Lecture 10

1 Introduction to, and Motivation for, Dirac Notation

Reasons to use this notation:

- Lingua Fronca amongst physicists
- Used in most books
- Emphasizes things our wavefunctions don't

However:

- It's equivalent - just a language
- We'll stick with wavefunctions in 8.04
- Today is a Thesaurus

Note: We've developed a lot of formalism. Today is the last pure formalism lecture - henceforth we'll be applying these tools to solve specific problems: Hydrogen, solids, classical mechanics, etc. Before we get started, let's disentangle some math from physics.

STUFF MISSING HERE

2 Reminder about Hermitian operators

Recall that if $\psi(x)$ is a function and $\hat{A}$ is an operator then $\hat{A}f(x)$ is another function. Recall too the definition of the "dot product" on functions,

$$\langle f|g\rangle = \int dx f^*(x)g(x) \tag{1}$$

Consider:

$$\langle f|\hat{A}g\rangle = \int dx f^*(x)(\hat{A}g(x)) \tag{2}$$

Define the operator $\hat{A}^\dagger$ to be the operator such that

$$\langle \hat{A}^\dagger f | g \rangle = \langle f | \hat{A} g \rangle \tag{3}$$

ie

$$\int dx (\hat{A}^\dagger f(x))^* g(x) = \int dx f^*(x) (\hat{A} g(x)) \tag{4}$$

$\hat{A}^\dagger$ is called the **adjoint** of $\hat{A}$.

Example. $\hat{Q} = \frac{\partial}{\partial x}$. What is $\hat{Q}^\dagger$?

$$\langle f | \hat{Q} g \rangle = \int dx\, f^*(x)\, \frac{\partial g}{\partial x}(x) \tag{5}$$

$$= \int dx\, (\frac{\partial (f^* g)}{\partial x} - \frac{\partial f^*}{\partial x} g(x)) \tag{6}$$

$$= \int dx\, (\frac{\partial}{\partial x} f^*) g \tag{7}$$

Which, by definition,

$$= \langle \hat{Q}^\dagger f | g \rangle \tag{8}$$

Therefore,

$$\hat{Q}^\dagger = -\frac{\partial}{\partial x} \tag{9}$$

which means that

$$\langle \hat{Q}^\dagger f | g \rangle = -\hat{Q} \tag{10}$$

Example. $\hat{p} = -i\hbar \frac{\partial}{\partial x}$

$$\langle f | \hat{p} g \rangle = \int dx f^* (-i\hbar \frac{\partial g}{\partial x}) \tag{11}$$

$$= \int dx (i\hbar) \frac{\partial f^*(x)}{\partial x} g(x) \tag{12}$$

$$= \int dx (-i\hbar \frac{\partial f^*(x)}{\partial x})^* g(x) \tag{13}$$

$$= \langle \hat{p}^\dagger f | g \rangle \tag{14}$$

Therefore,

$$\hat{p}^\dagger = -i\hbar \frac{\partial}{\partial x} \tag{15}$$

and

$$\hat{p}^\dagger = \hat{p} \tag{16}$$

Example. $\hat{x} = x$

$$\langle f|\hat{x}g\rangle = \int dx f^*(x) x g(x) \tag{17}$$

$$= \int dx (xf(x))^* g(x) \tag{18}$$

$$= \langle \hat{x}^\dagger f|g\rangle \tag{19}$$

Therefore,

$$\hat{x}^\dagger = x \tag{20}$$

and

$$\hat{x}^\dagger = \hat{x} \tag{21}$$

Note, therefore, that:

$$\hat{x}^\dagger = \hat{x} \tag{22}$$

and

$$\hat{p}^\dagger = \hat{p} \tag{23}$$

Definition: Operators which are self-adjoint - in other words, $\hat{A}^\dagger = \hat{A}$ - are called Hermitian.

Who cares about Hermitian operators? YOU DO!

Fact 1: All eigenvalues of a Hermitian operator are real.

Proof. Suppose $\hat{A}^\dagger = \hat{A}$, and $\hat{A}\phi_a = a\phi_a$. Then

$$(\hat{A}\phi_a)^* = a^* \phi_a^* \tag{24}$$

$$\int dx (\hat{A}\phi_a)^* \phi_a = \int \phi_a^* \hat{A}\phi_a \tag{25}$$

$$a^* \int dx \phi_a^* \phi_a = a \int \phi_a^* \phi_a \tag{26}$$

So,

$$a^* = a \tag{27}$$

For example, $\hat{x} = \hat{x}^\dagger$, $\hat{p} = \hat{p}^\dagger$, and $\hat{E} = \hat{E}^\dagger$.

