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SEMIFUZZY GAMES*

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A new type of decomposition of static games is defined in order to describe cases where overt behavior only is cooperative to a degree. The basic idea is to split the original game into a set of subgames defined by sets of decisions made by the players. In each subgame, different coalitions may form (e.g., two chain stores can cooperate in one city and not in another). To motivate the concept we give an example where such schizophrenic behaviour naturally arise between two firms competing on advertising and quantity. The new type of games are then compared to Aubin's fuzzy games and it is demonstrated that the core of these games with or without side-payments is identical to the core of the original games. We finally show how this type of games degenerates to fuzzy games under certain conditions. The concept of semifuzzy games therefore offers us a new way to think about fuzzy games, which allows us to avoid the interpretational difficulties associated with other approaches.

Keywords: Fuzzy games, Core, Characteristic function.

Introduction

Since Chamberlin [5], economists have argued that firms often collude on price and compete on advertising. G.M. and Toyota share the development costs of a new engine, while Volvo and Renault cooperate in another, meanwhile all four firms compete for car sales. The U.S. and the U.S.S.R. play the nuclear war game cooperatively and yet fight limited wars [15] in places like Korea and Vietnam. These examples illustrate the widespread practical incidence of partial cooperation, situations where players make some of a set of simultaneous decisions cooperatively and some noncooperatively.[1]

So far, our only tool for describing partial cooperation in static games is the fuzzy games [3] which are defined over fuzzy subsets of the set of players. Formally, if $N = \{1, \dots, n\}$ is the set of players, a fuzzy game with side payments is defined by its characteristic function $V : [0, 1]^n \to R$, assumed to be positively

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[1] In a setting where decisions are made sequentially, we can use standard dynamic game theory to describe alternating periods of cooperation and competition (e.g. [17]). Our concern here is with situations where cooperation and competition take place simultaneously. Such situations may of course occur at each stage of dynamic games.

homogeneous and assign zero value to the empty set. This function, which was introduced to simplify the proofs of several known results on the core, has two, for descriptive purposes, unfortunate properties.

The first of these is best brought out by an example. In a game with three players (a, b, c), assume the (ordinary) value function: $W(a, b) = W(a, c) = W(b, c) = 4$, $W(a, b, c) = 5$. If each player participates 50% in a two person coalition with each of the two others, the total payoff to these three fuzzy coalitions should be $2+2+2=6>5$ (!). The catch is, that sets of fuzzy coalitions like $(1/2, 1/2, 0), (1/2, 0, 1/2), (0, 1/2, 1/2)$ are infeasible, since each of these coalitions assume the existence of an isolated adversary which cannot be found in any of the other coalitions. (One may think of this as three people trying to schedule meetings during four hours or as three firms competing/colluding in four markets). Introducting appropriate constraints on the domain of V would of course deprive it of its nice properties and therefore render fuzzy games useless as a tool for mathematical convenience.

The other problem with descriptive uses of fuzzy games is, that they typically offer very limited insights about the way in which partial coalitions achieve their payoff. In an M market duopoly with price and advertising, a 50% coalition could mean collusion in some markets or that price and/or advertising or a combination thereof always was set between the competitive and the monopolistic levels. All that matters is that the payoffs come out right.

Let us emphasize that this critique applies only to the use of fuzzy games as a framework for understanding games with partial cooperation, a use which was not intended by the fathers of the concept. We have no quarrels with its mathematical usefulness.

The purpose of this paper is to propose an alternative tool for describing and analyzing partial cooperation in static games. Intuitively, the basic idea is to allow the players to be 'schizophrenic' and define 'semifuzzy' games over subsets of the sets of all decisions made by all players. Based on appropriate restrictions on the characteristic functions of such games, we can derive some simple relationships to ordinary and fuzzy games.

