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THE DYNAMICS OF PRICES AND MARKET SHARES OVER THE PRODUCT LIFE CYCLE*

BIRGER WERNERFELT
J. L. Kellogg Graduate School of Management, Northwestern University, Evanston, Illinois 60201

We analyze a duopoly through a differential game, in which the players set prices as functions of time. Under reasonable assumptions, we find that prices first decline, then increase. The market share of the biggest firm grows initially but decreases later. It is demonstrated that a firm may growth maximize early, but never late, in the product life cycle. Finally we show that only the low price firm will pay for informative advertising, whereas both firms will pay for persuasive advertising, though less if their market shares are very different. (MARKETING; PRICING; COMPETITION)

1. Introduction

During the last 20 years, some simple prescriptions originally due to the Boston Consulting Group (BCG) (although pioneered by Hirschmann 1964), have become increasingly important in the practice of management. Recent research (Haspeslagh 1981) indicates that most Fortune 500 firms use the framework suggested by BCG or some variation of it (see also *Fortune* September 24, 1979 and October 5, 1981). It is therefore pertinent to understand the logic behind the BCG argument. Although a more detailed exposition is given below, the basic idea is that firms should growth maximize early in the product life cycle, in order to create permanent cost or demand advantages, over competitors.

In the marketing literature on penetration pricing, most attempts to analyze the BCG framework have taken place in a monopolistic setting (e.g., Dolan and Jeuland 1981, Jeuland and Dolan 1980). While the results are interesting in the sense that they yield explicit price paths, the impact of competition is unclear. Might it for example not pay off for a single firm to go "against the tide" and "sell" market share early when everyone else wants to buy only to start buying later when others harvest? The question is important because pure monopolies are rare and most applications of the BCG ideas take place in oligopolistic settings. Due to the mathematical difficulties in the solution of dynamic games, only numerical results have been derived for the oligopolistic case (Clarke and Dolan 1982).

In two recent papers in the economics literature, Spence has examined some key elements of the BCG-prescriptions in the context of dynamic game theory. In the first paper (Spence 1979), early growth maximization was rationalized through irreversibility of investments combined with a growth constraint. This was done in the context of an open loop equilibrium of an implicit model of competition without returns to scale or learning curve effect in the cost functions. In the conclusion, the need for further analysis of those effects is noted. In the second paper (Spence 1981), the same results are found as a consequence of learning curve effects in an explicit model of competition between identical firms. Also these results, which mainly are numerical, are obtained in an open loop model. Neither of his models allows us to examine the implications of differences in the market shares of the firms.

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The present model gives richer results than the above, in the sense that it uses an open loop framework to develop analytical results about prices and their dependence on market shares in an explicit model of competition with economies of scale, learning on both demand and supply side and growth constraints. Like the previous literature, we get precise results about price paths, but now in an oligopolistic setting. In particular, we find that prices decrease early and increase later in the product life cycle. Like Spence, we look at a dynamic game, but now allow the players to be different, such that we can explore how the BCG prescriptions differ depending on a firm's market share. We here find that bigger firms gain early and lose later in the product life cycle. Compared to the Spence models the increased mathematical complexities force even more simplifying assumptions on the functional forms used and restricts the validity of the model to markets with relatively stable structural traits. (Similar limitations apply to the Dolan–Jeuland models.) Since asymmetric equilibria and analysis of continuous time games still are novel, we feel that the sometimes very restrictive assumptions are defensible as a first step.

After our analysis of dynamic pricing policies, we extend the model to analyze two types of advertising. First, we look at informative advertising (also known as awareness advertising), undertaken to draw attention to a brand and secondly we look at persuasive advertising (also known as positioning advertising), undertaken to favorably influence consumer perception of the brand.

2. The BCG Prescriptions

The reasoning behind the BCG prescriptions is simple and intuitively appealing. Assume, for purposes of argument immediately below, the following four premises to be true:

1. All costs follow "experience curve" reductions, where experience is a measure of accumulated, firm-specific sales.
2. In order to grow, a firm must, even in a friendly capital market, be able to retain some earnings.
3. Market shares exhibit declining price sensitivity over time.
4. Growth rates for individual products decline over time.[1]

BCG has developed an extensive body of data purporting to back up premise 1, while premises 2, 3, and 4 are easier for most practitioners to evaluate.