Since all observables are real, all operators corresponding to observables must be Hermitian.

Fact 2: If an operator $\hat{A}$ is Hermitian, any function can be expanded as a superposition of eigenfunctions of $\hat{A}$. This expansion is unique up to phases.

$$\hat{A}\phi_n = a_n \phi_n \tag{28}$$

$$\langle \phi_n | \phi_m \rangle = \delta_{nm} \tag{29}$$

$$\psi(x) = \sum_n c_n \phi_n(x) \tag{30}$$

Proving Fact 2 is hard; we'll take it as a mathematical fact.

Example. Energy eigenstates of the ∞ well.

$$\phi_n(x) = \sqrt{\frac{2}{L}} sin(\frac{n\pi x}{L}) \tag{31}$$

$$\hat{E}\phi_n = E_n \phi_n \tag{32}$$

$$\langle \phi_n | \phi_m \rangle = \delta_{nm} \tag{33}$$

Ditto the harmonic oscillator:

$$\phi_n = N_n e^{-\frac{x^2}{2a^2}} H_n(\frac{x}{a}) \tag{34}$$

$$\langle \phi_n | \phi_m \rangle = \delta_{nm} \tag{35}$$

Expanding into eigenstates:

$$\psi(x) = \sum_n c_n \phi_n \tag{36}$$

Ditto position eigenstates:

$$\hat{x}\delta(x - x_0) = x_0 \delta(x - x_0) \tag{37}$$

$$\langle \delta(x - x_n) | \delta(x - x_m) \rangle = \delta_{x_n - x_m} \tag{38}$$

Expanding into eigenstates:

$$\psi(x) = \int d\tilde{x}\psi(\tilde{x})\delta(\tilde{x} - x) \tag{39}$$

Ditto momentum eigenstates:

$$\hat{p}\frac{1}{\sqrt{2\pi}}e^{ikx} = \hbar k\frac{1}{\sqrt{2\pi}}e^{ikx} \tag{40}$$

$$\langle\frac{1}{\sqrt{2\pi}}e^{ikx}|\frac{1}{\sqrt{2\pi}}e^{ik'x}\rangle = \delta_{k-k'} \tag{41}$$

$$\psi(x) = \int dk(\frac{1}{\sqrt{2\pi}}e^{ikx})\tilde{\psi}(k) \tag{42}$$

This is Fourier transform.

Another name for Fact 2 is the **Spectral Theorem**. It says that, given $\hat{A}$, one can find a basis of functions ϕ_n so that

$$\hat{A}\phi_n = a_n\phi_n \tag{43}$$

$$\langle\phi_n|\phi_m\rangle = \delta_{nm} \tag{44}$$

$$\psi(x) = \sum_n c_n\phi_n(x) \tag{45}$$

for any $\psi(x)$. This means we can find some analog of Fourier transform for any operator, not just momentum.

3 Dirac Notation

3.1 Recall Vector Spaces

- A vector space is a set of vectors: $\{\vec{v}\}$ =V
- You can add vectors to get another vector in V: Given $\vec{v_1}$, $\vec{v_2}$,

$$\exists\,(\vec{v_1} + \vec{v_2}) \in V \tag{46}$$

- Multiplying by a constant gives another vector in V

$$\alpha\vec{v} \in V \tag{47}$$

- You can take a dot product: $\vec{v}\cdot\vec{v}$ = length squared, and

$$\vec{v}\cdot\vec{w} \in \mathbb{R} \tag{48}$$

- You can expand any vector in an orthonormal basis:

$$\vec{v} = \sum_n v_n \hat{e}_n \tag{49}$$

and

$$\hat{e}_n \cdot \hat{e}_m = \delta_{nm} \tag{50}$$

so

$$v_n = \hat{e}_n \cdot \vec{v} \tag{51}$$

- Given an orthonormal basis, you can completely specify a vector $\vec{v}$ by giving its coordinates $\{v_n\}$:

$$\{v_n\} \simeq \vec{v} \tag{52}$$

Key: $\vec{v}$ is a geometric object defined without any basis. Like my arm. A basis just lets us identify that object with a list of numbers.

Think about our wavefunctions.