The key aspect in the definition of semifuzzy games can be illustrated through three examples. The first example involves a game played in each of 20 cities such that payoffs in each of the 20 games are dependent only on the actions taken in that game. (The replicated economies of Debreu and Scarf [7] is a similar situation.) In this type of game we could look at all possible coalitions in each of the 20 games individually and define the set of semifuzzy coalitions as the set of ordered 20 game sequences of one game coalitions. So the value function would have 20 sets of one game coalitions as its domain. (This will be made more precise below.)

When the game cannot be divided into independent subgames, the situation gets somewhat more complicated. Assume that in the game above, a player's actions in a given pair of cities are linked by a common resource constraint, such that he or she has the choice between vigorous effort in one city and moderate effort in both cities. In this case, cooperation in one place and noncooperation in another seems to be an awkard scenario, since the characteristic function would depend on the way the two coalitions shared the scarce resource, while authority over this resource is unclear. We will therefore confine our decomposition to subgames with independent sets of feasible actions. Interdependent payoffs, as in, e.g., the price–advertising decomposition, do not pose this type of problem. We have finally made the choice not to include seemingly 'unreasonable' coalitions in which some players participate in fewer subgames than other players. In a duopoly, for example, we exclude a situation where our firm sets both price and advertising cooperatively, whereas the other prices noncooperatively.

It turns out, that the core of the game in which all partial coalitions are allowed is equal to the core of the ordinary game defining it. The reason lies, intuitively, in the super-additivity of the characteristic function. Because of the super-additivity, the core constraints from partial coalitions are less tight than those from the full coalitions. Furthermore, under suitable conditions, if the game can be decomposed into many decisions, we can find a partitioning of these such that any fraction of a full coalitions payoff can be approximated by colluding only in that fraction of the decisions. These results are obtained for both the sidepayment case and the case without side payments. In fact the difference between these two cases almost disappears for this class of games.

Before we proceed in our presentation of semifuzzy games we would like to point to some interesting parallels between this concept and another reformulation used in some studies of telephone ratesetting.[2] In their pathbreaking application of cooperative nonatomic game theory, Raanan [14] and Billera, Heath, and Raanan [4] consider symmetric games in which each player has the same large set of decision variables. They then reformulate these games to consider each increment in each decision variable by each player a 'player' in the reformulated game. Each member of the set of decision variables thus constitute a class of 'players'. This differs from our analysis in two ways. First, Raanan et al. are more parsimonious than we are in the sense that we look at each decision variable of each player as a separate 'player', where they consolidate across players into types. Secondly, the 'players' in their analysis are not known before the game is played and a distribution of decisions has emerged. By contrast, our reformulation is defined directly on the structure of the original game.

To motivate the concept of semifuzzy games and outline a promising area of application, we will now look at a simple example where two firms naturally are led to competitive advertising and cooperative quantity setting. After these more intuitive considerations we will proceed to the formal part of the paper. Section 2 will contain the relevant definitions while some properties of semifuzzy games are derived in Section 3. A short discussion is provided in Section 4 and a numerical illustration is contained in the Appendix.

1. Example

The phenomenon of tacit collusion in games of the prisoners' dilemma type has traditionally been analyzed in the setting of repeated (ordinary) games. While

[2] We are grateful to the referee for alerting us to this parallel.

several models are available [8, 11, 1, 13], we will here use the simple framework developed by Telser [17].

We look at a symmetric two-person game in which the players can cooperate or compete. If both cooperate, the payoff to each is π^+, if both compete they each get π^0 and if one competes while the other cooperates the former gets π^*. Assume further that the game is repeated an unknown number of times such that the expected number of games remaining $\mu+1$ is independent of the number of games already played. If firms believe that cooperation, once violated, will never be reestablished, the expected payoff from violating an agreement is given by $\pi^*+\mu\pi^0$, whereas keeping the agreement yields the expected payoff $(\mu+1)\pi^+$. Accordingly, cooperation will prevail if $\pi^*-\pi^+\leq\mu(\pi^+-\pi^0)$.