Let us look at BCG's practical conclusions from the above four points:

1. If you have a big market share and have moved far down the experience curve, you will have lower unit costs than your competition. (Some empirical evidence is provided by Schoeffler, Buzzell, and Heany 1974.)
2. This condition can be reached by an all out effort in the early stages of the life cycle, especially if you are stronger (financially) than your competitors.
3. In the late stages, market shares stabilize and total market growth slows down; investments are thus nonrewarding, and you should channel funds to younger industries.

The purpose of this paper is to provide a rational reconstruction of the recommendation that firms should maximize growth early in the product life cycle and take profits home later—even if all other firms do the same. Furthermore, we will investigate the way the optimal price paths depend on the market share of the firm.

[1] Points 1, 2 and 4 can be found in a slightly less precise form on p. 164 in Henderson (1979), whereas point 3 appears on p. 163. We are not, at this stage, concerned with the necessity or sufficiency of these points, but merely with BCG's use of them.

3. Analysis

3.1. *The Model*

For purposes of expositional ease we will look at a duopoly where both firms, following the second BCG premise, operate under a financial constraint which limit their ability to grow unless their current profits are high enough. We will use the simple form:

$$\dot{K}_{it} \leqslant b_0 \Pi_{0it} - b_1 K_{it}, \qquad i = 1,2, \tag{1}$$

where b_0, b_1 are two positive scalars, and K_{it} and Π_{0it} are the capital requirements (for inventories, accounts receivable and machinery) and operating profits, respectively, of the ith firm at time t. We use the notation $\dot{x} \equiv dx/dt$.[2] This constraint implies that the capital market is somewhat imperfect and demands that some fraction of the cash required for growth is internally generated. As we will see in Theorem 1, this allows a large firm to support more growth as long as economies of scale give it a higher relative profit.

Suppose further that there are TM_t customers in the market ($\dot{TM}_t \geqslant 0$, so the market does not decline) and that the ith firm has a share MS_{it} of these, each of which is consuming at a positive level y_{it}, so total demand is broken up as $\sum_{i=1}^{2} TM_t MS_{it} y_{it}$.

We will assume the capital required to produce and sell $TM_t MS_{it} y_{it}$ units to be given by a conventional power function (see e.g. Intriligator 1978, Chapter 8).

$$K_{it} = a_1 (TM_t MS_{it} y_{it})^{\phi}, \qquad i = 1,2, \tag{2}$$

where a_1, ϕ are positive scalars and $0 < \phi \leqslant 1$, such that returns to scale may be present.

As defined, the operating profits of each firm are

$$\Pi_{0it} \equiv TM_t MS_{it} y_{it} (p_{it} - c_{it}), \qquad i = 1,2, \tag{3}$$

where p_{it}, c_{it} are price and average unit costs, respectively, of the ith firm. Substituting (2) into (1) and rearranging terms we get

$$\frac{\dot{MS}_{it}}{MS_{it}} + \frac{\dot{y}_{it}}{y_{it}} + \frac{\dot{TM}_t}{TM_t} + b \leqslant a (TM_t MS_{it} y_t)^{1-\phi} (p_{it} - c_{it}), \qquad i = 1,2, \tag{4}$$

where $a = b_0 / a_1 \phi$ and $b = b_1 / \phi$.

The cost function is assumed to exhibit returns to scale and, following the first BCG premise, experience curve effects. Again here we, as does BCG 1972, use a power function to model returns to scale and experience curves.

$$c_{it} = c_{0t} (TM_t MS_t y_{it})^{\alpha} \epsilon_{it}^{\beta}, \qquad i = 1,2, \tag{5}$$

where c_{0t} is a declining industry-shared factor, α, β are nonpositive scalars; and $\epsilon_{it} - 1$ is the experience of the ith firm at t, such that

$$\dot{\epsilon}_{it} = TM_t MS_{it} y_{it}, \qquad i = 1,2, \tag{6}$$

where $\epsilon_{i0} = 1$. We should note that our choice of the loglinear forms in (2) and (5) was made for computational convenience only. Our results will remain essentially unchanged if (2) and (5) are exchanged with other forms with the same signs of the first and second derivatives. (Proofs available upon request.)

