Lecture 9

1 Review of Harmonic Oscillator Energy Eigenstates

Last time, we went over a brute force method for calculating the eigenstates of the harmonic oscillator. We started with the Schrödinger Equation:

$$\hat{E}\phi_E = E\phi_E \tag{1}$$

$$\left[-\frac{\hbar^2}{2m}\partial_x^2 + \frac{m\omega^2}{2}x^2\right]\phi_E = E\phi_E \tag{2}$$

and found that the eigenstates lie in a spectrum, with energy eigenvalues spaced by $\hbar\omega$.

In the position basis,

$$\phi_N = N_N e^{-\frac{x^2}{2a^2}} H_N\left(\frac{x}{a}\right) \tag{3}$$

where

$$N_N = \sqrt{\frac{1}{2^N \pi a N!}} \tag{4}$$

and H_N are the Hermite polynomials. For reference, here are the first four:

$$H_0 = 1; \quad H_1 = 2u; \quad H_2 = 4u^2 - 2; \quad H_3 = 8u^3 - 12u$$

Recall that the discrete nature of the energy eigenvalues came from imposing normalizability. However, our brute force approach did not explain why the tower of energy eigenvalues is evenly-spaced. To answer this, let's start again, using what is generally called "The Operator Method."

2 Energy Eigenstates and Eigenvalues Using the Operator Method

2.1 Factorizing the Energy Operator

First, let's go back to the operator we want to find eigenstates for: the $\hat{E}$ operator:

$$\hat{E} = \frac{\hat{p}^2}{2m} + \frac{m\omega_0^2}{2}\hat{x}^2 \tag{5}$$

Before we do anything, let's put this in dimensionless form:

$$\hat{E} = \hbar\omega_0 \left(\frac{\hat{p}^2}{2m\hbar\omega_0} + \frac{\hat{x}^2}{\left(\frac{2\hbar}{m\omega_0}\right)} \right) \tag{6}$$

We make the following substitutions:

$$x_0 = \sqrt{\frac{2\hbar}{m\omega_0}} \qquad p_0 = \sqrt{2m\hbar\omega_0}$$

And check dimensions:

$$[\hbar\omega_0] = E \qquad [p_0^2 = 2m\hbar\omega_0] = M^2L^2T^{-2} \qquad \left[x_0^2 = \frac{2\hbar}{m\omega_0}\right] = L^2$$

The expression for the $\hat{E}$ operator can then be rewritten:

$$\hat{E} = \hbar\omega_0 \left[\left(\frac{\hat{p}}{p_0}\right)^2 + \left(\frac{\hat{x}}{x_0}\right)^2 \right] \tag{7}$$

If this were a polynomial of complex numbers, we could simply factor this form as $c^2 + d^2 = (c - id)(c + id) = a^*a$. Can we do this with operators?

$$\begin{aligned}
\left(\frac{\hat{x}}{x_0} - i\frac{\hat{p}}{p_0}\right)\left(\frac{\hat{x}}{x_0} + i\frac{\hat{p}}{p_0}\right) &= \left(\frac{\hat{x}}{x_0}\right)^2 + \left(\frac{\hat{p}}{p_0}\right)^2 + i\left[\frac{\hat{x}}{x_0}, \frac{\hat{p}}{p_0}\right] && (8)\\
&= \left(\frac{\hat{x}}{x_0}\right)^2 + \left(\frac{\hat{p}}{p_0}\right)^2 + \frac{i}{x_0p_0}[\hat{x}, \hat{p}] && (9)\\
&= \left(\frac{\hat{x}}{x_0}\right)^2 + \left(\frac{\hat{p}}{p_0}\right)^2 + \frac{i}{2\hbar}(i\hbar) && (10)\\
&= \left(\frac{\hat{x}}{x_0}\right)^2 + \left(\frac{\hat{p}}{p_0}\right)^2 - \frac{1}{2} && (11)
\end{aligned}$$

Not quite - but almost! Accounting for the slight modification from factorization, Equation 7 can be rewritten as follows:

$$\hat{E} = \hbar\omega_0 \left[\left(\frac{\hat{x}}{x_0} - i\frac{\hat{p}}{p_0} \right) \left(\frac{\hat{x}}{x_0} + i\frac{\hat{p}}{p_0} \right) + \frac{1}{2} \right] \tag{12}$$

