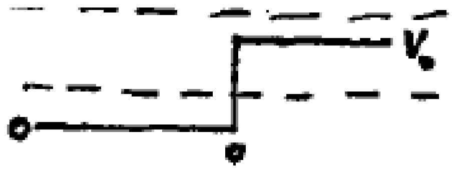
Lecture 12

1 Potential Step

1.1 From last time



Classical intuition: when $E > V_0$, the particle should not bounce. When $E < V_0$, the particle should bounce.

The question: What are $\phi_E(x)$?

This is easier than the finite well! Study the bouncing case first: $E < V_0$

For $x < 0$:

$$\phi_E(x) = Ae^{ikx} + Be^{-ikx} \tag{1}$$

where $k = \sqrt{\frac{2mE}{\hbar^2}}$

For $x > 0$:

$$\phi_E(x) = Ce^{\alpha x} + De^{-\alpha x} \tag{2}$$

where $\alpha = \sqrt{\frac{2m}{\hbar^2}(V_0 - E)}$

Normalization: $\phi(x \to \infty) \to 0 \Rightarrow C = 0$

Continuity:

- $\phi(0) = A + B = D$
- $\phi'(0) = ik(A - B) = -\alpha D$

Thus,

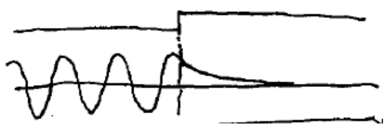
- $D = \frac{2k}{k+i\alpha} A$
- $B = \frac{k-i\alpha}{k+i\alpha} A$

Thus, we arrive at the following wavefunction:

$$\phi_E(x) = Ae^{ikx} + \frac{k - i\alpha}{k + i\alpha} e^{-ikx} \tag{3}$$

on the left, and

$$\phi_E(x) = A\frac{2k}{k + i\alpha} e^{-\alpha x} \tag{4}$$



Pause. What's the meaning of this? Suppose that $\psi(x, 0) = \phi_{E<V_0}(x)$. Then

$$\psi(x, t) = Ae^{i(kx-\omega t)} + Be^{-i(kx+\omega t)} \tag{5}$$

on the left, and

$$\psi(x, t) = De^{-(\alpha x+i\omega t)} \tag{6}$$

on the right. Thus, A is the amplitude of the right-moving incident wave, B is the amplitude of the left-moving reflected wave, and D is the amplitude of the right-moving transmitted wave, which decays to zero.

Note: The reflection incurs a phase shift; the amount of shift depends on V_0.

More precisely, we need a measure of "stuff going right". Define the probability density:

$$\rho = |\psi(x)|^2 \Rightarrow \mathbb{P}(a, b) = \int_a^b /!dx \rho(x) \tag{7}$$

Fact: $\mathbb{P}(-\infty, \infty) = 1$.

Question: What is $\frac{\partial \rho(x)}{\partial t}(x)$?

$$\frac{\partial \rho(x)}{\partial t} = \frac{\partial}{\partial t}(\psi * \psi) = \psi \frac{\partial \psi *}{\partial t} + \psi * \frac{\partial \psi}{\partial t} \tag{8}$$

Note that:

$$\partial_t \psi = \frac{1}{i\hbar}(-\frac{\hbar^2}{2m}\partial_x^2 \psi + V(x)\psi) \tag{9}$$

$$\partial_t \psi^* = -\frac{1}{i\hbar}(-\frac{\hbar^2}{2m}\partial_x^2 \psi^* + V(x)\psi^*) \tag{10}$$

So, the expression for $\frac{\partial \rho(x)}{\partial t}(x)$ becomes:

$$\frac{1}{i\hbar}(\psi^*[-\frac{\hbar^2}{2m}\partial_x^2 \psi + \cancel{V(x)\psi}] - \psi[-\frac{\hbar^2}{2m}\partial_x^2 \psi^* + V(x)\psi^*]) \tag{11}$$

$$= \frac{\hbar}{2mi}(\psi^* \partial_x^2 \psi - \psi \partial_x^2 \psi^*) \tag{12}$$

$$= \frac{\hbar}{2mi}\partial_x(\psi^* \partial_x \psi - \psi \partial_x \psi^*) \tag{13}$$

$$= -\partial_x J \tag{14}$$

So,

$$J = \frac{\hbar}{2mi}(\psi^* \partial_x \psi - \psi \partial_x \psi^*) \tag{15}$$

Statement of Conservation of Probability:

$$\frac{\partial \rho}{\partial t} = \frac{\partial J}{\partial x} \tag{16}$$

$$\frac{dP(x_a, x_b)}{dt} = J(x_a) - J(x_b) \tag{17}$$

Note: In a stationary state, $\cdot \rho = 0 \Rightarrow J = \mathit{constant}$!

