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A Time-certified Predictor-corrector IPM Algorithm for Box-QP

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***Abstract*—Minimizing both the *worst-case* and *average* execution times of optimization algorithms is equally critical in real-time optimization-based control applications such as model predictive control (MPC). Most MPC solvers have to trade off between certified *worst-case* and practical *average* execution times. For example, our previous work [1] proposed a full-Newton path-following interior-point method (IPM) with data-independent, simple-calculated, and *exact* $O(\sqrt{n})$ iteration complexity, but not as efficient as the heuristic Mehrotra's predictor–corrector IPM algorithm (which sacrifices global convergence). This letter proposes a new predictor–corrector IPM algorithm that preserves the same certified $O(\sqrt{n})$ iteration complexity while achieving a $5\times$ speedup over [1]. Numerical experiments and codes that validate these results are provided.**

***Index Terms*—Box-constrained quadratic program, iteration complexity, interior-point method, model predictive control.**

I. Introduction

THIS paper considers a scaled box-constrained quadratic program (Box-QP) with time-varying data $(H(t), h(t))$ as follows,

$$\begin{aligned}\min_{z\in\mathbb{R}^n}\ & \frac{1}{2}z^\top H(t)z + z^\top h(t)\\ \text{s.t.}\ & -\mathbf{1}_n \le z \le \mathbf{1}_n,\end{aligned} \tag{1}$$

where $H(t) \in \mathbb{R}^{n\times n}$ is symmetric positive semi-definite. Without loss of generality, we assume that the box constraints are scaled to $[-\mathbf{1}_n, \mathbf{1}_n]$.

Time-varying Box-QP (1) often arises from real-time model predictive control (MPC) problems. For example, input-constrained MPC [1], ℓ_1-penalty soft-constrained MPC [2], and model-penalized MPC [3] can be formulated as a Box-QP (1). While a shorter average computation time for MPC is desirable, the most critical factor is determining the worst-case computation time. Because the MPC solver has to return the optimal solution before the next feedback sampling time, referred to as the **execution time certificate**. Recently, the **execution time certificate** of MPC (reduced to certifying the worst-case number of iterations if each iteration requires the same fixed number of floating-point operations) has attracted significant scholarly interest and remains a vibrant research area [1], [2], [4]–[9]. A breakthrough is made by our previous work [1], which proposed *data-independent, simple-calculated*, and *exact (not worst-case)* iteration complexity: $\left\lceil \frac{\log(\frac{2n}{\epsilon})}{-2\log(\frac{\sqrt{2n}}{\sqrt{2n}+\sqrt{2}-1})} \right\rceil + 1$ ($O(\sqrt{n})$-order) for Box-QP (1). Then, it was applied to certifying the execution time of nonlinear MPC via the real-time-iteration scheme [10] and Koopman operator [11], respectively. Furthermore, [9] proposed exact iteration complexity: $\left\lceil \frac{\log(\frac{n+1}{\epsilon})}{-\log(1-\frac{0.414213}{\sqrt{n+1}})} \right\rceil$ for general convex QP. The algorithms in [1], [9] solve a linear system of equations (with $O(n^3)$) at each iteration, thus resulting $O(n^{3.5})$ time complexity. In [2], the solution of linear equations is replaced by multiple rank-1 updates, resulting in the first *implementable* QP algorithm with $O(n^3)$ time complexity (with $\mathcal{N}_{\text{iter}} = O(\sqrt{n})$ and $\mathcal{N}_{\text{rank}-1} = O(n)$).

However, those time-certified interior-point-method (IPM) algorithms proposed in [1], [2], [9] are still not practically competitive: their certified computation times are typically longer than those of state-of-the-art QP solvers, which, in contrast, do not provide an execution time certificate. For example, the **heuristic** Mehrotra's predictor-corrector IPM algorithm [12], renowned for its computational efficiency (empirically exhibiting $O(\log n)$ iteration complexity), has become the foundation of most IPM-based optimization software, yet its global convergence and theoretical iteration complexity bound remain unknown, and it even may diverge in some examples [13, see p. 411], [14]. This naturally raises the question:

> *Can we design a practically efficient predictor–corrector IPM-based Box-QP algorithm that also achieves the best-known certified iteration complexity of $O(\sqrt{n})$?*

A. Contributions

This paper gives a positive answer by proposing a predictor-corrector IPM algorithm with a certified, data-independent, and easily computable worst-case iteration bound, $N_{\max} = \left\lceil \frac{\log(\frac{2n}{\epsilon})}{-2\log\left(1-\frac{0.2348}{\sqrt{2n}}\right)} \right\rceil$ ($O(\sqrt{n})$-order), while empirically exhibiting much faster iteration complexity (such as $O(\log n)$ or $O(n^{0.25})$).

Although the proof framework is an extension of the predictor-corrector IPM framework from linear programs [15] to Box-QP (1), the proof is more complicated than the original linear programming case, as the direction vectors are not orthogonal in the Box-QP case. More importantly, our proposed predictor–corrector IPM algorithm is **implementable**

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since a Box-QP admits cost-free initialization [1], whereas the algorithm in [15] is not, as it assumes the availability of a strictly feasible initial point (usually requires solving another LP, known as the *phase I* stage [16, Sec. 11.4]); consequently, no numerical experiments are reported in [15].

Compared with our previous time-certified Box-QP algorithm [1], the proposed algorithm demonstrates a $5\times$ speedup.

