disturbance from a norm-bounded set of disturbances. Using the dual norm, the appropriate result is

$$\sup_{\|\tilde{d}\|_p\le 1}\ \limsup_{t\to\infty} |u_i(t)| = \|[\tilde{\boldsymbol{V}}]_i\tilde{\boldsymbol{\Lambda}}^{\dagger}(j\omega)\tilde{\boldsymbol{U}}^T\boldsymbol{P}_d(j\omega)\|_q \tag{45}$$

where $1/p + 1/q = 1$. The following is the test for norm-bounded disturbances:

Condition 6b:

$$\|[\tilde{\boldsymbol{V}}]_i\tilde{\boldsymbol{\Lambda}}^{\dagger}(j\omega)\tilde{\boldsymbol{U}}^T\boldsymbol{P}_d(j\omega)\|_q < 1 \tag{46}$$

Condition $6b$ is interpreted in the same manner as for condition $6a$. Constrained control is recommended for processes in which condition $6b$ is violated for some i.

The frequency-domain conditions for multivariable processes are not as general as those for the steady-state case, in that the frequency-domain conditions apply to each manipulated variable individually. Checking the frequency-domain conditions for each manipulated variable is equivalent to checking the ∞-norm on the vector of the manipulated variables. In contrast, the steady-state conditions allow general Holder p-norms on the manipulated variables. The ∞-norm is general enough for most practical applications.

All of the conditions of this section which take into account controllability of the disturbance directions have no counterparts in [10] or in any other earlier paper or thesis.

5 Assessment based on time-domain models

While frequency-domain conditions are a natural way to take dynamics into account, they do depend on the assumption that the disturbance is oscillatory, and only address the 'long-time' behaviour. This Section derives conditions applicable to disturbances with more general time-domain behaviour, and considers all time $t > 0$.

5.1 Multi-input/multi-output (MIMO): fully controllable case

As before, assume that $\boldsymbol{P}(s)$ and $\boldsymbol{P}_d(s)$ are stable transfer functions, with the manipulated variables given by (29). For a fixed disturbance $\boldsymbol{d}(s)$, the following equations hold:

$$\sup_{t>0} \|\boldsymbol{u}(t)\|_p = \sup_{t>0} \|\mathcal{L}^{-1}\{\boldsymbol{P}^{\dagger}(s)\boldsymbol{P}_d(s)\boldsymbol{d}(s)\}\|_p \tag{47}$$

$$\sup_{t>0} |u_i(t)| = \sup_{t>0} |\mathcal{L}^{-1}\{[\boldsymbol{P}^{\dagger}(s)]_i\boldsymbol{P}_d(s)\boldsymbol{d}(s)\}|\ \ \forall i \tag{48}$$

where $\mathcal{L}^{-1}$ is the inverse Laplace-transform operator. This leads to the following conditions. Depending on the norm, constrained control is recommended for processes in which one of the following conditions is violated:

Condition 7a:

$$\sup_{t>0} \|\mathcal{L}^{-1}\{\boldsymbol{P}^{\dagger}(s)\boldsymbol{P}_d(s)\boldsymbol{d}(s)\}\|_p < 1 \tag{49}$$

Condition 7a':

$$\sup_{t>0} |\mathcal{L}^{-1}\{[\boldsymbol{P}^{\dagger}(s)]_i\boldsymbol{P}_d(s)\boldsymbol{d}(s)\}|_p < 1\ \ \forall i \tag{50}$$

For sets of norm-bounded disturbances, numerous conditions can be derived depending on the signal norms or seminorms, such as the power seminorm, the spectral-density norm, the 2-norm (the 'integral-squared-error norm'), and the ∞-norm (the 'max norm'). The most useful norm for characterising actuator constraints is the ∞-norm:

$$\|\boldsymbol{u}(t)\|_\infty = \sup_{t>0}\max_i |u_i(t)| \tag{51}$$

For disturbances $\boldsymbol{d}$ in which each element is bounded by one for all time $t > 0$, the maximum value of the ith manipulated variable is [26]

$$\sup_{\|\boldsymbol{d}(t)\|_\infty \le 1} \sup_t |u_i(t)| = \|\mathcal{L}^{-1}\{[\boldsymbol{P}^\dagger(s)]_i \boldsymbol{P}_d(s)\}\|_1 \tag{52}$$

where $\|g(t)\|_1 \equiv \int_0^\infty \sum_j |g_j(t)| dt$. Constrained control is recommended for processes in which the following condition is violated for any i:

Condition 8a:

$$\|\mathcal{L}^{-1}\{[\boldsymbol{P}^\dagger(s)]_i \boldsymbol{P}_d(s)\}\|_1 < 1 \tag{53}$$

The 2-norm on the manipulated variables is also considered since it has a frequency-domain connection:

$$\|\boldsymbol{u}(t)\|_2^2 \equiv \int_0^\infty \|\boldsymbol{u}(t)\|_2^2 dt \tag{54}$$

For disturbances bounded by the 2-norm, the worst-case 2-norm of the vector of manipulated variables is [26]

$$\sup_{\|\boldsymbol{d}(t)\|_2 \le 1} \|\boldsymbol{u}(t)\|_2 = \|\boldsymbol{P}^\dagger(s)\boldsymbol{P}_d(s)\|_\infty \equiv \sup_{\omega \in \mathcal{R}} \bar{\sigma}\{\boldsymbol{P}^\dagger(j\omega)\boldsymbol{P}_d(j\omega)\} \tag{55}$$

where $\mathcal{R}$ is the set of real numbers and $\bar{\sigma}$ is the maximum singular value. Constrained control is recommended for processes in which the following condition is violated:

Condition 9a:

$$\sup_{\omega \in \mathcal{R}} \bar{\sigma}\{\boldsymbol{P}^\dagger(j\omega)\boldsymbol{P}_d(j\omega)\} < 1 \tag{56}$$

5.2 Multi-input/multi-output (MIMO): general case

The control moves that achieve zero output error in the controllable disturbance directions, while manipulating only in controllable manipulated variable directions, are

$$\boldsymbol{u}(s) = -\tilde{\boldsymbol{V}}\tilde{\boldsymbol{\Lambda}}^{-1}(s)\tilde{\boldsymbol{U}}^T\boldsymbol{P}_d(s)\boldsymbol{d}(s) \tag{57}$$

[This follows a similar development as used to derive (42).] If $\tilde{\boldsymbol{\Lambda}}(s)$ is nonminimum phase, it is replaced by a physically realisable stable inverse

$$\boldsymbol{u}(s) = -\tilde{\boldsymbol{V}}\tilde{\boldsymbol{\Lambda}}^\dagger(s)\tilde{\boldsymbol{U}}^T\boldsymbol{P}_d(s)\boldsymbol{d}(s) \tag{58}$$

For a fixed disturbance $\boldsymbol{d}(s)$, the following equations hold:

$$\sup_{t>0} \|\boldsymbol{u}(t)\|_p = \sup_{t>0} \|\tilde{\boldsymbol{V}}\mathcal{L}^{-1}\{\tilde{\boldsymbol{\Lambda}}^\dagger(s)\tilde{\boldsymbol{U}}^T\boldsymbol{P}_d(s)\boldsymbol{d}(s)\}\|_p \tag{59}$$

$$\sup_{t>0} |u_i(t)| = \sup_{t>0} |[\tilde{\boldsymbol{V}}]_i\mathcal{L}^{-1}\{\tilde{\boldsymbol{\Lambda}}^\dagger(s)\tilde{\boldsymbol{U}}^T\boldsymbol{P}_d(s)\boldsymbol{d}(s)\}| \quad \forall i \tag{60}$$

Depending on the norm, constrained control is recommended for processes in which one of the following conditions is violated:

Condition 7b:

$$\sup_{t>0} \|\tilde{\boldsymbol{V}}\mathcal{L}^{-1}\{\tilde{\boldsymbol{\Lambda}}^\dagger(s)\tilde{\boldsymbol{U}}^T\boldsymbol{P}_d(s)\boldsymbol{d}(s)\}\|_p < 1 \tag{61}$$

Condition 7b′:

$$\sup_{t>0} |[\tilde{\boldsymbol{V}}]_i\mathcal{L}^{-1}\{\tilde{\boldsymbol{\Lambda}}^\dagger(s)\tilde{\boldsymbol{U}}^T\boldsymbol{P}_d(s)\boldsymbol{d}(s)\}| < 1 \quad \forall i \tag{62}$$