Now, define two new operators, $\hat{a}$ and $\hat{a}^\dagger$:

$$\hat{a} = \left(\frac{\hat{x}}{x_0} + i\frac{\hat{p}}{p_0} \right) \tag{13}$$

$$\hat{a}^\dagger = \left(\frac{\hat{x}}{x_0} - i\frac{\hat{p}}{p_0} \right) \tag{14}$$

The energy operator is then simplified to:

$$\hat{E} = \hbar\omega_0 \left[\hat{a}^\dagger \hat{a} + \frac{1}{2} \right] \tag{15}$$

$\hat{a}^\dagger$ looks kind of like a "complex conjugate," but $\hat{a}$ is an operator. So what does it mean to take the "complex conjugate" of an operator?

2.2 Math Aside: Hermitian Operators

Given any linear operator $\hat{\Theta}$, we can always build a related operator $\hat{\Theta}^\dagger$ in the following way:

$$\int_{-\infty}^{\infty} \mathrm{d}x \, f^*(x) \, (\hat{\Theta}^\dagger \, g(x)) \equiv \int_{-\infty}^{\infty} \mathrm{d}x \, (\hat{\Theta} \, f(x))^* \, g(x) \tag{16}$$

This is true for all $f(x)$ and $g(x)$.

We define the **Hermitian adjoint**: $\hat{\Theta}^\dagger$ is the Hermitian adjoint of $\hat{\Theta}$.

Note: What is the adjoint of the complex number α?

$$\begin{aligned} \int f^*(\alpha^\dagger\, g(x)) &\equiv \int (\alpha\, f)^* g & (17)\\ &= \int \alpha^* f^* g & (18)\\ &= \int f^*(\alpha^* g) & (19) \end{aligned}$$

For complex numbers, the adjoint is the same thing as the complex conjugate.

Check at home: $(\hat{\Theta}^\dagger)^\dagger = \hat{\Theta}$

Example. What is $(\partial_x)^\dagger$?

$$\begin{aligned} \int f^* \left(\partial_x^\dagger\, g(x)\right) &\equiv \int (\partial_x\, f)^* g & (20)\\ &= \int (\partial_x^* f^*) g & (21)\\ &= -\int f^* \partial_x g & (22)\\ &= \int f^*(-\partial_x g) & (23) \end{aligned}$$

Where integration by parts was used in the third step. Therefore:

$$(\partial_x)^\dagger = -\partial_x \tag{24}$$

Example. What is $\hat{x}^\dagger$?

$$\begin{aligned} \int f^*(\hat{x}^\dagger\, g(x)) &\equiv \int (\hat{x}\, f)^* g & (25)\\ &= \int (xf)^* g & (26)\\ &= \int x f^* g & (27)\\ &= \int f^*(xg) & (28)\\ &= \int f^*(\hat{x} g) & (29) \end{aligned}$$

Therefore:

$$\hat{x}^\dagger = x \tag{30}$$

Definition: A Hermitian operator is one that is its own Hermitian adjoint: $\hat{\theta} = \hat{\theta}^\dagger$

Note: For complex numbers, $\alpha^\dagger = \alpha$, and for Hermitian numbers, $\alpha^* = \alpha$, Hermitian numbers must be **real.**

Note: If $\hat{x}$ is Hermitian, $\hat{x}$ has real eigenvalues.

Math fact: Hermitian operators have only real eigenvalues.

Example. What is $\hat{p}^\dagger$?

$$\begin{aligned}
\int f^*(\hat{p}^\dagger\, g) &= \int (\hat{p}\, f)^* g && (31)\\
&= \int (-i\hbar\partial_x f)^* g && (32)\\
&= \int i\hbar(\partial_x f^*) g && (33)\\
&= \int f^*(-i\hbar\partial_x g) && (34)\\
&= \int f^*(\hat{p} g) && (35)
\end{aligned}$$

Where integration by parts was used to get the fourth line. Therefore:

$$\hat{p}^\dagger = p \tag{36}$$

$\hat{p}$ is Hermitian, and has real eigenvalues.

Physical fact: All observables are real, and all observables corresponding to observables must be Hermitian operators.

2.3 More on $\hat{a}$ and $\hat{a}^\dagger$

A reminder that we have established the following definition:

$$\hat{a} = \frac{\hat{x}}{x_0} + i\frac{\hat{p}}{p_0} \tag{37}$$

Is $\hat{a}$ Hermitian?