II. Feasible Predictor-Corrector IPM Algorithm

According to [16, Ch 5], the Karush–Kuhn–Tucker (KKT) condition of Box-QP (1) is the following nonlinear equations,

$$
\begin{aligned}
H(t)z + h(t) + \gamma - \theta &= 0, && \text{(2a)}\\
z + \phi - \mathbf{1}_n &= 0, && \text{(2b)}\\
z - \psi + \mathbf{1}_n &= 0, && \text{(2c)}\\
(\gamma, \theta, \phi, \psi) &\geq 0, && \text{(2d)}\\
\gamma \odot \phi &= 0, && \text{(2e)}\\
\theta \odot \psi &= 0, && \text{(2f)}
\end{aligned}
$$

where γ, θ are the Lagrangian variables of the lower and upper bound, respectively, and ϕ, ψ are the slack variables of the lower and upper bound, respectively. $\odot$ represents the Hadamard product, i.e., $\gamma \odot \phi = \mathrm{col}(\gamma_1\phi_1, \gamma_2\phi_2, \cdots, \gamma_n\phi_n)$.

Path-following primal–dual IPMs are categorized into two types: *feasible* and *infeasible*, distinguished by whether the initial point satisfies Eqns. (2a)–(2d). For the complementarity constraints (2e)–(2f), *feasible* path-following IPMs require the initial point to lie in a narrow neighborhood. To demonstrate this, let us denote the feasible region by $\mathcal{F}$, i.e.,

$$\mathcal{F} = \{(z, \gamma, \theta, \phi, \psi) : \text{(2a)–(2c)}, (\gamma, \theta, \phi, \psi) \geq 0\} \quad (3)$$

and the set of strictly feasible points by

$$\mathcal{F}^+ \triangleq \{(z, \gamma, \theta, \phi, \psi) : \text{(2a)–(2c)}, (\gamma, \theta, \phi, \psi) > 0\}. \quad (4)$$

We also consider the neighborhood

$$\mathcal{N}(\beta) \triangleq \left\{ (z, \gamma, \theta, \phi, \psi) \in \mathcal{F}^+ : \left\| \begin{bmatrix} \gamma \odot \phi \\ \theta \odot \psi \end{bmatrix} - \mu \mathbf{1}_{2n} \right\|_2 \leq \beta\mu \right\}, \quad (5)$$

where the duality measure $\mu \triangleq \frac{\gamma^\top \phi + \theta^\top \psi}{2n}$ and $\beta \in [0, 1]$. *Feasible* path-following IPMs require the initial point satisfying

$$(z^0, \gamma^0, \theta^0, \phi^0, \psi^0) \in \mathcal{N}(\beta), \quad (6)$$

and computing such a point is typically expensive for general strictly convex QPs.

A. Cost-free initialization for Feasible IPMs

Inspired by our previous work [1], which was the first to point out that Box-QP supports cost-free initialization for feasible IPMs, this letter proposes the following initialization to ensure $(z^0, \gamma^0, \theta^0, \phi^0, \psi^0) \in \mathcal{N}(\beta)$.

Remark 2.1: For $h = 0$, the optimal solution of Box-QP (1) is $z^* = 0$. For $h \neq 0$, first scale the objective as

$$\min_z \tfrac{1}{2} z^\top (2\lambda H) z + z^\top (2\lambda h),$$

which does not affect the optimal solution and can ensure the initial point lies in $\mathcal{N}(\beta)$ if $\lambda = \frac{\beta}{\sqrt{2}\|h\|_2}$. Then (2a) is replaced by

$$2\lambda H z + 2\lambda h + \gamma - \theta = 0,$$

and the initialization strategy for Box-QP (1) is

$$z^0 = 0,\ \gamma^0 = \mathbf{1}_n - \lambda h,\ \theta^0 = \mathbf{1}_n + \lambda h,\ \phi^0 = \mathbf{1}_n,\ \psi^0 = \mathbf{1}_n, \quad (7)$$

which clearly places this initial point in $\mathcal{N}(\beta)$ by its definition in Eqn. (6) (for example, $\left\| \begin{bmatrix} \gamma^0 \odot \phi^0 \\ \theta^0 \odot \psi^0 \end{bmatrix} - \mu \mathbf{1}_{2n} \right\|_2 = \beta\mu$, where $\mu = 1$). In particular, this letter chooses $\beta = \frac{1}{4}$, then $\lambda = \frac{1}{4\sqrt{2}\|h\|_2}$.

B. Algorithm descriptions

For simplicity, we introduce $v \triangleq \mathrm{col}(\gamma, \theta) \in \mathbb{R}^{2n}$, $s \triangleq \mathrm{col}(\phi, \psi) \in \mathbb{R}^{2n}$. According to Remark 2.1, we have $(z, v, s) \in \mathcal{N}(\beta)$. Then, all the search directions $(\Delta z, \Delta v, \Delta s)$ (for both predictor and corrector steps) are obtained as solutions of the following system of linear equations:

$$
\begin{aligned}
(2\lambda H)\Delta z + \Omega \Delta v &= 0, && \text{(8a)}\\
\Omega^\top \Delta z + \Delta s &= 0, && \text{(8b)}\\
s \odot \Delta v + v \odot \Delta s &= \sigma\mu \mathbf{1}_{2n} - v \odot s, && \text{(8c)}
\end{aligned}
$$

where $\Omega = [I, -I] \in \mathbb{R}^{n \times 2n}$, σ is chosen 0 in predictor steps and 1 in corrector steps, respectively, and $\mu \triangleq \frac{v^\top s}{2n}$ denotes the duality measure.