Let us now apply this to a Cournot duopoly where two firms set outputs x_1, x_2 and face a market price of $P=1-\frac{2}{3}(x_1+x_2)$. Assuming costs to be zero, it is trivial to find Nash profits as $\pi^0=\frac{1}{6}$. Cooperative profits are $\pi^+=\frac{3}{16}$, whereas a firm which cheats on a cooperating colleague will receive $\pi^*=\frac{27}{128}$. If we now assume that $\mu=\frac{17}{16}$, we can conclude that cooperation is impossible. So far, there is no problem.

The problems appear if we try to take the model one step further to study a situation where firms set both output and advertising. Assume that the price goes up by y_1+y_2 if the firms spend y_1^2 and y_2^2, respectively, on advertising. If firms compete, cooperate or cheat on both parameters we can find the resulting profits as $\frac{5}{12}$, $\frac{3}{4}$, and $\frac{143}{100}$, respectively. Accordingly, we will not have total cooperation. However, if firms compete on advertising, the profits from competing, cooperating and cheating on quantity are $\frac{5}{12}$, $\frac{3}{7}$, and $\frac{108}{245}$, respectively.[3] In this case quantity-cooperation is a self-enforcing agreement (still assuming $\mu=\frac{17}{16}$). So we have reconstructed a rational analogy to Chamberlin's [5] much acclaimed conjecture, but not in a setting covered by ordinary game theory.

As this and the other examples cited in the introduction show, many instances of collusion are of this partial nature. It is therefore crucial to know the properties of games where such schizophrenic behavior is allowed. We will here call such games 'semifuzzy'.

2. Definitions

Let us look at an n-person game with super-additive characteristic function $W:\{0,1\}^n\to A$, where A is an n-dimensional space of multi-utilities in the no side-payment case and the non-negative real line in the side-payment case and the empty set is mapped into the origin. Assume that the actions of each player can be decomposed into m identical mutually exclusive and collectively exhaustive factor-sets, such that the set of feasible actions in one factor-set is independent of the choice made in another factor-set. We will call these decompositions 'scenes'. Note that all games have at least one scene.

Given a decomposition into m scenes, we can think of a player as participating in (perhaps) different coalitions on each scene. Denote by $\{0,1\}_i^n$ the set of

[3] It turns out that quantity competition prevents cooperative advertising.

n-vectors with 1 in the i-th argument and 0 or 1 everywhere else. $S_i \equiv (S_{i1}, \ldots, S_{im}) \in (\{0, 1\}_i^n)^m$ can then represent an ordered set of coalitions in which player i participates. We will now assume the existence of a rule (e.g. no side-payment) which assigns payoffs to each player as a function of the coalitions formed and the strategies played by all players on all scenes. Let A_i be the i-th argument of A in the no side-payment case and the non-negative real line in the side-payment case. We can now use the rule above to define the functions $V_i : (\{0, 1\}_i^n)^m \to A_i$, $i \in N$, by the maximum payoff i can guarantee himself given that he participates in a given ordered set of coalitions.[4] V_i is the characteristic function for this 'n-person, m-scene semifuzzy game'.

In the no side-payment case, we can use V_i to define the imputations of the semifuzzy game as the set of payoffs which weakly dominates the payoff which $V \equiv (V_1, \ldots, V_n)$ assigns to individual play on all scenes and the payoff which V assigns to forming the grand coalition on all scenes. As usual the core is the set of imputations which no coalition can block. Furthermore we assume that V is super-additive on each scene. That is, for a given set of coalitions on all scenes other than j, for all disjoint $(S_j, T_j) \in (\{0, 1\}^n)^2$:

$$\sum_{i \in S_j \cup T_j} V_i(S_j \cup T_j, \cdot) \geq \sum_{i \in S_j} V_i(S_j, \cdot) + \sum_{i \in T_j} V_i(T_j, \cdot).$$

The definitions for the side-payment case proceed analogously.