[2] This constraint is identical to a constraint demanding dividends to exceed or equal $(i_e - a_2 \dot{K}/K)E$, where i_e is the market return on risk-free equity, E is equity, the debt-equity ratio is constant and $a_2 < 1$ following Lintner (1962). If i_d is the interest rate on debt, D, $b_0 = (1 + D/E)(1 - a_2)^{-1}$ and $b_1 = (D/E i_d + i_e)(1 - a_2)^{-1}$.

The market share dynamics require more explanation. The number of customers switching from brand 1 to brand 2 will depend on three factors: The number of consumers currently buying brand 1, the intensity with which these consider brand switching and the intensity with which these who consider actually do switch. (Of course the flow from brand 2 to brand 1 can be looked at in analogous terms.) The intensity with which consumers consider brand switching should grow, here linearly, with the number of consumers already buying brand 2, and, following the third BCG premise, decline over time. This can be seen as a result of learning by doing on the part of consumers who develop brand loyalties as they become better at using the two brands. (Stigler and Becker 1977 develop a theoretical argument to this effect although in a more static model. In our model brand switching should ideally depend on expectations about price changes as well as on current prices.) The intensity of actual brand switching will be treated as a function of price differences only, such that greater price differences correspond to more consumers switching. Ideally this should be a nonnegative function. Following Phelps and Winter (1971) we use a linear function through the origin as a first order approximation to a more general function which satisfies the constraint $MS_{jt} + MS_{it} = 1$. According to the above we write the market share dynamics as

$$\dot{MS}_{it} = MS_{jt}\left(\tfrac{1}{2} f(t) MS_{it}\right)(p_{jt} - p_{it}) - MS_{it}\left(\tfrac{1}{2} f(t) MS_{jt}\right)(p_{it} - p_{jt}), \qquad i \neq j, \quad i = 1,2$$

or

$$\dot{MS}_{it} = f(t) MS_{it} MS_{jt}(p_{jt} - p_{it}), \qquad 0 \leqslant MS_{i0} = 1 - MS_{j0} \leqslant 1; \qquad i = 1,2; \quad i \neq j, \tag{7}$$

where the declining speed coefficient $f(t)$ converges $(\dot{f}(t) \geqslant 0)$ to a nonnegative level.

One should note that a form as (7) can be derived from a Markov model of brand switching (see, e.g., Styan and Smith, Jr. 1964) where the choice propensities are proportional to the market share (e.g., the visibility) of the new brand and the price difference in its favor. Furthermore the decline of $f(t)$ can be derived from nonstationarities due to, for example, linear learning effects (Carman 1966). Such behavioural models are sufficient for (7) but not necessary.

While the particular functional form of (7) and the preceding interpretation of the brand switching process are highly stylized the general properties of the function should be noncontroversial. In particular it seems natural that lower prices result in higher market shares and that this effect is lower in more mature businesses (Buzzell and Wiersema 1980, Lynk 1981). In order to maintain computational ease for our initial analysis, we have temporarily excluded the other elements of the marketing mix (advertising, promotion, quality, etc.). Given the intuition behind our results (given below), it is reasonable to posit that one can develop models where a low price is analogous to "high marketing effort." For example, two types of advertising are analyzed in §4. Advertising and promotion are treated explicitly in a related paper which also includes empirical tests of the theory (Fornell, Robinson, and Wernerfelt 1984).

We assume a certain adjustment period in the consumption level of consumers; thus,

$$\dot{y}_{it} = g(p_{it}, y_{it}), \qquad 0 = g(p_{i0}, y_{i0}), \qquad i = 1,2, \quad \text{where} \tag{8}$$

$$\frac{\partial g}{\partial p}, \frac{\partial g}{\partial y}, \frac{\partial^2 g}{\partial p\, \partial y}, \frac{\partial^2 g}{\partial p^2} < 0 \quad \text{and} \quad \frac{\partial^2 g}{\partial y^2} > 0;$$

also this can be explained through the ingraining habit argument of Stigler and Becker (1977), if one applies the habits to consumption levels. As we saw after the increase in oil prices in the early '70s, it takes time to adjust lifestyle to new price levels. Note that a customer of firm i sets his consumption level as a function of p_i only. So the irrelevant price p_j $(j \neq i)$ does not matter. Relative prices only matter in the flow of market shares.