Remark 2.2: Eqns. (8a) and (8b) imply that

$$\Delta v^\top \Delta s = \Delta v^\top (-\Omega^\top \Delta z) = \Delta z^\top (2\lambda H) \Delta z \geq 0,$$

which is critical in the following iteration complexity analysis. Note that in [15], $\Delta v^\top \Delta s = 0$, thus making our analysis different and more complicated than the linear program case. By letting

$$
\begin{aligned}
&\Delta\gamma = \sigma\mu \frac{1}{\phi} - \gamma + \frac{\gamma}{\phi}\Delta z,\ \Delta\theta = \sigma\mu\frac{1}{\psi} - \theta - \frac{\theta}{\psi}\Delta z,\\
&\Delta\phi = -\Delta z,\ \Delta\psi = \Delta z,
\end{aligned} \quad (9)
$$

Eqn. (8) can be reduced into a more compact system of linear equations,

$$\left(2\lambda H + \mathrm{diag}\left(\frac{\gamma}{\phi}\right) + \mathrm{diag}\left(\frac{\theta}{\psi}\right)\right)\Delta z = \sigma\mu\left(\frac{1}{\phi} - \frac{1}{\psi}\right) + \gamma - \theta. \quad (10)$$

The proposed feasible adaptive-step predictor-corrector IPM algorithm for Box-QP (1) is first described in Algorithm 1. In the next Subsection, we prove that Algorithm 1 converges to the ϵ-optimal solution ($v^\top s \leq \epsilon$) in the worst-case number of iterations

$$N_{\max} = \left\lceil \frac{\log\left(\frac{2n}{\epsilon}\right)}{-2\log\left(1 - \frac{0.2348}{\sqrt{2n}}\right)} \right\rceil, \quad (11)$$

which is an $O(\sqrt{n})$-order iteration complexity. In practice, Algorithm 1 exhibits $O(n^{0.25})$ or $O(\log n)$-order iteration complexity due to the conservativeness of our proof.

Algorithm 1 Time-certified predictor-corrector IPM for Box-QP (1)

Input: Given a strictly feasible initial point $(z^0, v^0, s^0) \in \mathcal{N}(1/4)$ from Remark 2.1 and a desired optimal level ϵ. Then the worst-case iteration bound is $N_{\max} = \left\lceil \frac{\log(\frac{2n}{\epsilon})}{-2\log\left(1-\frac{0.2348}{\sqrt{2n}}\right)} \right\rceil$.

for $k = 0, 1, 2, \cdots, N_{\max} - 1$ **do** label*=0., ref=0

1) if $(v^k)^\top s^k \leq \epsilon$, then break;
2) Compute the predictor direction $(\Delta z_p, \Delta v_p, \Delta s_p)$ by solving Eqn. (8) with $(z, s, v) = (z^k, v^k, s^k)$, $\sigma \leftarrow 0$, and $\mu \leftarrow \mu^k = \frac{(v^k)^\top s^k}{2n}$ (involving Eqns. (10) and (9));
3) $\Delta\mu_p \leftarrow \frac{(\Delta v_p)^\top \Delta s_p}{2n}$;
4) $\alpha^k \leftarrow \min\left(\frac{1}{2}, \sqrt{\frac{\mu^k}{8\|\Delta v_p \odot \Delta s_p - \Delta\mu_p \mathbf{1}_{2n}\|}}\right)$;
5) $\hat{z}^k \leftarrow z^k + \alpha^k \Delta z_p$, $\hat{v}^k \leftarrow v^k + \alpha^k \Delta v_p$, $\hat{s}^k \leftarrow s^k + \alpha^k \Delta s_p$;
6) Compute the corrector direction $(\Delta z_c, \Delta v_c, \Delta s_c)$ by solving Eqn. (8) with $(z, v, s) = (\hat{z}^k, \hat{v}^k, \hat{s}^k)$, $\sigma \leftarrow 1$, and $\mu \leftarrow \hat{\mu}^k = \frac{(\hat{v}^k)^\top \hat{s}^k}{2n}$ (involving Eqns. (10) and (9));
7) $z^{k+1} \leftarrow \hat{z}^k + \Delta z_c$, $v^{k+1} \leftarrow \hat{v}^k + \Delta v_c$, $s^{k+1} \leftarrow \hat{s}^k + \Delta s_c$;

end

Output: z^{k+1}.

C. Convergence and worst-case iteration complexity

The analysis is carried out separately for the predictor and corrector steps. We first present the common results that apply to both.

Lemma 2.3: Let $(z, v, s) \in \mathcal{F}^+$ and let $(\Delta z, \Delta v, \Delta s)$ be the solution of Eqn. (8). Then,

$$\|\Delta v \odot \Delta s\| \leq \frac{\sqrt{2}}{4}\|r\|^2, \tag{12}$$

where $r \triangleq \frac{1}{\sqrt{v \odot s}}(\sigma\mu\mathbf{1}_{2n} - v \odot s)$.

Proof: Dividing both sides of Eqn. (8c) by $\sqrt{v \odot s}$, results in

$$\sqrt{\frac{s}{v}} \odot \Delta v + \sqrt{\frac{v}{s}} \odot \Delta s = r. \tag{13}$$

Let us denote

$$p \triangleq \sqrt{\frac{s}{v}} \odot \Delta v, \;\; q \triangleq \sqrt{\frac{v}{s}} \odot \Delta s, \tag{14}$$

we have $p + q = r$, $\Delta v \odot \Delta s = p \odot q$, $\Delta v^\top \Delta s = p^\top q$. By Remark 2.2, $\Delta v^\top \Delta s \geq 0$, so $p^\top q \geq 0$. Then, we have

$$\sum_{p_i q_i \geq 0} p_i q_i \geq -\sum_{p_i q_i < 0} p_i q_i, \;\; i = 1, \cdots, 2n.$$