For disturbances $\boldsymbol{d}$ in which each element is bounded by one for all time $t > 0$, the maximum value of the ith manipulated variable is [26]

$$\sup_{\|\boldsymbol{d}(t)\|_\infty \le 1} \sup_{t>0} |u_i(t)| = \|[\tilde{\boldsymbol{V}}]_i\mathcal{L}^{-1}\{\tilde{\boldsymbol{\Lambda}}^\dagger(s)\tilde{\boldsymbol{U}}^T\boldsymbol{P}_d(s)\}\|_1 \tag{63}$$

Constrained control is recommended for processes in which the following condition is violated for any i:

Condition 8b:

$$\|[\tilde{\boldsymbol{V}}]_i\mathcal{L}^{-1}\{\tilde{\boldsymbol{\Lambda}}^\dagger(s)\tilde{\boldsymbol{U}}^T\boldsymbol{P}_d(s)\}\|_1 < 1 \tag{64}$$

For disturbances bounded by a 2-norm, the worst-case 2-norm of the vector of manipulated variables is

$$\sup_{\|\boldsymbol{d}(t)\|_2 \le 1} \|\boldsymbol{u}(t)\|_2 = \|\tilde{\boldsymbol{V}}\tilde{\boldsymbol{\Lambda}}^\dagger(s)\tilde{\boldsymbol{U}}^T\boldsymbol{P}_d(s)\|_\infty \equiv \sup_{\omega \in \mathcal{R}} \bar{\sigma}\{\tilde{\boldsymbol{V}}\tilde{\boldsymbol{\Lambda}}^\dagger(j\omega)\tilde{\boldsymbol{U}}^T\boldsymbol{P}_d(j\omega)\} \tag{65}$$

where $\bar{\sigma}$ is the maximum singular value. Constrained control is recommended for processes in which the following condition is violated:

Condition 9b:

$$\sup_{\omega \in \mathcal{R}} \bar{\sigma}\{\tilde{\boldsymbol{V}}\tilde{\boldsymbol{\Lambda}}^\dagger(j\omega)\tilde{\boldsymbol{U}}^T\boldsymbol{P}_d(j\omega)\} < 1 \tag{66}$$

All of the conditions of this Section, which take into account violations in the actuator constraints at all future times, have no counterparts in [10] or in any other paper or thesis.

6 Case study: paper-machine control

The model predictive control of paper machines has been under active investigation [28–31]. Here the conditions are applied to a paper-machine-simulation model constructed from industrial input–output data [32]. The industrial paper machine has 650 measurements and 130 slice-lip actuators. The constraints and the noise in the measurements are also quantified from the input–output data. The controllable directions are thus specified by the noise, interactions among actuators, the constraints and the quantity of data. Fig. 2 shows all the pseudosingular values (Λ_{ii}) for this paper-machine model. It can be observed that, while most pseudosingular values are much larger than zero, some pseudosingular values are close to zero and are uncontrollable [9, 19].

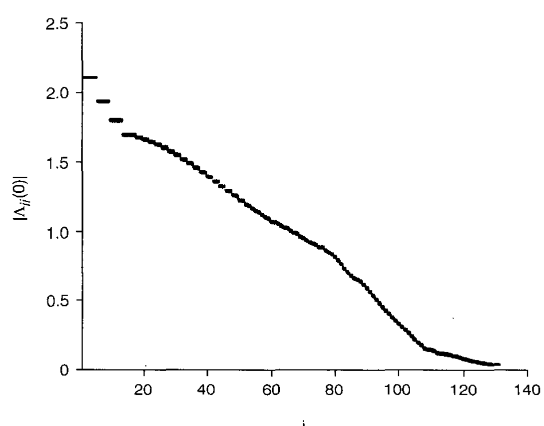

Fig. 2 *Pseudosingular values at steady state*

Table 1: Magnitude of u with respect to different disturbances and norms (condition 1)

	1-norm	2-norm	∞-norm
$\hat{d}_1$	324.83	29.08	4.11
$\hat{d}_2$	3.50	0.60	0.30
$\hat{d}_3$	16.59	3.87	2.32