Assume that each firm tries to find the "best" piecewise continuous strategy $p_{it} \equiv \hat{p}_i(t)$, $i = 1,2$; $j \neq i$. The aim is to maximize discounted operating profit:

$$\underset{\hat{p}_i}{\mathrm{MAX}} \int_0^T e^{-\rho t} TM_t MS_{it} y_{it}(p_{it} - c_{it})\,dt + B_1 MS_{iT} + B_2 y_{iT} + B_3 \epsilon_{iT}, \qquad i = 1,2, \quad (9)$$

subject to (6), (7) and (8), where ρ, B_1, B_2, B_3 are nonnegative scalars. The salvage value can be seen as a first order approximation to a more general function.

By looking at feedback or closed loop strategies, we would allow the players to revise their plans in response to developments in the state variables. In order to keep the mathematics at a more manageable level, as a first step we here confine ourselves to open loop solutions. This was done for computational ease and also served to avoid making even more restrictive assumptions than those already needed. Since the essential competitive interdependence takes place through the price dynamics, we do capture the most important "gaming" aspects. As shown by Fudenberg and Tirole (1983) there may, however, be substantial differences between open and closed loop solutions. In particular, behaviour in closed loop equilibria may be even more aggressive than what we see here. This remains an important area for future research.

From now on, with no loss of generality, subscript 1 will refer to the initially larger firm, whereas the smaller firm is indicated by subscript 2.

3.2. *Growth Maximization*

Let us first look at the situation where (3) is binding for both firms. In this case, in the presence of some financial economies of scale ($\phi < 1$), operational economies of scale ($\alpha < 0$) or experience curves ($\beta < 0$), we can find

THEOREM 1. *The big firm can keep gaining market share in the growth maximizing game.*

PROOF. See Appendix.

Given the constraint (1), it is of course very intuitive that the larger firm is able to finance a higher growth rate if some economies of scale are present.

It also seems intuitive that declining growth rates would result in declining prices for growth maximizing firms. In order to get this result we do, however, need to keep prices relatively effective, so the decline in $f(t)$ cannot be too steep compared to the decline in $\dot{TM}(t)/TM(t)$. Furthermore, we need the two firms to be of at least comparable competitive strength, such that no firm loses market share too fast. Clearly, in the presence of economies of scale, a firm which experiences declining unit sales would have to change increasing prices in order to keep the unit sales decrease at a minimum and still maintain financial integrity. More specifically we will assume

$$\dot{MS}_{it}/MS_{it} + \dot{TM}_t/TM_t \geqslant 0. \quad \text{(A1)}$$

(A1) literally demands that no firm can lose market share at a rate faster than the market growth. Given the model, this means that prices and thus market shares cannot be too different, such that a firm loses share too fast.

$$\frac{d(\dot{TM}_t/TM_t)}{dt} \leqslant \frac{\dot{TM}_t}{TM_t}\,\frac{\dot{f}(t)}{f(t)}. \quad \text{(A2)}$$

(A2) corresponds roughly to the fourth BCG premise, saying that the rate of decline in $f(t)$ is less than the growth rate in TM_t, while simultaneously $\dot{TM}_t$ is negative, conditions which should hold true in all but very young industries.[3] (Note that the assumption about declining growth rate is made only on the number of customers in the market, not on sales, which depend on the price charged.) According to the above we can get $dp_i/dt < 0$. So we have:

THEOREM 2. *Under* (A1)–(A2), *prices decline in the growth maximizing game.*

PROOF. See Appendix.

3.3. *Profit Maximization*

It is well known (see, e.g., Buzzell and Wiersema 1981 or Scherer 1970, pp. 217–218) that larger firms often lose market share (slowly) in mature businesses where financial constraints are immaterial to competitive activities. We derive this result in our model

THEOREM 3. *Late in the life cycle of the unconstrained game, the market share of the big firm goes down.*

PROOF. See Appendix.

The mechanism at work here is that the marginal costs of a price cut are larger for the larger firm, whereas the marginal utility is the same for both firms late in the life cycle, when the net present value of investments in market share, volume, and experience are the same for both firms. (See also Fudenberg and Tirole 1984.)

The fact that a firm which can monopolize an industry quite often chooses not to do so is well known to marketing researchers (see e.g., Buzzell 1980)—and also to BCG (Henderson 1979, pp. 90–94). Theoretical explanations of the phenomenon, however, have been rather poor, concentrating either on chance, segmentation, or fear of regulation. In the above model, the explanation emerges directly from the economics of the problem. (Wernerfelt 1984 is an attempt to extend the result beyond the duopolistic setting into a new theory of the size distribution of firms.)