We can obtain the result as follows:

$$\|\Delta v \odot \Delta s\|^2 = \|p \odot q\|^2 = \sum_{i=1}^{2n} (p_i q_i)^2 = \sum_{p_i q_i \geq 0} (p_i q_i)^2 + \sum_{p_i q_i < 0} (p_i q_i)^2$$

$$\leq \left(\sum_{p_i q_i \geq 0} p_i q_i\right)^2 + \left(\sum_{p_i q_i < 0} p_i q_i\right)^2 \leq 2\left(\sum_{p_i q_i \geq 0} p_i q_i\right)^2$$

(we have $4p_i q_i = (p_i + q_i)^2 - (p_i - q_i)^2$)

$$\leq 2\left(\sum_{p_i q_i \geq 0} \frac{1}{4}(p_i + q_i)^2\right)^2 \leq 2\left(\sum_i \frac{1}{4}(p_i + q_i)^2\right)^2 = 2\left(\frac{1}{4}\|r\|^2\right)^2.$$

That is, $\|\Delta v \odot \Delta s\| \leq \frac{\sqrt{2}}{4}\|r\|^2$, which completes the proof. ■

In Algorithm 1, both pairs $(\Delta z_p, \Delta v_p, \Delta s_p)$ and $(\Delta z_c, \Delta v_c, \Delta s_c)$ are obtained by solving Eqn. (8b) and thus by Remark 2.2 we have

$$\Delta v_p^\top \Delta s_p \geq 0, \;\; \Delta v_c^\top \Delta s_c \geq 0. \tag{15}$$

The key of Algorithm 1 is that, during the predictor step, the iterate transitions from $(z^k, v^k, s^k) \in \mathcal{N}(1/4)$ to $(\hat{z}^k, \hat{v}^k, \hat{s}^k) \in \mathcal{N}(1/2)$, and during the corrector step, it returns from $(\hat{z}^k, \hat{v}^k, \hat{s}^k) \in \mathcal{N}(1/2)$ to $(z^{k+1}, v^{k+1}, s^{k+1}) \in \mathcal{N}(1/4)$, which will be proved in Lemmas 2.4 and 2.5, respectively.

Lemma 2.4: (**Analysis of Predictor Step**): Consider Algorithm 1. If $(z^k, v^k, s^k) \in \mathcal{N}(1/4)$ and the predictor step applies the step-size

$$\alpha^k = \min\left(\frac{1}{2}, \sqrt{\frac{\mu^k}{8\|\Delta v_p \odot \Delta s_p - \Delta\mu_p \mathbf{1}_{2n}\|}}\right), \tag{16}$$

with $\mu^k = \frac{(v^k)^\top s^k}{2n}$ and

$$\Delta\mu_p \triangleq \frac{\Delta v_p^\top \Delta s_p}{2n}. \tag{17}$$

Then, $(\hat{z}^k, \hat{v}^k, \hat{s}^k) \in \mathcal{N}(1/2)$ holds.

Proof: For any step-size $\alpha \in [0, 1]$, let us define

$$v^k(\alpha) \triangleq v^k + \alpha\Delta v_p, \;\; s^k(\alpha) \triangleq s^k + \alpha\Delta s_p,$$

and

$$\mu^k(\alpha) \triangleq \frac{(v^k(\alpha))^\top s^k(\alpha)}{2n}.$$

By Eqn. (8c) and the choice $\sigma = 0$ in the predictor step, we have (by $\Delta v_p^\top \Delta s_p = 2n\Delta\mu_p$):

$$\mu^k(\alpha) = (1-\alpha)\mu^k + \alpha^2 \frac{(\Delta v_p)^\top \Delta s_p}{2n} = (1-\alpha)\mu^k + \alpha^2 \Delta\mu_p, \tag{18}$$

and by $\Delta v_p^\top \Delta s_p \geq 0$ ($\Delta\mu_p \geq 0$), $\alpha \geq 0$, we have

$$\mu^k(\alpha) \geq (1-\alpha)\mu^k. \tag{19}$$

Next,

$$\begin{aligned}
&\left\|v^k(\alpha) \odot s^k(\alpha) - \mu^k(\alpha)\mathbf{1}_{2n}\right\| \\
&= \left\|(1-\alpha)(v^k \odot s^k - \mu^k \mathbf{1}_{2n}) + \alpha^2(\Delta v_p \odot \Delta s_p - \Delta\mu_p \mathbf{1}_{2n})\right\| \\
&\leq (1-\alpha)\|v^k \odot s^k - \mu^k \mathbf{1}_{2n}\| + \alpha^2\|\Delta v_p \odot \Delta s_p - \Delta\mu_p \mathbf{1}_{2n}\| \\
&\leq \frac{1-\alpha}{4}\mu^k + \alpha^2\|\Delta v_p \odot \Delta s_p - \Delta\mu_p \mathbf{1}_{2n}\|,
\end{aligned}$$

and by the choice of α^k in Eqn. (16), for any $\alpha \in [0, \alpha^k]$

$$0 \leq \alpha \leq \frac{1}{2} \text{ and } 8\alpha^2\|\Delta v_p \odot \Delta s_p - \Delta\mu_p \mathbf{1}_{2n}\| \leq \mu^k,$$

thus we have

$$\begin{aligned}
&\left\|v^k(\alpha) \odot s^k(\alpha) - \mu^k(\alpha)\mathbf{1}_{2n}\right\| \\
&\leq \frac{1-\alpha}{4}\mu^k + \frac{1}{8}\mu^k \leq \frac{1-\alpha}{4}\mu^k\left(1 + \frac{1}{2(1-\alpha)}\right) \\
&\leq \frac{1-\alpha}{2}\mu^k \leq \frac{1}{2}\mu^k(\alpha),
\end{aligned}$$

which can imply that for all $\alpha \in [0, \alpha^k]$: $v^k(\alpha) \odot s^k(\alpha) \geq \frac{1}{2}\mu^k(\alpha)\mathbf{1}_{2n}$, and further by continuity: $v^k(\alpha) > 0$ and $s^k(\alpha) > 0$. This completes the proof of $(\hat{z}^k, \hat{x}^k, \hat{z}^k) \in \mathcal{N}(1/2)$. ■

Lemma 2.5: (**Analysis of Corrector Step**): Consider Algorithm 1. If $(\hat{z}^k, \hat{x}^k, \hat{z}^k) \in \mathcal{N}(1/2)$, then $(z^{k+1}, v^{k+1}, s^{k+1}) \in \mathcal{N}(1/4)$.