Well, $\hat{x}$ and $\hat{p}$ are, but i$\hat{p} \sim \partial_x$ is NOT. So NO, $\hat{a}$ is not Hermitian. Let's calculate its adjoint:

$$\begin{aligned}
\int f^*(\hat{a}^\dagger\, g) &= \int (\hat{a}\, f)^* g & (38)\\
&= \int \left(\frac{\hat{x}}{x_0}\, f + i\frac{\hat{p}}{p_0}\, f\right)^* g & (39)\\
&= \int \left(\frac{\hat{x}}{x_0}\, f\right)^* g - i\int \left(\frac{\hat{p}}{p_0}\, f\right)^* g & (40)\\
&= \int f^* \left(\frac{\hat{x}}{x_0} g\right) - i\int f^* \left(\frac{\hat{p}}{p_0}\, g\right) & (41)\\
&= \int f^* \left(\frac{\hat{x}}{x_0} g - i\frac{\hat{p}}{p_0}\right) g & (42)
\end{aligned}$$

Therefore:

$$\left(\frac{\hat{x}}{x_0} + i\frac{\hat{p}}{p_0}\right)^\dagger = \left(\frac{\hat{x}}{x_0} - i\frac{\hat{p}}{p_0}\right) \tag{43}$$

Which confirms the original definition of $\hat{a}^\dagger$.

Note: $\hat{a} \neq \hat{a}^\dagger$ so $\hat{a}$ is not Hermitian, does not have pure real eigenvalues, and does not correspond to an observable.

Now, the question is: why are we bothering with $\hat{a}$ and $\hat{a}^\dagger$?

First of all, it puts the Hamiltonian into a simple form:

$$\hat{E} = \hbar\omega_0 \left[\hat{a}^\dagger\hat{a} + \frac{1}{2}\right] \tag{44}$$

Letting $\hat{N} = \hat{a}^\dagger\hat{a}$, this can be condensed further to

$$\hat{E} = \hbar\omega_0(\hat{N} + \frac{1}{2}) \tag{45}$$

Also, we can write the energy eigenvalues as follows, given an eigenvalue N of $\hat{N}$:

$$E_N = \hbar\omega_0(N + \frac{1}{2}) \tag{46}$$

Secondly, they satisfy the coolest commutation relation in the universe!

$$\begin{aligned}
\left[\hat{a}, \hat{a}^\dagger\right] &= \left[\frac{\hat{x}}{x_0} + i\frac{\hat{p}}{p_0}, \frac{\hat{x}}{x_0} - i\frac{\hat{p}}{p_0}\right] && (47)\\
&= \frac{i}{x_0 p_0}[\hat{p}, \hat{x}] - \frac{i}{x_0 p_0}[\hat{x}, \hat{p}] && (48)\\
&= \frac{i}{x_0 p_0}(-i\hbar - (i\hbar)) && (49)\\
&= 1 && (50)
\end{aligned}$$

Note:

$$\left[\hat{a}^\dagger, \hat{a}\right] = -1 \tag{51}$$

$$[\hat{a}, \hat{a}] = 0 \tag{52}$$

$$\left[\hat{a}^\dagger, \hat{a}^\dagger\right] = 0 \tag{53}$$

First, we combine expression 44 with the commutation relation we just derived:

$$\begin{aligned}
\left[\hat{E}, \hat{a}\right] &= \left[\hbar\omega\, \hat{a}^\dagger\hat{a}^\dagger, \hat{a}\right] - \left[\frac{1}{2}\hbar\omega, \hat{a}\right] && (54)\\
&= \hbar\omega\left(\hat{a}^\dagger\hat{a}\hat{a} - \hat{a}\hat{a}^\dagger\hat{a}\right) && (55)\\
&= \hbar\omega\left[\hat{a}^\dagger, \hat{a}\right]\hat{a} && (56)\\
&= -\hbar\omega\hat{a} && (57)
\end{aligned}$$

Thus,

$$[\hat{E}, \hat{a}] = -\hbar\omega\hat{a} \tag{58}$$

Similarly,

$$\begin{aligned}
[\hat{E}, \hat{a}^\dagger] &= \hbar\omega[\hat{a}^\dagger\hat{a}, \hat{a}^\dagger] && (59)\\
&= \hbar\omega(\hat{a}^\dagger\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}^\dagger\hat{a}) && (60)\\
&= \hbar\omega\hat{a}^\dagger[\hat{a}, \hat{a}^\dagger] && (61)\\
&= \hbar\omega\hat{a}^\dagger && (62)
\end{aligned}$$