The goal is to determine whether a constrained control algorithm such as model predictive control (MPC) should be used. Unless stated otherwise, the number of controllable directions, which is set by the measurement noise level and the quantity of experimental data, is assumed to be $r = 107$. Here three particular disturbances are considered, to determine which disturbances can be suppressed with unconstrained control, and which disturbances are best handled by MPC. The three disturbances are:

$$\hat{d}_1 = [1 \ \cdots \ 1]^T \quad \text{(uniform)} \tag{67}$$

$$\hat{d}_2 = \underbrace{[0 \ \cdots \ 0 \ 1 \ 0 \ \cdots \ 0]^T}_{\text{1 in the 60th position}} \quad \text{(spike)} \tag{68}$$

$$\hat{d}_3 = \underbrace{[0 \ \cdots \ 0 \ 1 \ \cdots \ 1 \ 0 \ \cdots \ 0]^T}_{\text{1 in the 60th to 80th positions}} \quad \text{(square)} \tag{69}$$

The magnitudes of the manipulated variables required to achieve zero steady-state error for the three disturbances are reported in Table 1. The spike disturbance $\hat{d}_2$ is the easiest to handle, followed by the square disturbance $\hat{d}_3$, followed by the uniform disturbance $\hat{d}_1$. This is expected from physical considerations, since it is very difficult in practice (sometimes impossible [7]) to suppress uniform disturbances using CD control, given that the total flow rate is usually controlled rather tightly by a separate MD controller. Another observation is that the magnitude of the manipulated variable is very different depending on which norm is used to measure its magnitude. While past studies used the 2-norm [12, 21], the ∞-norm is typically a more accurate measure to describe the acceptable manipulated variable moves. Based on the ∞-norm, the spike disturbance can be suppressed using unconstrained control, whereas the other two disturbances cannot be suppressed with the allowed manipulated variable moves.

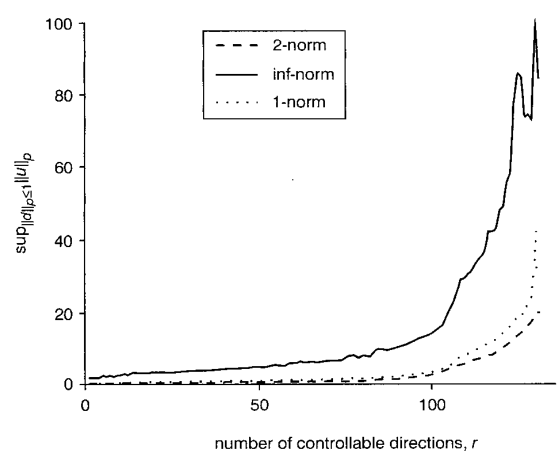

Fig. 3 *Magnitude of the manipulated variables required to obtain zero steady-state error for norm-bounded disturbances (the right-hand side of condition 2) for a paper-machine model*

Now the disturbances will be treated as norm-bounded, and the number of controllable directions will be varied to explore this effect on the magnitude of the desired control action. Fig. 3 shows that the control effort tends to increase as more directions are controllable, with the increase being nearly monotonic. Thus, MPC becomes increasingly beneficial as the number of controllable directions is increased,