3. Properties

By defining partial coalitions on the set of scenes, semifuzzy games are more explicit about the mechanisms through which partial coalitions are realized than is the case for fuzzy games. On the other hand, we lose the link to the payoff offered by the fuzzy game concept. Similarly, by defining V on $(\{0, 1\}_i^n)^m$, we rule out the existence of inconsistent simultaneous partial coalitions. The domain of V is of course no longer a convex compact set, so we lose all the nice analytical properties of fuzzy games.

With respect to the core of semifuzzy games, it may be tempting to follow the intuition of Aubin [3], according to which we 'shrink' the core by allowing more coalitions to form. (See, however, Artzner et al. [2].) While clearly $\text{Core}(V) \leq \text{Core}(W)$, we can in fact show:

Theorem 1S. *In a side-payment game:*

$$\text{Core}(V) = \text{Core}(W).$$

Proof. Note first that V_i is constructed such that $\sum_{i \in S} V_i(S, \ldots, S) = W(S)$ for $S \in \{0, 1\}^n$. We can use the super-additivity of V_i to get that for any set of m

[4] Because the other members in S_i may vary from scene to scene, it is not possible to define $V(S_i)$ as the payoff to the whole set of coalition members. Accordingly, our framework is most natural in the no side-payment case.

coalitions, $(S_1, \ldots, S_m) \in \{0, 1\}^{nm}$, if $T = \bigcup_{j=1}^{m} S_j$, then $\sum_{i\in T} V_i(S_1, \ldots, S_m) \leq W(T)$. So these partial coalitions cannot lead to stronger core constraints than the corresponding coalitions in W. Thus $\mathrm{Core}(W) \leq \mathrm{Core}(V)$. □

Theorem 1N. *In a no side-payment game:*

$$\mathrm{Core}(V) = \mathrm{Core}(W).$$

The proof proceeds as above except for the fact that we avoid the summations.

Theorems 1S and 1N have very wide implications. In particular, they tell us that any results about the core of ordinary games (existence, convexity, compactness, etc.) transfer immediately to any associated semifuzzy games.

Let us now prove some limit theorems on semifuzzy games. Assume that a game can be decomposed into countably many scenes, each of which has the property that the marginal value of the grand coalition (call G) on that scene, over no coalitions (call 1) on that scene, goes to zero as $m \to \infty$. In order to formalize this we define $V_i^p : (\{0, 1\}_i^n)^{p-1} \times \{0, 1\}_i^n \to B$, such that $V_i^p(T_0, S)$ is the value to player i of being in coalitions T_0 the first $p-1$ scenes, in coalitions S in the p-th scene and alone on the remaining $m-p$ scenes. We thus assume:

$$\forall i \in N, \forall p \leq m, \forall T_0 \in (\{0, 1\}_i^n)^{p-1}: \qquad V_i^p(T_0, G) - V_i^p(T_0, 1) \to 0 \quad \text{for } m \to \infty. \tag{A1}$$

This type of assumption would apply to cases where countably many subgames are played simultaneously, for example over a geographical space.

If we define $\bar{V}_i^p : \{0, 1\}^n \to B$, by $\bar{V}_i^p(S) = V_i^p(S^{p-1}, S)$, we can now prove:

Theorem 2S. *In a side-payment game which satisfies* (A1):

$$\forall S \in \{0, 1\}^n, \forall \lambda \in [0, 1], \exists p \leq m: \quad \sum_{i \in S} \bar{V}_i^p(S) \to \lambda W(S) \quad \text{for } m \to \infty.$$

So as in fuzzy games, any fractional payoff can be realized for a given coalition.