The time dependence of prices in the unconstrained game depends on two counteracting forces. On the one hand, the discounting will induce impatience and experience and scale effects will drive marginal costs down and, on the other hand, the incentive to "invest in market share" or, really, experience and brand loyalty, goes down as the product matures. In order to get the expected result, that prices eventually go up, we need to assume that, relative to the marginal profit margin, neither the terminal value of having a low p (a high y) nor the interest rate are too high. In particular we need to assume:

$$e^{-\rho t}TM_t MS_{it}(p_{it} - c_{it}[1+\alpha]) > -B_2\frac{\partial g}{\partial y} \quad \text{and} \tag{A3}$$

$$\frac{\dot{TM}_t}{TM_t} + f(t)(1 - MS_{it})(p_{jt} - c_{it}[1+\alpha]) + \dot{y} > \rho. \tag{A4}$$

Clearly a high terminal value on charging low prices late in the product life cycle would create an (unrealistic) incentive to do so. Similarly, if the interest rate is too high, the incentive to invest in market share earlier in the product life cycle goes down, and we do not get any price effect of the time change in this incentive. Given these assumptions we have:

[3]Both here and later, I prefer early interpretable to weaker assumptions. I furthermore make my arguments on both prices, where statements about average price or leader price would require fewer assumptions.

THEOREM 4. *Under* (A3)–(A4) *we get inflation late in the profit maximizing game.*

PROOF. See Appendix.

Note however that the presence of $\dot{y}$ in (A4) means that prices cannot increase too fast. (While (A3) and (A4) can be made to look offensive by substitution of Π_{0it}, they nevertheless involve the endogenous p. More basic assumptions could yield the same results, but would be very hard to interpret.)

The underlying mechanism is that firms take home profits from earlier investments in sales size or, really, experience effects and brand loyalty (see also Dolan and Jeuland 1981).

3.4. *Constrained Profit Maximization*

Let us now use the results from the two simple games to analyze the constrained profit maximizing game. We easily find the result predicted by BCG:

THEOREM 5. *Under* (A1)–(A4), *if the firm ever is financially constrained, it should maximize growth early in the product life cycle and take profits home later.*

PROOF. See Appendix.

From this viewpoint, growth maximization can be seen as a means to profit maximization. Of course for many big diversified firms, the single business self-financing constraint (1) will not apply, such that they can grow at a higher rate than smaller single business firms, thereby gaining competitive advantage, by operating an internal capital market, exactly as prescribed by BCG (see Henderson 1979, p. 166). In a perfect external capital market no firms will be subject to (1), and other asymmetries, such as order of entry, will also affect relative performance.

Further analysis of the results would confirm that the BCG premises constitute an overkill, in the sense that the growth constraint and either of the trends 1 and 3 (experience curves and declining price sensitivity) alone would be sufficient.

4. Advertising

In this section, we sequentially analyze two types of advertising and see how the firms should change advertising policies over the product life cycle. Denote advertising by A_i, $i = 1,2$ and let the two constants $\gamma \in R_+$ and $\lambda \in (0,1)$ measure the cost and the marginal effectiveness of advertising respectively. The operating profit of a firm will here be given by

$$\Pi_{0it} = TM_t MS_{it} y_{it}(p_{it} - c_{it}) - \gamma A_{it}.$$

We stay in an open loop model and all symbols retain their meaning.

We first study advertising that does not influence tastes but instead makes the product or its price more visible to the consumer such that price and advertising are complements. This is the position of Stigler and Becker (1977). A related viewpoint (Hauser and Shugan 1983) is that advertising can be studied as two components, that which makes the consumer aware of the product and that which persuades them of the product's relative benefits. Looking at the first component and again using a simple power function, the extended version of (7) is:

$$\dot{MS}_{it} = \tfrac{1}{2} f(t) MS_{jt}\left(MS_{it} + A_{it}^{\lambda}\right)(p_{jt} - p_{it}) - \tfrac{1}{2} f(t) MS_{it}\left(MS_{jt} + A_{jt}^{\lambda}\right)(p_{it} - p_{jt}),$$
$$i \neq j, \quad i = 1,2. \qquad (10)$$

For this model we can show

THEOREM 6. *Only the low price firm engages in awareness advertising to inform, and late in the life cycle it advertises more the smaller it is.*

PROOF. See Appendix.