Proof: For any step-size $\alpha \in [0,1]$, let us define

$$v^{k+1}(\alpha) \triangleq \hat{v}^k + \alpha\Delta v_c,\ \ s^{k+1}(\alpha) \triangleq \hat{s}^k + \alpha\Delta s_c,$$
$$\mu^{k+1}(\alpha) \triangleq \frac{(v^{k+1}(\alpha))^\top s^{k+1}(\alpha)}{2n},$$

then by Eqn. (8a) and the choice $\sigma = 1$ in the corrector step, we have (by $(\hat{v}^k)^\top \hat{s}^k = 2n\hat{\mu}^k, \Delta v_c^\top \Delta s_c = 2n\Delta\mu_c$):

$$\begin{aligned}\mu^{k+1}(\alpha) &= \frac{(\hat{v}^k)^\top \hat{s}^k + \alpha(\hat{\mu}^k 2n - \hat{v}^k \hat{s}^k) + \alpha^2 \Delta v_c^\top \Delta s_c}{2n} \\ &= \hat{\mu}^k + \alpha^2 \frac{\Delta v_c^\top \Delta s_c}{2n},\end{aligned} \tag{20}$$

where $\hat{\mu}^k = \frac{(\hat{v}^k)^\top \hat{s}^k}{2n}$. By $\Delta v_c^\top \Delta s_c \geq 0$ and $\alpha \geq 0$, we have

$$\mu^{k+1}(\alpha) \geq \hat{\mu}^k. \tag{21}$$

Next, let us define

$$\Delta\mu_c \triangleq \frac{\Delta v_c^\top \Delta s_c}{2n}, \tag{22}$$

and we have

$$\begin{aligned}&\left\|v^{k+1}(\alpha) \odot s^{k+1}(\alpha) - \mu^{k+1}(\alpha)\mathbf{1}_{2n}\right\| \\ &= \left\|(1-\alpha)(\hat{v}^k \odot \hat{s}^k - \hat{\mu}^k \mathbf{1}_{2n}) + \alpha^2(\Delta v_c \odot \Delta s_c - \Delta\mu_c \mathbf{1}_{2n})\right\| \\ &\leq (1-\alpha)\|\hat{v}^k \odot \hat{s}^k - \hat{\mu}^k e\| + \alpha^2 \|\Delta v_c \odot \Delta s_c - \Delta\mu_c \mathbf{1}_{2n}\| \\ &\leq \frac{(1-\alpha)\hat{\mu}^k}{2} + \alpha^2 \|\Delta v_c \odot \Delta s_c - \Delta\mu_c \mathbf{1}_{2n}\|.\end{aligned}$$

Focusing on the term $\|\Delta v_c \odot \Delta s_c - \Delta\mu_c \mathbf{1}_{2n}\|$,

$$\begin{aligned}&\|\Delta v_c \odot \Delta s_c - \Delta\mu_c \mathbf{1}_{2n}\| \\ &= \sqrt{\sum_{i=1}^{2n}(\Delta v_{c,i}\Delta s_{c,i})^2 - 2\Delta\mu_c(\Delta v_c^\top \Delta s_c) + 2n(\Delta\mu_c)^2} \\ &= \sqrt{\sum_{i=1}^{n}(\Delta v_{c,i}\Delta s_{c,i})^2 - 2n(\Delta\mu_c)^2} \leq \|\Delta v_c \odot \Delta s_c\|.\end{aligned}$$

By Lemma 2.3 and the choice $\sigma = 1$ in the corrector step, we have

$$\|\Delta v_c \odot \Delta s_c\| \leq \frac{\sqrt{2}}{4}\left\|\frac{1}{\sqrt{\hat{v}^k \odot \hat{s}^k}}(\hat{\mu}^k \mathbf{1}_{2n} - \hat{v}^k \odot \hat{s}^k)\right\|^2.$$

Because $(\hat{z}^k, \hat{x}^k, \hat{z}^k) \in \mathcal{N}(1/2)$, we have $\|\hat{v}^k \odot \hat{s}^k - \hat{\mu}^k \mathbf{1}_{2n}\| \leq \frac{1}{2}\hat{\mu}^k$, so for each $i = 1, \cdots, 2n$

$$\frac{1}{2}\hat{\mu}^k \leq \hat{v}_i^k \hat{s}_i^k \leq \frac{3}{2}\hat{\mu}^k \Leftrightarrow \left(\min(\sqrt{\hat{v}^k \odot \hat{s}^k}\right)^2 = \frac{1}{2}\hat{\mu}^k$$

Thus, we have

$$\begin{aligned}&\left\|\frac{1}{\sqrt{\hat{v}^k \odot \hat{s}^k}}(\hat{\mu}^k \mathbf{1}_{2n} - \hat{v}^k \odot \hat{s}^k)\right\|^2 \\ &\leq \frac{1}{(\min(\sqrt{\hat{v}^k \odot \hat{s}^k}))^2}\|\hat{\mu}^k \mathbf{1}_{2n} - \hat{v}^k \odot \hat{s}^k\|^2 \\ &\leq \frac{2}{\hat{\mu}^k}(\frac{1}{2}\hat{\mu}^k)^2 = \frac{1}{2}\hat{\mu}^k.\end{aligned} \tag{23}$$