Proof. Assume $S = G$, for any λ, we can find a p such that

$$\sum_{i \in G} \bar{V}_i^{p-1}(G) \leq \lambda W(G) \leq \sum_{i \in G} \bar{V}_i^p(G).$$

By (A1), the theorem is true for $S = G$. For $S \subset G$, we can again find a p such that

$$\sum_{i \in S} \bar{V}_i^{p-1}(S) \leq \lambda W(S) \leq \sum_{i \in S} \bar{V}_i^p(S) \leq \sum_{i \in S} V_i^p(S, G),$$

using (A1) twice, the thoerem thus also holds for $S \subset G$. □

Let W_i denote the i-th argument of the characteristic function of a game without side payments.

Theorem 2N. *In a no side-payment game which satisfies* (A1):

$$\forall i \in N, \forall S \in \{0, 1\}^n, \forall \lambda \in [0, 1], \exists p \leq m: \quad \bar{V}_i^p(S) \to \lambda W_i(S) \quad \textit{for } m \to \infty.$$

The proof proceeds as above except for the fact that the different players $i \in S$ now may have different p's. Of course if the scenes are identical this will not happen.

Also Theorems 2S and 2N point to a wide range of results. In particular, we can employ completely analog methods to show how semifuzzy games may be used to proxy differentiable extensions of value functions of ordinary games [12, 6, 3]. Furthermore, we now have a logically consistent way of thinking about fuzzy games, since the difficulties mentioned in Section 1 disappear if we think of a fuzzy game as the limit of a semifuzzy game.

4. Discussion

We have defined a new class of games suitable for understanding the very common occurrence of partial cooperation in static games. Allowing the players to play cooperatively or competitively on individual decision parameters does not change the core of the game. Also, under suitable conditions any fraction of the payoff to a full coalition and thus a fuzzy game can be approximated. The main contribution of the present paper is to show that we can allow partial cooperation in our models without making any additional restrictions on the core of a given game. Our second set of theorems might be useful in establishing some limit results in particular applications of the semifuzzy game concept. More importantly they do, however, allow us to think about fuzzy games in a way in which the interpretational difficulties mentioned in Section 1 do not occur.

Our definition excludes games where the set of feasible actions in one subset of the decision variables is dependent on the actions in its complement. Since such situations are very common, e.g., where a resource constraint is involved, it would be useful to develop a characterization of such games. Within the class of games we consider, it should be possible to develop more insights than those in our theorems, particularly for certain special cases. For example, it might be interesting to look at games composed of identical subgames.

Another avenue for further research lies in the application of the concept of semifuzzy games rather than the theorems in this paper. In this context one should note that contrary to other concepts in cooperative game theory, semifuzzy games do not have nice market representations. The reason is, that the mathematically convenient assumption that each coalition has a strictly positive endowment of each good is in direct conflict with the idea underlying semifuzzy games. As suggested in the example of Section 1, games of the prisoners' dilemma type, particularly in economics, political science and biology seems a much more promising application. In economics, the allegation that multiple point competition can facilitate partial cooperation has a long history [5, 17] and little supporting formal theory.[5] In political science, the use of selected areas to signal threats or the use of selected weapons as deterrence have long been part of the received theory [15, 10]. Finally, there would seem to be many possible applications in the rapidly developing area of mathematical biology.

[5] This has recently been confirmed in empirical studies by Heggestad and Rhoades [9] and Scott [16].

In short, the semifuzzy game concept seems to describe several common phenomena and in particular may have the potential to increase our understanding of instances of partial cooperation.

Appendix. Numerical illustration

Consider an ordinary game between players a and b, with value function $W(1, 0) = W(0, 1) = 2$, $W(1, 1) = 6$. If this game decomposes into two identical scenes, the value function for the semifuzzy game could look like

$$V_a(1, 0; 1, 0) = V_b(0, 1; 0, 1) = 2,$$
$$V_a(1, 1; 1, 0) = V_b(1, 1; 0, 1) = V_a(1, 0; 1, 1) = V_b(0, 1; 1, 1) = 2\tfrac{1}{2},$$
$$V_a(1, 1; 1, 1) = V_b(1, 1; 1, 1) = 3.$$

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