The result implies that when advertising complements a low price, it is the big firm which advertises price early in the product life cycle (when its price is lower), and the small firm which does it later in the product life cycle.

If we analyze that component of advertising which is a substitute for price in the sense that it molds consumer tastes, we can reformulate (see Hauser and Shugan 1983, and J. K. Galbraith 1958, pp. 155–156) (7) as

$$\dot{MS}_{it} = f(t)MS_{it}(1 - MS_{it})\left(A_{it}^{\lambda} - A_{jt}^{\lambda} + p_{jt} - p_{it}\right), \qquad i \neq j, \quad i = 1,2. \tag{11}$$

In this case we get

THEOREM 7. *Late in the life cycle both firms spend the same amounts on persuasive advertising and more the more similar they are.*

PROOF. See Appendix.[4]

The result for the case where advertising is a substitute for price is due to two factors. First, unlike price competition, advertising competition does not offer the larger firm greater incentives to keep advertising high or low. Secondly, the consumers switch brands fastest when the firms are of similar size, such that the rewards to advertising (and price) incentives are greatest at that point.

If advertising in reality has both of the above components, our results coincide with casual empiricism. Industry leaders tend to advertise a lot early in the life cycle and advertising tends to go up when competitors are of more comparable size. Again a more detailed analysis of advertising can be found in Fornell, Robinson and Wernerfelt (1984).

5. Summary and Future Directions

The above model is a mathematical representation of the BCG hypothesis. Analysis based on the model reconstructs the BCG argument for early growth maximization. In the presence of experience curves, declining price sensitivity, and declining growth rates, growth maximization early in the product life cycle can be a means to profit maximization. It is further demonstrated that the larger firm will gain market share early in the life cycle and lose some of it later. Finally, prices will decline early and may increase late. This means that both large and relatively small firms, if they expect their competitors to behave optimally, should invest as much as possible in market share early in the product life cycle and harvest it later. See Figure 1. Furthermore if you are a relatively big firm, you should be able to increase your market share early in the product life cycle and harvest it later. By implication if you are a relatively small firm you should expect to lose some market share early in the product life cycle and take some of it back later. See Figure 2.

Based on the examples in Scherer (1970) and Buzzell and Wiersema (1980) this pattern seems to be followed in many industries. Given the huge amount of unorganized empirical and anecdotal evidence about different parts of the BCG theory, it seems pertinent that at least some theoretical work is done, such that we can interpret and organize our empirical results.

Even though we reproduce the BCG result, the complex nature and strong assumptions of the model leave several open questions, for future research. Within the context of the present model, one such question is under which conditions a firm which has left the growth constraint will ever go back. Another interesting issue is which firm is the first to leave the growth constraint for good.

[4]In this latter model, Theorems 3 and 4 remain valid exactly as stated, because (11) reduces to (7) for optimal advertising late in the life cycle. For the model (10), the proofs change a bit.

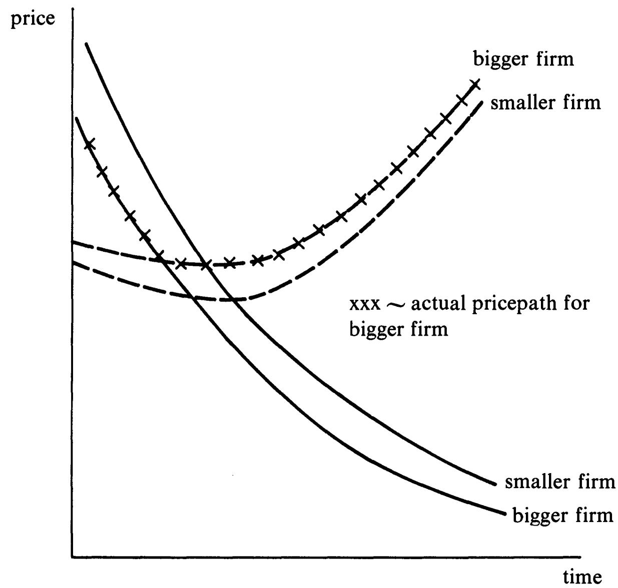

FIGURE 1. Growth (---) and Profit (· · ·) Maximizing Pricepaths for a Duopoly.