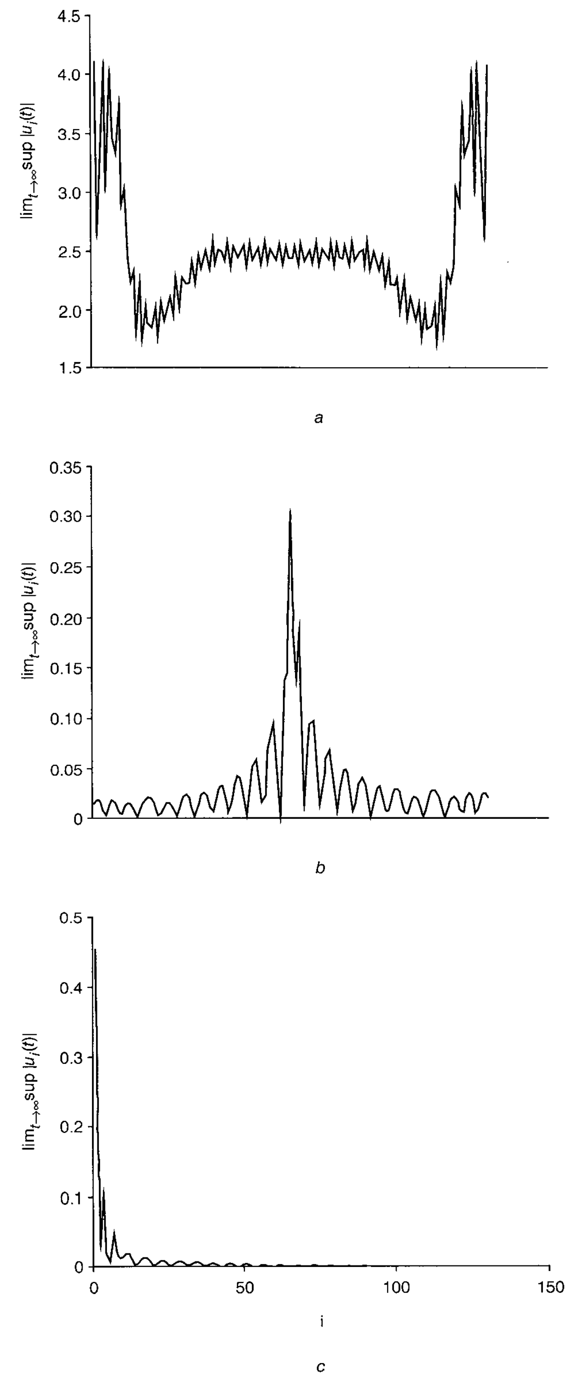

Fig. 4 *Manipulated variable magnitudes needed to provide zero steady-state error in the controllable directions (condition 5c with $\omega = 0$)*
a $\hat{d}_1$
b $\hat{d}_2$
c $\hat{d}_4$

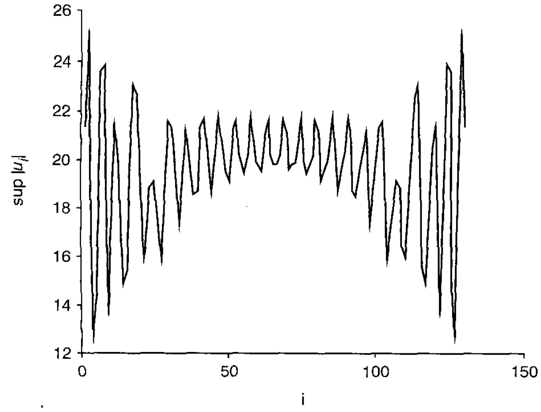

Fig. 5 *Worst-case manipulated-variable magnitudes needed to provide zero steady-state error for norm-bounded disturbances (condition 6b with $\omega = 0$)*

which occurs when the measurement noise is smaller or the quantity of identification data is larger, so that the model is more accurate. Since the left hand side of condition 2 is greater than 1 for most values of r, MPC may provide better steady-state performance than an unconstrained controller implemented on this particular paper machine.

If the manipulated variables were less constrained, as would occur if another type of actuator such as a water jet or steam spray were used, then whether MPC would be beneficial would depend on the level of measurement noise and the quantity of data. For example, if the magnitude of allowed manipulated variables moves were relaxed to $\|\boldsymbol{u}\|_2 \le 10$, then MPC would be beneficial if the number of controllable directions were larger than 119 (see Fig. 3).

The previous conditions take into account the closed-loop performance only at steady-state. In general, conditions that take dynamic behaviour into account may suggest that MPC should be used when unconstrained control may be adequate at steady-state. For the industrial paper machine model considered here [32], the dynamics of each element was first-order plus time delay, with a time delay of 2 minutes and a time constant of 0.533 minutes. In the paper machine, however, the dynamic plots do not give significantly new information due to the paper machine's simple dynamics (plots not shown for brevity). However, it is interesting to plot the largest magnitude of each

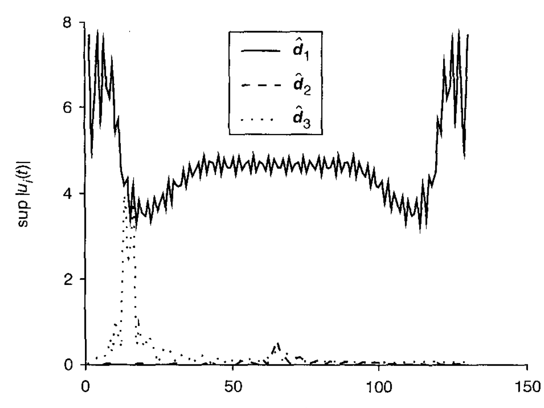

Fig. 6 *Manipulated-variable magnitudes needed to provide zero steady-state error in the controllable directions, maximised over all $t > 0$, for three step disturbances with directions $\hat{d}_1$, $\hat{d}_2$ and $\hat{d}_3$ (condition 7b′)*