Then, by Eqn. (21), for any step-size $\alpha \in [0,1]$, we have

$$\begin{aligned}&\|v^{k+1}(\alpha) \odot s^{k+1}(\alpha) - \mu^{k+1}(\alpha)\mathbf{1}_{2n}\| \\ &\leq \frac{(1-\alpha)\hat{\mu}^k}{2} + \alpha^2 \|\Delta v_c \odot \Delta s_c\| \\ &\leq \frac{(1-\alpha)\hat{\mu}^k}{2} + \alpha^2 \frac{\sqrt{2}}{4}\frac{1}{2}\hat{\mu}^k \leq \frac{(1-\alpha)\hat{\mu}^k}{2} + \alpha^2 \frac{2}{8}\hat{\mu}^k \\ &= \frac{(\alpha-1)^2+1}{4}\hat{\mu}^k \leq \frac{(\alpha-1)^2+1}{4}\mu^{k+1}(\alpha) \leq \frac{1}{2}\mu^{k+1}(\alpha).\end{aligned}$$

which can imply that for all $\alpha \in [0,1]$

$$v^{k+1}(\alpha) \odot s^{k+1}(\alpha) \geq \frac{1}{2}\mu^{k+1}(\alpha)\mathbf{1}_{2n}$$

and namely by continuity $v^{k+1}(\alpha) > 0$ and $s^{k+1}(\alpha) > 0$.

Moreover, in the special case that $\alpha = 1$, we have $\frac{(\alpha-1)^2+1}{4} = \frac{1}{4}$ and it proves that

$$\|v^{k+1} \odot s^{k+1} - \mu^{k+1}\mathbf{1}_{2n}\| \leq \frac{1}{4}\mu^{k+1},$$

which completes the proof. ■

Lemma 2.6: Consider Algorithm 1. At the k-th iteration, the following inequalities

$$\Delta\mu_p \leq \frac{1}{4}\mu^k, \tag{24a}$$
$$\hat{\mu}^k \leq \left(1 - \frac{\alpha^k}{2}\right)^2 \mu^k, \tag{24b}$$
$$\Delta\mu_c \leq \left(1 - \frac{\alpha^k}{2}\right)^2 \frac{1}{16n}\mu^k \tag{24c}$$

hold.

Proof: Taking the 2-norm on both sides of Eqn. (13) (by the definition in Eqn. (14)) results in

$$4\Delta v^\top \Delta s + \|p-q\|^2 = \|r\|^2,$$

which implies that (by the definition in Eqn. (12))

$$4\Delta v^\top \Delta s \leq \|r\|^2 = \left\|\frac{1}{\sqrt{v \odot s}}(\sigma\mu\mathbf{1}_{2n} - v \odot s)\right\|^2. \tag{25}$$

Regarding the inequality (24a), the predictor step adopts $\sigma \leftarrow 0$ and $(z,v,s) = (z^k, v^k, s^k)$, this yields

$$\begin{aligned}\Delta v_p^\top \Delta s_p &\leq \frac{1}{4}\left\|\frac{1}{\sqrt{v^k \odot s^k}}(0\mu^k \mathbf{1}_{2n} - v^k \odot s^k)\right\|^2 \\ &= \frac{1}{4}(v^k)^\top s^k \Leftrightarrow \Delta\mu_p \leq \frac{1}{4}\mu^k,\end{aligned}$$

which completes the proof of the inequality (24a).

Regarding the inequality (24b), by Eqn. (18) and the statement *i)*, we have

$$\begin{aligned}\hat{\mu}^k = (1-\alpha^k)\mu^k + (\alpha^k)^2\Delta\mu_p &\leq (1-\alpha^k)\mu^k + \frac{(\alpha^k)^2}{4}\mu^k \\ &= \left(1 - \frac{\alpha^k}{2}\right)^2 \mu^k,\end{aligned}$$

which completes the proof of the inequality (24b).

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Regarding the inequality (24c) the corrector step adopts $(z, v, s) = (\hat{z}^k, \hat{v}^k, \hat{s}^k)$, $\sigma \leftarrow 1$, and $\mu \leftarrow \hat{\mu}^k$, and by Eqn. (23), this yields

$$\Delta v_c^\top \Delta s_c \leq \frac{1}{4}\left\|\frac{1}{\sqrt{\hat{v}^k \odot \hat{s}^k}}(\hat{\mu}^k \mathbf{1}_{2n} - \hat{v}^k \odot \hat{s}^k)\right\|^2 \leq \frac{1}{8}\hat{\mu}^k.$$

That is, by the inequality (24b), we have

$$\Delta\mu_c \leq \frac{1}{16n}\hat{\mu}^k \leq \left(1 - \frac{\alpha^k}{2}\right)^2 \frac{1}{16n}\mu^k,$$

which completes the proof of the inequality (24c). ■

Theorem 2.7: Let $\{(z^k, v^k, s^k)\}$ be generated by Algorithm 1. Then

$$\mu^{k+1} \leq \left(1 - \frac{0.2348}{\sqrt{2n}}\right)^2 \mu^k. \tag{26}$$

Furthermore, Algorithm 1 requires at most

$$N_{\max} = \left\lceil \frac{\log(\frac{2n}{\epsilon})}{-2\log\left(1 - \frac{0.2348}{\sqrt{2n}}\right)} \right\rceil. \tag{27}$$