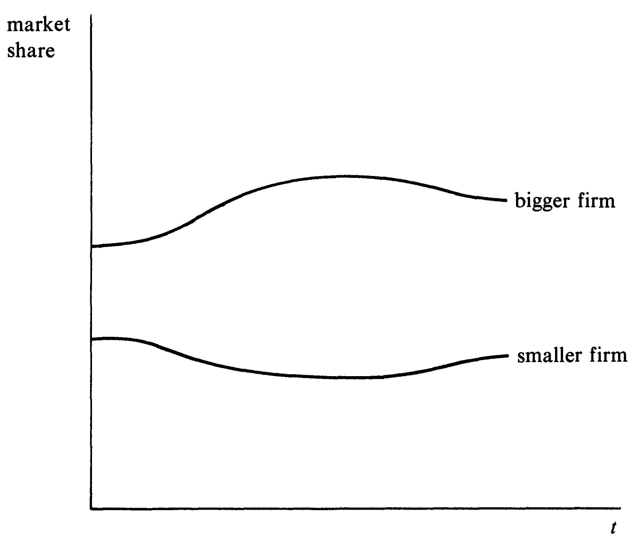

FIGURE 2. Market Share Paths for a Duopoly.

If we go beyond the formulation used here, two limitations seem to be that the analysis presented here is for one market only, and that the outlined path is the Nash equilibrium path. If the firm participates actually or potentially in more than one market, we may get a different behavior. It might be, for example, that the bigger firms would want to withdraw cash early in order to support divisions in younger or more promising markets. It might also be that firms competing simultaneously in more markets would want to disturb each other's cash transfer plans.

Even more important are the limitations of the open loop Nash equilibrium concept. In practice one quite often sees the establishment of some sort of price leadership such that the largest firm, which could become a monopolist if it wanted to, sets the price and gets the highest unit profit. Note, however, that under the stated assumptions, regardless of when and how we depart from the Nash path and enter price leadership, *the profits per unit will be greater for the biggest firms in all phases: growth maximization, unconstrained profit maximization,* and *price leadership.*

Two developments of the model—the extension to n firms, and the introduction of firm-specific scaling factors on the cost function—will presumably be relatively easy. The ease with which uncertainty can be dealt with will clearly depend on the way it is introduced. On the surface, one would expect it to enhance the advantage of the larger firm, but to leave the qualitative results unchanged. Finally, there are three issues which are too important to be neglected and yet seem difficult to treat formally: those of entry, exit and heterogeneous products.

Appendix

The two-person, general sum differential game defined by (4), (6), (7), (8), and (9) leads to the Hamiltonian (see e.g. Basar and Olsder 1982, Chapter 6.5):

$$H_i(t, MS_{it}, MS_{jt}, \mu_{it}, \nu_{it}, \eta_{it}, \hat{p}_1, \hat{p}_2)$$
$$= e^{-\rho t} TM_t MS_{it} y_{it}(p_{it} - c_{it}) + \sum_{j=1}^{2} \left(\mu_{it}^j f(t) MS_{jt} \left[\sum_{k=1}^{2} MS_{kt}(p_{kt} - p_{jt}) \right] + \nu_{it}^j g(p_{jt}, y_{jt}) + \eta_{it}^j TM_t MS_{jt} y_{jt} \right); \quad i = 1, 2,$$

where $\mu_{it}, \nu_{it}, \eta_{it}$ are the three 2 vectors with typical arguments $\mu_{it}^j, \nu_{it}^j, \eta_{it}^j, j = 1, 2$. These, of course, form the dual system. Note now how $H[\cdot]$ is concave in p_{it} and how we can find the optimal policy at a given point, $\hat{p}_{it}$, independent of competitor action.[5] The individual results are proved below.

Proof of Theorem 1. If we denote the implied growth-maximizing price strategies, against whatever $\hat{p}_2, \hat{p}_1$ the other firm plays, by $p_1^0(\hat{p}_2), p_2^0(\hat{p}_1)$, (3) gives for $i = 1, 2, j \neq i$:

$$f(t)(1 - MS_{it})(p_{jt} - p_{it}^0) + \frac{1}{y_{it}} g(p_{it}^0, y_{it}) + \frac{\dot{TM}_t}{TM_t} + b - a(TM_t MS_{it} y_{it})^{1-\phi}(p_{it}^0 - c_{it}) = 0. \tag{I}$$

From this we see immediately that unless there are no economies of scale present ($\alpha = \beta = 0$, $1 - \phi = 0$), $p_{1t}^0(p_2^0) < p_{2t}^0(p_1^0)$. Q.E.D.

