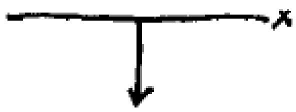
Lecture 12

1 Delta Potential Well: $V(x) = -V_0\delta(x)$



Question: Is there a bound state? ie is there a normalizable state with $E < 0$?

Hint: This is like a narrow, deep finite well...and a finite well always has a bound state, as we have just seen.

For $x < 0$:

$$\phi_E(x) = Ae^{\alpha x} + Be^{-\alpha x} \tag{1}$$

For $x > 0$:

$$\phi_E(x) = Ce^{\alpha x} + De^{-\alpha x} \tag{2}$$

Boundary conditions:

- At x = ∞, $\phi \to 0 \Rightarrow C = 0$
- At x = $-\infty$, $\phi \to 0 \Rightarrow B = 0$
- At x = 0, ϕ is continuous $\Rightarrow A = D$

$$\Delta\phi'(0) = \frac{-2mV_0}{\hbar^2}\phi(0) \Rightarrow -A\alpha - (+\alpha A) = \frac{-2mV_0}{\hbar^2}A \Rightarrow \alpha = \frac{mV_0}{\hbar^2} \tag{3}$$

From this, we get that

$$\phi_E(x) = e^{\alpha x} \tag{4}$$

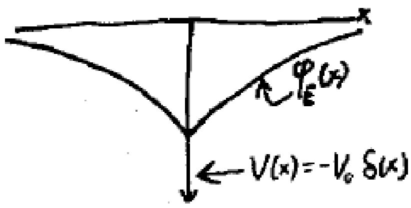
for $x < 0$, and

$$\phi_E(x) = e^{-\alpha x} \tag{5}$$

for $x > 0$, where $\alpha = \frac{mV_0}{\hbar^2}$ and $E = \frac{-\hbar^2\alpha^2}{2m} = -\frac{mV_0^2}{2\hbar^2}$.

Note: Up to normalization, this is unique! $\Rightarrow$ There is exactly one bound state! It has this form:

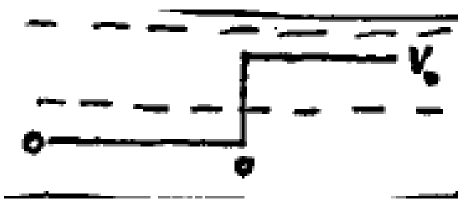
$$\phi_E = \sqrt{\frac{mV_0}{\hbar^2}} e^{-\frac{mV_0}{\hbar^2}|x|} \tag{6}$$



So much for bound states! Time for scattering.

2 Scattering

2.1 The Potential Step



Classical intuition: when $E > V_0$, the particle should not bounce. When $E < V_0$, the particle should bounce.

The question: What are $\phi_E(x)$?

This is easier than the finite well! Study the bouncing case first: $E < V_0$

For $x < 0$:

$$\phi_E(x) = Ae^{ikx} + Be^{-ikx} \tag{7}$$

where $k = \sqrt{\frac{2mE}{\hbar^2}}$

For $x > 0$:

$$\phi_E(x) = Ce^{\alpha x} + De^{-\alpha x} \tag{8}$$

where $\alpha = \sqrt{\frac{2m}{\hbar^2}(V_0 - E)}$

Normalization: $\phi(x \to \infty) \to 0 \Rightarrow C = 0$

Continuity:

- $\phi(0) = A + B = D$
- $\phi'(0) = ik(A - B) = -\alpha D$

Thus,

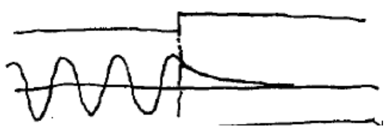
- $D = \frac{2k}{k+i\alpha} A$
- $B = \frac{k-i\alpha}{k+i\alpha} A$

Thus, we arrive at the following wavefunction:

$$\phi_E(x) = Ae^{ikx} + \frac{k - i\alpha}{k + i\alpha} e^{-ikx} \tag{9}$$

on the left, and

$$\phi_E(x) = A\frac{2k}{k + i\alpha} e^{-\alpha x} \tag{10}$$



Pause. What's the meaning of this? Suppose that $\psi(x, 0) = \phi_{E<V_0}(x)$. Then

$$\psi(x, t) = Ae^{i(kx - \omega t)} + Be^{-i(kx + \omega t)} \tag{11}$$

on the left, and

$$\psi(x,t) = De^{-(\alpha x + i\omega t)} \tag{12}$$

on the right. Thus, A is the amplitude of the right-moving incident wave, B is the amplitude of the left-moving reflected wave, and D is the amplitude of the right-moving transmitted wave, which decays to zero.

More precisely, we need a measure of "stuff going right". Define the probability density:

$$\rho = |\psi(x)|^2 \Rightarrow \mathbb{P}(a,b) = \int_a^b /!dx\rho(x) \tag{13}$$

Fact: $\mathbb{P}(-\infty,\infty) = 1$.

Question: What is $\frac{\partial\rho(x)}{\partial t}(x)$?

$$\frac{\partial\rho(x)}{\partial t} = \frac{\partial}{\partial t}(\psi * \psi) = \psi\frac{\partial\psi*}{\partial t} + \psi * \frac{\partial\psi}{\partial t} \tag{14}$$

Note that:

$$\partial_t\psi = \frac{1}{i\hbar}(-\frac{\hbar^2}{2m}\partial_x^2\psi + V(x)\psi) \tag{15}$$

$$\partial_t\psi^* = -\frac{1}{i\hbar}(-\frac{\hbar^2}{2m}\partial_x^2\psi^* + V(x)\psi^*) \tag{16}$$

So, the expression for $\frac{\partial\rho(x)}{\partial t}(x)$ becomes:

$$\frac{1}{i\hbar}(\psi^*[-\frac{\hbar^2}{2m}\partial_x^2\psi + \cancel{V(x)\psi}] - \psi[-\frac{\hbar^2}{2m}\partial_x^2\psi^* + V(x)\psi^*]) \tag{17}$$

$$= \frac{\hbar}{2mi}(\psi^*\partial_x^2\psi - \psi\partial_x^2\psi^*) \tag{18}$$

$$= \frac{\hbar}{2mi}\partial_x(\psi^*\partial_x\psi - \psi\partial_x\psi^*) \tag{19}$$

$$= -\partial_x J \tag{20}$$

So,

$$J = \frac{\hbar}{2mi}(\psi^* \partial_x \psi - \psi \partial_x \psi^*) \tag{21}$$

Statement of Conservation of Probability:

$$\frac{\partial \rho}{\partial t} = \frac{\partial J}{\partial x} \tag{22}$$

$$\frac{dP(x_a, x_b)}{dt} = J(x_a) - J(x_b) \tag{23}$$