- Can add wavefunctions:

$$\psi_1(x) + \psi_2(x) = \psi(x) \tag{53}$$

- Can multiply by a constant and get a fine wavefunction:

$$e^{i\theta}\psi(x) = \psi'(x) \tag{54}$$

- Can take a dot product:

$$\langle \psi_1 | \psi_2 \rangle = \int dx \psi_1^*(x) \psi_2(x) \in \mathbb{C} \tag{55}$$

- Can expand any continuous function in an orthonormal basis of eigenfunctions of a Hermitian operator:

$$\psi(x) = \sum_n \phi_n(x) c_n \tag{56}$$

where $\langle \phi_n | \phi_m \rangle = \delta_{nm}$, $c_n = \langle \phi_n | \psi \rangle$, $\hat{A}\phi_n = a_n \phi_n$

- Given a continuous function, you can completely specify an abstract object "state" in terms of a set of numbers: $\psi(x) \simeq$ "state"

What's the connection?

3.2 Change of Notation

$$\vec{v} \to |v\rangle \tag{57}$$

$$\vec{v} \cdot \vec{w} \to \langle v | w \rangle \tag{58}$$

$\langle v | v \rangle$ = "Length squared"

3.3 Let coefficients be $\mathbb{C}$: $\langle v|w\rangle \in \mathbb{C}$

We want $\langle v|v\rangle$ to be real, because this represents a length squared. Thus, we require:

$$\langle v|w\rangle = (\langle w|v\rangle)^* \tag{59}$$

Translation into this notation, now with a $\mathbb{C}$ dot product:

- $\{|v\rangle\} = V$
- You can add kets to get another ket in V: Given $|v\rangle$, $|w\rangle$,

$$(|v\rangle + |w\rangle) = |v+w\rangle \in V \tag{60}$$

- Multiplying by a constant gives another vector in V

$$\alpha\,|v\rangle = |\alpha v\rangle \in V \tag{61}$$

- You can take an inner product:

$$\langle v|w\rangle \in \mathbb{C} \tag{62}$$

$$\sqrt{\langle v|w\rangle} = ||v|| = \langle w|v\rangle^* \tag{63}$$

- You can expand any ket in an orthonormal basis:

$$\exists\,\{|e_n\rangle\} \tag{64}$$

$$\langle e_n|e_m\rangle = \delta_{nm} \tag{65}$$

  So,

$$|v\rangle = \sum_n v_n\,|e_n\rangle \tag{66}$$

$$v_n = \langle e_n|v\rangle \tag{67}$$

- Given $\{|e_n\rangle\}$, $\{v_n\} \leftrightarrow |v\rangle$

3.4 Infinite-dimensional vector space

Now suppose our vector space is huge, so the basis has an index (n) which is $\mathbb{R}$, not $\mathbb{Z}$. This means $\sum_n \to \int dn$. Rather than write v_n, it's easier to think of $v(n)$, but means the same thing: expansion coefficient of $|v\rangle$ along $|e_n\rangle$:

- $\{|e_n\rangle\} = V$
- You can add kets to get another ket in V: Given $|v\rangle$, $|w\rangle$,

$$(|v\rangle + |w\rangle) = |v+w\rangle \in V \tag{68}$$

- Multiplying by a constant gives another vector in V

$$\alpha |v\rangle = |\alpha v\rangle \in V \tag{69}$$

- You can take an inner product:

$$\langle v|w\rangle \in \mathbb{C} \tag{70}$$

$$\sqrt{\langle v|w\rangle} = ||v|| = \langle w|v\rangle^* \tag{71}$$

- Basis: $\{|e_n\rangle\}$ st $\langle e_n|e_m\rangle = \delta(n-m)$

$$|v\rangle = \int v(n)\, |e_m\rangle \tag{72}$$

$$v(n) = \langle e_n|v\rangle = \int dn\, v(m)\, \langle e_n|e_m\rangle \tag{73}$$

$$= \int dn\, v(m)\, \delta(n-m) = v(n) \tag{74}$$

- Given $\{|e_n\rangle\}$, $v(n)$, can define

$$|v\rangle = \int dn\, v(n)\ |e_n\rangle \tag{75}$$

3.5 Apply to QM

- The state of a system is a vector in a vector space:

$$|\psi\rangle \in V \tag{76}$$

- Given any two states, $|\psi_1\rangle$, $|\psi_2\rangle$, the superposition $|\psi_1\rangle + |\psi_2\rangle = |\psi_3\rangle$ is also in V.
- Multiplying by any $\mathbb{C}$ number gives another state $\propto |\psi_1\rangle = |\psi_2\rangle$ also in V
- There is a dot product on states,

$$\langle \psi_1|\psi_2\rangle = \langle \psi_2|\psi_1\rangle^* \tag{77}$$

- You can find orthonormal bases on V, eg $\{|x\rangle\}$ st

$$\langle x_1|x_2\rangle = \delta(x_1 - x_2) \tag{78}$$

$$\psi(x) = \langle x|\psi\rangle \tag{79}$$

The "wavefunction" is the expansion coefficients of state $|\psi\rangle$ in $|x\rangle$ basis.

- Given a basis $\{|x\rangle\}$ and a set of expansion coefficients, $\psi(x)$, you can specify the state,

$$|\psi\rangle = \int \psi(x)\,|x\rangle \tag{80}$$

Key: The wavefunction, $\psi(x)$, is the expansion of the state, $|\psi\rangle$ in the position basis, $\{|x\rangle\}$.
Note: $|x\rangle$ basis is the eigenbasis of $\hat{x}$:

$$\hat{x}\,|x\rangle = x\,|x\rangle \tag{81}$$

Via the spectral theorem, we can find an orthonormal basis for any Hermitian operators.

Example:

$$\hat{p}\,|k\rangle = hk\,|k\rangle \tag{82}$$

$$\langle k_1|k_2\rangle = \delta(k_1 - k_2) \tag{83}$$

$$|\psi\rangle = \int dk\,\tilde{\psi}(k)\;|k\rangle \tag{84}$$

$$\tilde{\psi}(k) = \langle k|\psi\rangle \tag{85}$$

How does this relate to $\psi(x)$?

$$|\psi\rangle = \int dx\;\langle x|k\rangle\;|x\rangle \tag{86}$$

$$\hat{p}\,|k\rangle = \int dx\,\hat{P}\,\langle x|k\rangle\;|x\rangle = \int dx\,(-i\hbar\frac{\partial}{\partial x}\,\langle x|k\rangle)\,|x\rangle \tag{87}$$

3.6 New Object: the "Bra"

Suppose I have a vector $|v\rangle$ (a "ket"). I can define a "bra" $\langle v|$ as follows:

$\langle v|$ eats a vector $|w\rangle$ and spits out a number $\langle v|w\rangle$. Or, equivalently, $\langle v|$ is a function on vectors: $\langle v| : |w\rangle \to \langle v|w\rangle$.

Note:

$$(\langle v_1| + \langle v_2|)\,|w\rangle = \langle v_1|w\rangle + \langle v_2|w\rangle \tag{88}$$

$$= \langle v_1 + v_2 | w \rangle \tag{89}$$

So,

$$\langle v_1 | + \langle v_2 | = \langle v_1 + v_2 | \tag{90}$$

$$\langle \alpha v | w \rangle = \alpha^* \langle v | w \rangle \tag{91}$$

so

$$\alpha^* \langle v | = \langle \alpha v | \tag{92}$$

$\{\langle v|\}$ also form a vector space, called the "dual space" V^*. The vector space structure of V^* is inherited from V.

Note: $|v\rangle = \sum_n v_n |e_n\rangle$ and $\langle v| = \sum_m \langle e_m| v_m^*$, so

$$\langle v|v\rangle = (\sum_m \langle e_m| v_m^*)(\sum_n v_n |e_n\rangle) \tag{93}$$

$$= \sum_m \sum_n v_m^* v_n \langle e_m | e_n \rangle \tag{94}$$

$$= \sum_m \sum_n v_m^* v_n \delta_{mn} \tag{95}$$

$$= \sum_m v_m^* v_m = \sum_m |v_m|^2 \tag{96}$$

Note: $\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle = \langle \psi | \hat{A} \psi \rangle = \langle A^\dagger \psi | \psi \rangle$.

3.7 Final words

Most QM textbooks use this "bra-ket" notation. It is coordinate-independent, which is useful. Like using $\vec{v}$ rather than $\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$. This makes many calculations much easier and more compact. It generalizes to many systems we've not yet studied (you'll see more of this in 8.05). We will stick with wavefunctions $\psi(x)$ for the rest of 8.04. But now you should be able to use any QM text.

Enjoy!