Proof: By Eqn. (18) and (20)

$$\mu^{k+1} = \hat{\mu}^k + \Delta\mu_c = (1-\alpha^k)\mu^k + (\alpha^k)^2\Delta\mu_p + \Delta\mu_c$$

(by Lemma 2.6)

$$\leq (1-\alpha^k)\mu^k + \frac{(\alpha^k)^2}{4}\mu^k + \left(1 - \frac{\alpha^k}{2}\right)^2 \frac{1}{16n}\mu^k$$

$$\leq \left(1 - \frac{\alpha^k}{2}\right)^2 \left(1 + \frac{1}{16n}\right)\mu^k.$$

Because we have

$$\|\Delta v_p \odot \Delta s_p - \Delta\mu_p \mathbf{1}_{2n}\|$$

$$= \sqrt{\sum_{i=1}^{2n}(\Delta v_{p,i}\Delta s_{p,i})^2 - 2\Delta\mu_p \Delta v_p^\top \Delta s_p + 2n(\Delta\mu_p)^2}$$

$$= \sqrt{\sum_{i=1}^{2n}(\Delta v_{p,i}\Delta s_{p,i})^2 - 2n(\Delta\mu_p)^2} \leq \|\Delta v_p \odot \Delta s_p\|$$

(By Lemma 2.3 and the choice $\sigma = 0$)

$$\leq \frac{\sqrt{2}}{4}\left\|\frac{1}{\sqrt{v^k \odot s^k}}(-v^k \odot s^k)\right\|^2 = \frac{\sqrt{2}}{4}(2n)\mu^k,$$

the choice of α^k in Eqn. (16) can imply that for all $n \geq 1$

$$\alpha^k \geq \min\left(\frac{1}{2}, \frac{2^{0.25}}{2}\frac{1}{\sqrt{2n}}\right) \geq \frac{2^{0.25}}{2}\frac{1}{\sqrt{2n}},$$

which can imply that the following condition for all $n \geq 2$ ($n = 1$ no need for optimization)

$$\left(1 - \frac{\alpha^k}{2}\right)^2 \left(1 + \frac{1}{16n}\right)$$

$$\leq \left(1 - \frac{\alpha^k}{2}\right)\left(1 - \frac{2^{0.25}}{4}\frac{1}{\sqrt{2n}} + \frac{1}{16n} - \frac{2^{0.25}}{64n\sqrt{2n}}\right)$$

$$\leq \left(1 - \frac{\alpha^k}{2}\right)\left(1 - \frac{2^{0.25}}{4}\frac{1}{\sqrt{2n}} + \frac{1}{16n}\right)$$

$$= \left(1 - \frac{\alpha^k}{2}\right)\left(1 - \left(\frac{2^{0.25} - \frac{1}{2\sqrt{2n}}}{4}\right)\frac{1}{\sqrt{2n}}\right)$$

$$\leq \left(1 - \frac{2^{0.25}}{4}\frac{1}{\sqrt{2n}}\right)\left(1 - \frac{2^{0.25} - 0.25}{4}\frac{1}{\sqrt{2n}}\right)$$

$$\leq \left(1 - \frac{2^{0.25} - 0.25}{4}\frac{1}{\sqrt{2n}}\right)^2 = \left(1 - \frac{0.2348}{\sqrt{2n}}\right)^2$$

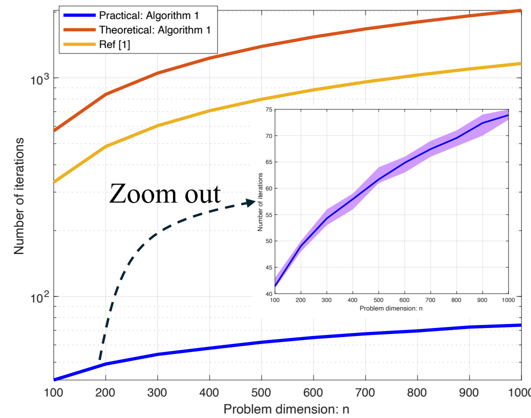

Fig. 1. Practical and theoretical iteration counts of Algorithm 1 compared with the exact (practical = theoretical) iteration counts from Ref. [1].

holds, which proves that $\mu^{k+1} \leq \left(1 - \frac{0.2348}{\sqrt{2n}}\right)^2 \mu^k$.

Based on the above and the initial value $\mu^0 = 1$, $\mu^k \leq \left(1 - \frac{0.2348}{\sqrt{2n}}\right)^{2k} \mu^0 = \left(1 - \frac{0.2348}{\sqrt{2n}}\right)^{2k}$. Hence $(v^k)^\top s_k \leq \epsilon$ holds if $2n\left(1 - \frac{0.2348}{\sqrt{2n}}\right)^{2k} \leq \epsilon$. Taking logarithms gives that $2k\log\left(1 - \frac{0.2348}{\sqrt{2n}}\right) + \log(2n) \leq \log\epsilon$, which holds if $k \geq N_{\max} = \left\lceil \frac{\log(\frac{2n}{\epsilon})}{-2\log\left(1 - \frac{0.2348}{\sqrt{2n}}\right)} \right\rceil$, which completes the proof. ■

Remark 2.8: Algorithm 1 exhibit $O(\log n)$ or $O(n^{0.25})$ iteration behavior in practice while the worst-case iteration bound $N_{\max}$ in Eqn . (27) (as $-\log(1 - 0.2348/\sqrt{2n}) \approx 0.2348/\sqrt{2n}$ when $0 < 0.2348/\sqrt{2n} < 1$) is $O(\sqrt{n})$ from the conservative estimate of the adaptive step-size $\alpha^k = \Omega(1/\sqrt{n})$ via Eqn. (12) in Lemma 2.3. Refs. [17] and [18] provide probabilistic proofs that, in linear programming cases, the adaptive step sizes satisfy $\alpha^k = \Omega(1/n^{0.25})$ and $\alpha^k = \Omega(1/\log n)$, respectively.