1.2 Currents

The question: What are the incident, reflected, and transmitted currents?

- $\psi_I = Ae^{i(kx-\omega t)} \Rightarrow J_I = \frac{\hbar k}{m}|A|^2$
- $\psi_R = Be^{-i(kx+\omega t)} \Rightarrow J_I = -\frac{\hbar k}{m}|B|^2$
- $\psi_T = De^{-\alpha x - i\omega t)} \Rightarrow J_T = 0$

Thus, for the incident and reflected cases, we have a constant flow; for the transmitted case, we have no flow (a steady state). We can therefore define:

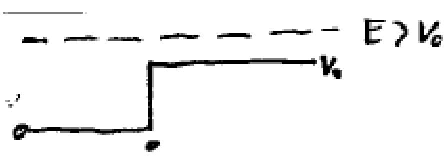
$$T = |\frac{J_T}{J_I}| \tag{18}$$

and

$$R = |\frac{J_R}{J_I}| \tag{19}$$

to get that $T = 0$, and $R = |\frac{k-i\alpha}{k+i\alpha}|^2 = \nVdash$

1.3 Example: No Classical Bounce



In other words, $E > V_0$

For $x < 0$:

$$\phi_E(x) = Ae^{ik_1x} + Be^{-ik_1x} \tag{20}$$

where $k_1 = \sqrt{\frac{2mE}{\hbar^2}}$

For $x > 0$:

$$\phi_E(x) = Ce^{ik_2x} + De^{-ik_2x} \tag{21}$$

where $\alpha = \sqrt{\frac{2m}{\hbar^2}(E - V_0)}$

Continuity:

- $A + B = C + D$
- $ik_1(A_B) = ik_2(C - D)$

There are two cases to consider. The first, D=0, means that the incident beam is from the left; we thus have a transmitted component to the right, and a reflected component to the left. Thesecond, A = 0, means that the incident beam is from the right; we thus have a transmitted component to the left, and a reflected component to the right.

Let us consider D = 0. The result:

$$C = \frac{2k_2}{k_1 + k_2} A \tag{22}$$

and

$$B = \frac{k_1 - k_2}{k_1 + k_2} A \tag{23}$$

We get the following values for current:

$$J_I = \frac{\hbar k_1}{m} |A|^2 \tag{24}$$

$$J_R = -\frac{\hbar k_1}{m} |\frac{k_1 - k_2}{k_1 + k_2}|^2 |A|^2 \tag{25}$$

and

$$J_T = \frac{\hbar k_2}{m} |\frac{2k_1}{k_1 + k_2}|^2 |A|^2 \tag{26}$$

We get the following values for the transmitted and reflected fractions:

$$R = |\frac{k_1 - k_2}{k_1 + k_2}|^2 \tag{27}$$

$$T = \frac{4k_1 k_2}{|k_1 + k_2|^2} \tag{28}$$

Note that $R + T = 1$.

It is illuminating to rewrite all this in terms of E, V_0:

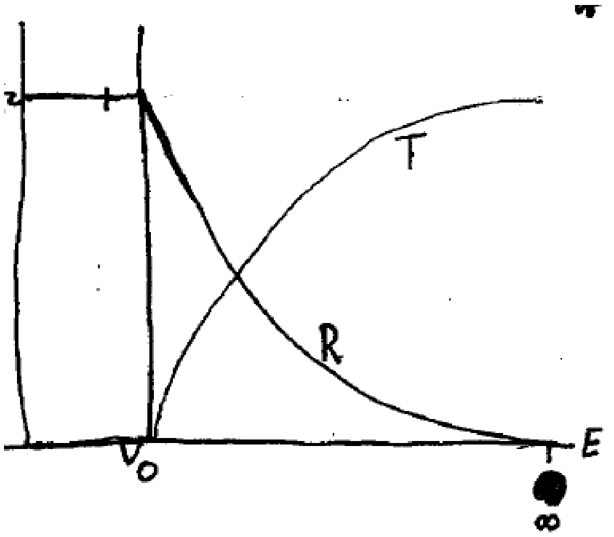
$$k_1^2 = \frac{2mE}{\hbar^2} \tag{29}$$

