Lecture 16

1 Last time

$$L^2 Y_{lm} = \hbar^2 l(l+1) Y_{lm} \tag{1}$$

$$L_z Y_{lm} = \hbar m Y_{lm} \tag{2}$$

where m ranges from $-l$ to l.

The top and bottom of the ladder are defined as follows:

$$L_+ Y_{ll} = 0, L_- Y_{l-l} = 0 \tag{3}$$

Okay, what do Y_{lm} look like?

1. Pick spherical

$$L_z = \frac{\hbar}{i}\partial_\phi \tag{4}$$

$$L_\pm = \hbar e^{\pm i\phi}(\partial_\theta \pm i \cot\theta \partial_\phi) \tag{5}$$

2. $L_z Y_{lm} = \hbar m Y_{lm}$

$$\partial_\phi Y_{lm} = i\, m\, Y_{lm} \tag{6}$$

$$Y_{lm} = P_{lm}(\theta) e^{im\theta} \tag{7}$$

3. Can determine $P_{lm}\theta$ from:

(a) $\hat{L}_+ P_{ll} = 0$ can use to determine P_{ll}

$$0 = L_+ Y_{ll} = [\hbar e^{i\phi}(\partial_\theta + i\cot\theta\partial_\phi)] P_{ll}(\theta) e^{il\phi} \tag{8}$$

$$(\partial_\theta - l\cot\theta) P_{ll}(\theta) = 0 \tag{9}$$

$$P_{ll} = (\sin\theta)^l \tag{10}$$

$$Y_{ll} = c_{ll}(\sin\theta)^l e^{il\phi} \tag{11}$$

(b) Get P_{lm} by soc. app. of $\hat{L}_-$ as we did for the harmonic oscillator

$$Y_{l,l-k} \propto (L_-)^k Y_{ll} = c_{lm}[\hbar e^{-i\phi}(\partial_\theta - i\cot\theta\partial_\phi)]^k (\sin\theta)^l e^{il\phi} \tag{12}$$

Some examples:

"s":

$$Y_{00} = \frac{1}{\sqrt{4\pi}} \tag{13}$$

"p":

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi} \tag{14}$$

$$Y_{10} = \sqrt{\frac{3}{8\pi}} \cos\theta \tag{15}$$

$$Y_{1-1} = \sqrt{\frac{3}{8\pi}} \sin\theta e^{-i\phi} \tag{16}$$

"d":

$$Y_{22} = \sqrt{\frac{15}{32\pi}} \sin^2\theta e^{2i\phi} \tag{17}$$

$$Y_{22} = \sqrt{\frac{15}{16\pi}}(3\cos^2\theta - 1) \tag{18}$$

$$Y_{22} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{-2i\phi} \tag{19}$$

Note: What about $Y_{\frac{1}{2},\frac{1}{2}}$, $Y_{\frac{1}{2},-\frac{1}{2}}$?