III. Numerical Examples[1]

A. Practical and theoretical behavior on random Box-QPs

We apply Algorithm 1 to random Box-QP problems with dimensions n ranging from 100 to 1000, and evaluate its practical iteration counts. These are then compared with the theoretical worst-case bound $N_{\max}$ in Eqn. (27) and with the results reported in [1]. Fig. 1 shows that Algorithm 1 practically behaves far less iterations than the $O(\sqrt{n})$-iteration-complexity results in Eqn. (27) and Ref. [1], which corresponds to Remark 2.8. Fig. 1 also shows that the practical number of iterations of Algorithm 1 exhibits very small variations for Box-QPs of the same dimension.

B. Nonlinear PDE-MPC case study

We apply Algorithm 1 to a nonlinear PDE-MPC problem from [11]. The considered PDE plant is the nonlinear

[1]The MATLAB code for Algorithm 1 and numerical examples are publicly available at `https://github.com/liangwu2019/PC_BoxQP`.

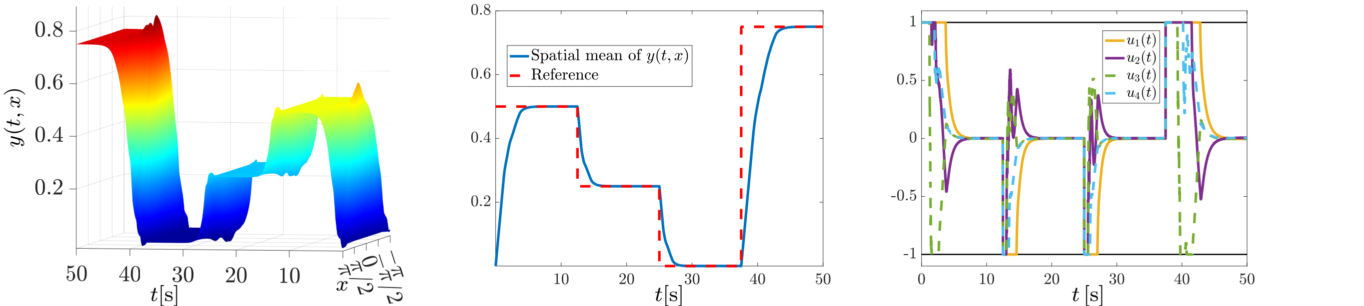

Fig. 2. Closed-loop simulation of the nonlinear KdV system with Box-QP based MPC controller – Tracking a piecewise constant spatial profile reference. Left: time evolution of the spatial profile $\boldsymbol{y(t,x)}$. Middle: spatial mean of the $\boldsymbol{y(t,x)}$. Right: the four control inputs.

Korteweg-de Vries (KdV) equation as follows,

$$\frac{\partial y(t,x)}{\partial t} + y(t,x)\frac{\partial y(t,x)}{\partial x} + \frac{\partial^3 y(t,x)}{\partial x^3} = u(t,x) \qquad (28)$$

where $x \in [-\pi, \pi]$ is the spatial variable. We consider the control input u to be $u(t,x) = \sum_{i=1}^{4} u_i(t)v_i(x)$, in which the four coefficients $\{u_i(t)\}$ are subject to the constraint $[-1,1]$, and $v_i(x)$ are predetermined spatial profiles given as $v_i(x) = e^{-25(x-m_i)^2}$, with $m_1 = -\pi/2$, $m_2 = -\pi/6$, $m_3 = \pi/6$, and $m_4 = \pi/2$. The objective is to adjust $u_i(t)$ so that $y(t,x)$ tracks a given reference. Following [11], data are generated from the KdV equation, a Koopman-based linear model is identified, and the resulting MPC problem is formulated as a Box-QP. With a prediction horizon of 10, the Box-QP dimension is $n = 40$, and the stopping tolerance is set to $\epsilon = 10^{-6}$. Table I shows that Algorithm 1 is approximately $5\times$ faster than the method in [1], primarily due to a much smaller iteration count ($202/29 \approx 10$) despite a per-iteration cost that is about twice as large. Figure 2 further demonstrates fast and accurate tracking of the spatial profile $y(t,x)$ with zero control-input violations.

TABLE I
COMPUTATION BEHVIOR OF ALGORITHM 1 AND THE METHOD IN REF. [1] IN THE PDE-MPC EXAMPLE

Methods	Number of iterations	Execution time [s][2]
Ref [1]	202	3.4×10^{-3}
Algorithm 1	Average: $\mathbf{29.2758 \pm 2.3843}$ Worst-case: 271	$\mathbf{6.6024 \times 10^{-4}}$

IV. CONCLUSION

This letter presents a significant improvement over our previous work [1] with a new predictor–corrector IPM algorithm, preserving the data-independent and simple-calculated $O(\sqrt{n})$-iteration-complexity, while achieving a $5\times$ speedup.

Limitation: Algorithm 1 and the heuristic Mehrotra's predictor–corrector IPM behave similarly in practical iteration complexity, but Algorithm 1 is slower because it solves two distinct linear systems at each iteration, whereas the latter does not. Future work will address this and then perform more numerical comparisons with other state-of-the-art solvers.

[2]The execution time results were based on MATLAB implementations running on a Mac mini with an Apple M4 Chip (10-core CPU and 16 GB RAM). Further speedup can be achieved via C-code implementation.

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