$$k_2^2 = \frac{2m(E - V_0)}{\hbar^2} \tag{30}$$

$$(\frac{k_2}{k_1})^2 = 1 - \frac{E}{V_0} \tag{31}$$

$$R = |\frac{1 - \frac{k_2}{k_1}}{1 + \frac{k_2}{k_1}}|^2 = |\frac{1 - \sqrt{1 - \frac{E}{V_0}}}{1 + \sqrt{1 - \frac{E}{V_0}}}|^2 \tag{32}$$

$$T = \frac{4\sqrt{1 - \frac{E}{V_0}}}{(1 + \sqrt{1 - \frac{E}{V_0}})^2} \tag{33}$$



Now, take the case where A = 0. By explicit construction, or by symmetry with the D = 0 case:

$$C = \frac{k_1 - k_2}{k_1 + k_2} A \tag{34}$$

and

$$B = \frac{2k_2}{k_1 + k_2} A \tag{35}$$

We get the following values for current:

$$J_I = -\frac{\hbar k_2}{m} |D|^2 \tag{36}$$

$$J_R = \frac{\hbar k_2}{m} | \frac{k_1 - k_2}{k_1 + k_2} |^2 |D|^2 \tag{37}$$

and

$$J_T = -\frac{\hbar k_1}{m} | \frac{2k_2}{k_1 + k_2} |^2 |D|^2 \tag{38}$$

We get the following values for the transmitted and reflected fractions:

$$R = | \frac{k_1 - k_2}{k_1 + k_2} |^2 \tag{39}$$

$$T = | \frac{4k_1 k_2}{(k_1 + k_2)^2} | \tag{40}$$

Check: $R + T = 1$.

We got the same R, T as in the uphill case!

textbfNote: Again, illuminating to rewrite in terms of E, V_0.

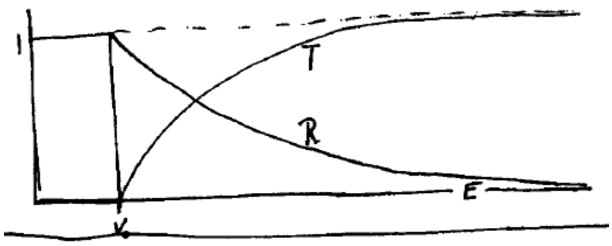
$$k_1^2 = \frac{2mE}{\hbar^2} \tag{41}$$

$$k_2^2 = \frac{2m(E - V_0)}{\hbar^2} \tag{42}$$

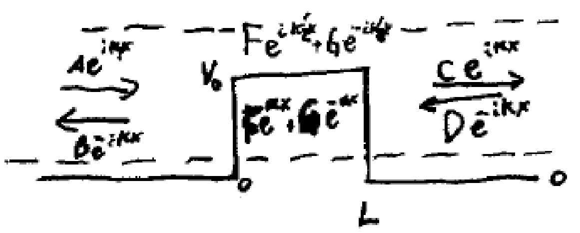
$$(\frac{k_2}{k_1})^2 = 1 - \frac{V_0}{E} \tag{43}$$

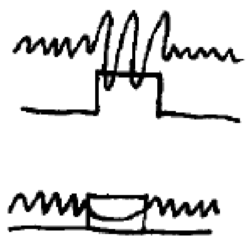
$$R = | \frac{1 - \frac{k_2}{k_1}}{1 + \frac{k_2}{k_1}} |^2 = | \frac{1 - \sqrt{1 - \frac{V_0}{E}}}{1 + \sqrt{1 - \frac{V_0}{E}}} |^2 \tag{44}$$

$$T = \frac{4\sqrt{1-\frac{V_0}{E}}}{(1+\sqrt{1-\frac{V_0}{E}})^2} \tag{45}$$



1.4 A more intricate example





As above, $T = |\frac{C}{A}|^2_{D=0}$ for particles incident from the left, and $T = |\frac{B}{D}|^2_{A=0}$ for particles incident from the right. Similarly, $TR = |\frac{B}{A}|^2_{D=0}$ for particles incident from the left, and $R = |\frac{C}{D}|^2_{A=0}$ for particles incident from the right.

Goal: Compute, B, C given A, D = 0. **Tool:** Four boundary conditions, two at x = 0 and 2 at x = L. We then determine G, H, B, C in terms of A, D.