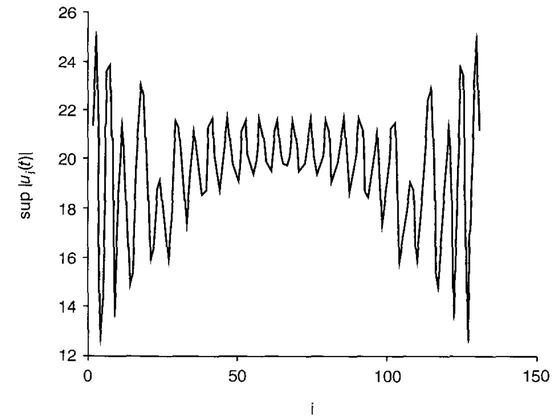

Fig. 7 *Worst-case manipulated variable magnitudes needed to provide zero closed-loop error in the controllable directions, for max–norm-bounded disturbances (condition 8b)*

manipulated variable across the machine, to determine which actuator constraints provide the greatest limitation on the closed-loop performance (see Fig. 4). For the uniform disturbance $\hat{\boldsymbol{d}}_1$, the actuators at the end of the paper machine provide the greatest limitation on being able to suppress the disturbance. This agrees with physical intuition for this paper machine, since the gains for the actuators near the edge are smaller than near its centre [32]. For the spike disturbance $\hat{\boldsymbol{d}}_2$ and square disturbance $\hat{\boldsymbol{d}}_3$, the actuators upstream from the disturbances are the most taxed, which also agrees with physical intuition. Disturbance $\hat{\boldsymbol{d}}_4$ is another spike disturbance, the same as $\hat{\boldsymbol{d}}_2$ but located at the edge of the paper machine. To suppress the spike disturbance near the edge of the paper machine requires larger actuator moves than the spike disturbance located more centrally. The implications of these results for actuator design are clear.

The largest manipulated variables needed to suppress norm-bounded disturbances are shown in Fig. 5. It is clear that all of the actuator moves are too constrained to suppress some of the norm-bounded disturbances.

Time-domain models provide the most complete assessment of whether constrained control may be beneficial for a particular process. The manipulated-variable magnitudes required to achieve perfect control within the controllable subspace for several step disturbances, which take all time into account, are plotted in Fig. 6. The spike disturbance $\hat{\boldsymbol{d}}_2$ can be suppressed using unconstrained control, whereas the other two disturbances require the actuators to hit the constraints. The worst-case manipulated-variable magnitudes for norm-bounded disturbances are shown in Fig. 7. This plot is identical to Fig. 5, which indicates that the dynamics of the paper machine are simple enough that a steady-state analysis is adequate.

The worst-case manipulated-variable magnitude needed to provide zero closed-loop error in the controllable directions, for 2-norm-bounded disturbances, is $\bar{\sigma} = 10.76$ (condition 9*b*). This implies that there are 2-norm disturbances whose suppression could benefit from constrained control.

7 Conclusions

It is especially important for large-scale systems such as polymer-film extruders and paper machines to determine whether the high cost of the modern control hardware and software required for implementing model predictive control is justified. Explicit computable conditions were derived that identify whether a particular large-scale system may benefit from constrained-control algorithms. These conditions are also useful for understanding the interactions between design and control for a particular system, especially for actuator placement and selection. The formulation considers the effects of measurement noise, process disturbances, model uncertainties, plant directionality and the quantity of experimental data. Application of the conditions to a paper machine resulted in general recommendations concerning which sheet and film processes are likely to benefit from constrained control, and insights into actuator design. Several levels of noise and model uncertainties are investigated to explore the relationship between model accuracy and the potential benefits provided by constrained control. It is shown that constrained control becomes increasingly desirable the higher the accuracy of the model.

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