Thus,

$$\left[\hat{E}, \hat{a}^\dagger\right] = \hbar\omega\hat{a}^\dagger \tag{63}$$

Here's why this is **so damn useful:** suppose we have a state ϕ_E with energy E, ie

$$\hat{E}\phi_E = E\phi_E \tag{64}$$

Consider the distinct state: $\hat{a}\phi_E$. The miracle is that it is also an energy eigenstate!

$$\begin{aligned}
\hat{E}(\hat{a}\phi_E) &= \hat{E}\hat{a}\phi_E - \hat{a}\hat{E}\phi_E + \hat{a}\hat{E}\phi_E && (65)\\
&= [\hat{E}, \hat{a}]\phi_E + \hat{a}\hat{E}\phi_E && (66)\\
&= -\hbar\omega\hat{a}\phi_E + \hat{a}E\phi_E && (67)\\
&= (E - \hbar\omega)\hat{a}\phi_E && (68)
\end{aligned}$$

Thus,

$$\hat{E}(\hat{a}\phi_E) = (E - \hbar\omega)(\hat{a}\phi_E) \tag{69}$$

$$\hat{a}\phi_E \propto \phi_{E-\hbar\omega} \tag{70}$$

$$\hat{a}(\hat{a}\phi_E) \propto \phi_{E-2\hbar\omega} \tag{71}$$

$$(\hat{a})^N\phi_E \propto \phi_{E-N\hbar\omega} \tag{72}$$

We have an evenly-spaced tower of energies! Note that proportionality is used because though acting with $\hat{a}$ gives a function with the correct eigenvalue, it may not normalized.

We can use $\hat{a}$ to build an evenly-spaced ladder of $\hat{E}$ eigenstates!

Now, what about $\hat{a}^\dagger$?

All that changes in the above is the sign of $\hbar\omega$ (since $[\hat{E}, \hat{a}] = -\hbar\omega$, but $[\hat{E}, \hat{a}^\dagger] = \hbar\omega$)

Thus,

$$(\hat{a})^\dagger \phi_E \propto \phi_{E+\hbar\omega} \tag{73}$$

$\hat{a}^\dagger$ raises E by $\hbar\omega$!

$$(\hat{a}^\dagger)^N \phi_E \propto \phi_{E+N\hbar\omega} \tag{74}$$

We can use $\hat{a}^\dagger$ to walk back UP our ladder of $\hat{E}$ eigenstates!

So, we have a ladder of $\hat{E}$ eigenstates spaced evenly by $\hbar\omega$.

Problem: this implies E can get negative! But we have

$$\langle E \rangle = \int dp |\tilde{\psi}(p)|^2 \frac{p^2}{2m} + \int dx |\tilde{\psi}(x)|^2 \frac{m\omega^2 x^2}{2} \geq 0 \tag{75}$$

So there MUST be a lowest state, ϕ_0, with least possible energy E_0. Let's try to satisfy the condition

$$\hat{a}\phi_0 = 0 \tag{76}$$

This gives:

$$\hat{a}\phi_0 = (\frac{\hat{x}}{x_0} + i\frac{\hat{p}}{p_0})\phi_0 = 0 \tag{77}$$

$$\partial_x + \frac{p_0}{\hbar x_0} x\phi_0 = 0 \tag{78}$$

$$(\partial_x + \frac{x}{a^2})\phi_0 = 0 \tag{79}$$

Thus,

$$\phi_0 = \theta_0 e^{-\frac{x^2}{2a^2}} \tag{80}$$

Easy way to find ϕ_0!!