For the case $E > V_0, D = 0$:

- For the boundary condition at x = 0: $A + B = F + G$ and $ik_1(A_B) = ik_2(F - G)$
- For the boundary condition at x = L: $Fe^{ik_2L} + Ge^{-ik_2L} = Ce^{ik_1L}$ and $ik_2(Fe^{ik_2L - Ge^{-ik_2L}}) - ik_1Ce^{ik_1L}$

After a bunch of algebra, we get

$$B = A\frac{(k_1^2 - k_2^2 \sin k_2L)}{2ik_1k_2\cos k_2L + (k_1^2+k_2^2)\sin k_2L} \tag{46}$$

$$C = A\frac{(ik_1k_2e^{-ik_1L})}{2ik_1k_2\cos k_2L + (k_1^2+k_2^2)\sin k_2L} \tag{47}$$

Thus,

$$T = |\frac{C}{A}|^2 = \frac{4k_1^2k_2^2}{4k_1^2k_2^2\cos^2 k_2L + (k_1^2+k_2^2)^2\sin^2 k_2L} \tag{48}$$

Finally:

$$T = \frac{1}{1+\frac{(k_1^2-k_2^2)^2}{4k_1^2k_2^2}\sin^2 k_2L} \tag{49}$$

To make our expression dimensionless, we make the substitution $k_1^2 = \frac{2m}{\hbar^2}E$, to get

$$T = \frac{1}{1+\frac{V_0^2}{4E(E-V_0)}\sin^2 k_2L} \tag{50}$$

Similarly, we have

$$R = \frac{sin^2 k_2L}{\frac{4k_1^2k_2^2}{(k_1^2-k_2^2)^2}+\sin^2 k_2L} \tag{51}$$

that we can make dimensionless using the substitution $k_2^2 = \frac{2m}{\hbar^2}(E-V_0)$, to get

$$R = \frac{sin^2 k_2L}{\frac{4E(E-V_0)}{V_0^2}+\sin^2 k_2L} \tag{52}$$

Note: When $k_2L = n\pi, T \to 1, R \to 0$. This is called resonance. This means that n wavlengths fit inside the barrier perfectly. Think of it in terms of number of scatterings.

Note: When $k_2L = (n+\frac{1}{2})\pi, T \to \frac{4E(E-V_0)}{(2E-V_0)^2}, R \to \frac{V_0^2}{(2E-V_0)^2}$.

$$T_{E>V_0} = \frac{1}{1+\frac{1}{4\epsilon(\epsilon-1)}\sin^2(\sqrt{g_0^2(\epsilon-1)})} \tag{53}$$

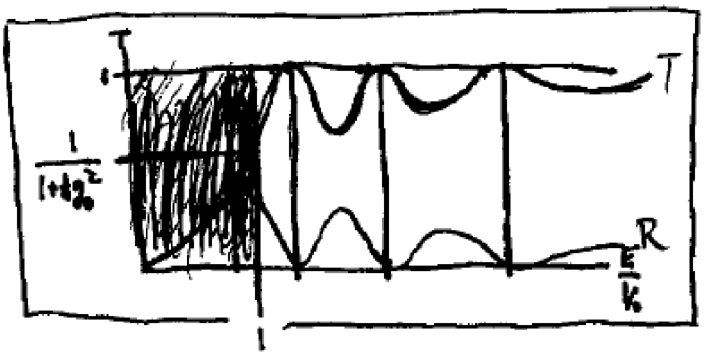
where

$$g_0^2 = \frac{2m}{\hbar^2} L^2 V_0 \tag{54}$$

and

$$\epsilon = \frac{E}{V_0} \tag{55}$$

In the limit $E \to V_0, \epsilon \to 1$



Now, what about $E < V_0$?

We use identical reasoning. In other words, define $ik_2 \to \alpha$, $\alpha^2 = \frac{2m}{\hbar^2}(V_0 - E)$.

$$T_{E<V_0} = \frac{1}{1 + \frac{1}{4\epsilon(1-\epsilon)} \sinh^2 \sqrt{g_0^2(1-\epsilon)}} \tag{56}$$

We have **tunneling** for $E < V_0$!! This is a purely quantum phenomenon!! Note: resonance for $E > V_0$ like light: perfect transmission for thin films.

Note: fix $E < V_0$, vary L. For large L, $T\ e^{-2\alpha L}$. This is very improbable!!!