What is the ground state energy?

$$\begin{aligned}
\hat{E}\phi_0 &= \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\phi_0 && (81)\\
&= \hbar\omega\left(\hat{a}^\dagger(\hat{a}\phi_0) + \frac{1}{2}\phi_0\right) && (82)\\
&= \hbar\omega\left(\hat{a}^\dagger(0) + \frac{1}{2}\phi_0\right) && (83)\\
&= \frac{1}{2}\hbar\omega\phi_0 && (84)
\end{aligned}$$

Thus,

$$E_0 = \frac{1}{2}\hbar\omega \tag{85}$$

By raising E_0 N-times with the raising operator $\hat{a}^\dagger$, we get

$$E_N = \hbar\omega(N + \frac{1}{2}) \tag{86}$$

Note, from Equations (86) and (44),

$$\hat{N}\phi_N = N\phi_N \tag{87}$$

Finally, let's get this normalization straight. Given that $\psi = \hat{a}^\dagger\phi_0$:

$$\begin{aligned}
\langle\psi\rangle &= \int(\hat{a}^\dagger\phi_0)^*\hat{a}^\dagger\phi_0 && (88)\\
&= \int\phi_0^*\hat{a}\hat{a}^\dagger\phi_0 && (89)\\
&= \int\phi_0([\hat{a},\hat{a}^\dagger] + \hat{a}^\dagger\hat{a})\phi_0 && (90)\\
&= \int|\phi_0|^2 && (91)\\
&= 1 && (92)
\end{aligned}$$

Therefore ψ is properly normalized and we have

$$\phi_1 = \hat{a}^\dagger \phi_0 \tag{93}$$

Similarly, let's now investigate $\psi = \hat{a}^\dagger \phi_1$

$$\begin{aligned}
\langle\psi\rangle &= \int (\hat{a}^\dagger \phi_1)^* \hat{a}^\dagger \phi_1 && (94)\\
&= \int \phi_1^* \hat{a}\hat{a}^\dagger \phi_1 && (95)\\
&= \int \phi_1 ([\hat{a}, \hat{a}^\dagger] + \hat{a}^\dagger \hat{a}) \phi_1 && (96)\\
&= 2 && (97)
\end{aligned}$$

Thus,

$$\phi_2 = \frac{1}{\sqrt{2}} \hat{a}^\dagger \phi_1 \tag{98}$$

In general,

$$\phi_N = \frac{1}{\sqrt{N!}} (\hat{a}^\dagger)^N \phi_0 \tag{99}$$

$$\hat{a}^\dagger \phi_N = \sqrt{N+1}\, \phi_{N+1} \tag{100}$$

$$\hat{a} \phi_N = \sqrt{N}\, \phi_{N-1} \tag{101}$$

Moreover, we can easily find the excited state eigenfunctions:

$$\phi_N = \frac{1}{\sqrt{N!}} \left(\sqrt{\frac{mw}{2\hbar}} x - \sqrt{\frac{\hbar}{2m\omega}} \partial_x \right)^N \phi_0 \tag{102}$$

Plug in the following, and you're done:

$$\phi_0 = \left(\frac{m\omega}{\pi\hbar} \right)^{\frac{1}{2}} e^{-\frac{m\omega}{2\hbar} x^2} \tag{103}$$

To close, let's go back to the magic moment when we found that

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1 \tag{104}$$

I claim that everything follows from this. In particular, we can immediately construct the number operator $\hat{N}$ such that

$$\left[\hat{N}, \hat{a}^\dagger\right] = \hat{a}^\dagger \tag{105}$$

and

$$\left[\hat{N}, \hat{a}\right] = -\hat{a} \tag{106}$$

and deduce that the number operator has eigenvalues lying in a ladder with unit spacing.

To finish, we need to relate $\hat{N}$ to an observable. For the harmonic oscillator, we find

$$\hat{E} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right) \tag{107}$$

where

$$E_N = \hbar\omega\left(N + \frac{1}{2}\right) \tag{108}$$

There is a ground state with $\hat{N} = 0$.

Note that, if we had had,

$$\hat{E} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 + \lambda\left(\frac{\hat{p}^4}{p_0^4} + \frac{\hat{x}^4}{x_0^4} + \frac{\hat{x}^2\hat{p}^2 + \hat{p}^2\hat{x}^2}{x_0^2 p_0^2}\right) \tag{109}$$

$$\hat{E} = \hbar\omega(\hat{N} + \lambda\hat{N}^2 + \delta) \tag{110}$$

Easy!

We would STILL know the answer,

$$E_N = \hbar\omega(N + \lambda N^2 + \delta) \tag{111}$$

Trivial!

The algebra, $\left[\hat{N}, \hat{a}\right] = -\hat{a}$, $\left[\hat{N}, \hat{a}^\dagger\right] = \hat{a}^\dagger$ does all the work!