Proof of Theorem 2. Let us use the implicit function theorem to get an idea about price dynamics along (4). Using the names $F_1(t, p_1, p_2)$ and $F_2(t, p_1, p_2)$ for the two equations in (I), we find:

$$\frac{dp_i}{dt} = \left[\frac{\partial F_j}{\partial p_j} - \frac{\partial F_j}{\partial p_i} \frac{\partial F_i}{\partial p_j} \left(\frac{\partial F_i}{\partial p_i} \right)^{-1} \right]^{-1} \left[\frac{\partial F_j}{\partial p_i} \frac{\partial F_i}{\partial t} \left(\frac{\partial F_i}{\partial p_i} \right)^{-1} - \frac{\partial F_j}{\partial t} \right], \qquad i = 1, 2.$$

It is trivial but tedious to see that the first factor is negative under our assumptions and that the second is positive under (A1) and (A2). Q.E.D.

Proof of Theorem 3. Denoting the price strategies which maximize profit, against whatever $\hat{p}_2, \hat{p}_1$ the other firm plays, in the unconstrained game by $p_1^*(\hat{p}_2), p_2^*(\hat{p}_1)$, we find from the Hamiltonian an implicit definition of p_1^* as:

$$e^{-\rho t} TM_t MS_{it} y_{it} - \mu_{it} f(t) MS_{it}(1 - MS_{it}) + \nu_{it}^i \frac{\partial g}{\partial p} = 0. \tag{II}$$

The dual dynamics are for μ_{it}, ν_{it}^i:[5]

$$\dot{\mu}_{it} = -e^{-\rho t} TM_t y_{it}\left[p_{it}^* + \eta_{it}^i e^{\rho t} - c_{it}(1 + \alpha)\right] - \eta_{it}^j TM_t y_{jt}$$
$$-f(t)\left[(p_{jt} - p_{it}^*)(1 - 2MS_{it})\right] \mu_{it}, \qquad \mu_{iT} = B_1. \tag{III}$$

$$\dot{\nu}_{it}^i = -e^{-\rho t} TM_t MS_{it}\left[p_{it}^* + \eta_{it}^i e^{\rho t} - c_{it}(1 + \alpha)\right] - \nu_{it}^i \frac{\partial g}{\partial y}, \qquad \nu_{iT}^i = B_2. \tag{IV}$$

[5] By Uchida's (1978) Theorem 2, a Nash equilibrium exists for the much more complex stochastic game which results from adding Wiener processes to (6), (7), and (8), and allowing strategies to depend on all state variables. The theorem does not allow state-dependent constraints on the policy variables, such as (4), whereas fixed constraints of, e.g., nonnegativity type are allowed. To apply the theorem one therefore has to reformulate the problem, such that "foregone growth" instead of price becomes a policy variable. In this formulation H_i is still concave in the own policy variable, while it is convex in the competitor policy variable. Unfortunately, neither this nor any other currently available theorem covers the deterministic game presented here.

In order to get a hold on these complex equations we can exploit the fact that the costate variables go arbitrarily close to their terminal values, as time comes close to T. We can therefore, for large t, reason on the three terms in

$$e^{-\rho t}TM_t MS_{it}y_{it} - B_1 f(t) MS_{it}(1 - MS_{it}) + B_2 \frac{\partial g}{\partial p} \simeq 0. \tag{V}$$

Considering everything as functions of either MS or p, we find that for large t, $p_{2t}^*(p_1^*) < p_{1t}^*(p_2^*)$. Q.E.D.

Proof of Theorem 4. We again use the implicit function theorem on each of the functions in (II) and take the total time derivative of all variables except the prices on the l.h.s. of (II):

$$\begin{aligned} e^{-\rho t}TM_t MS_{it}y_{it}\Bigg[-\rho + \frac{\dot{TM}_t}{TM_t} + f(t)(1 - MS_{it})\big(p_{jt} + \eta_{it}^i e^{\rho t} - c_{it}[1+\alpha]\big) \\ - \frac{1}{y_{it}}\frac{\partial g}{\partial p}\big(p_{it}^* + \eta_{it}^i e^{\rho t} - c_{it}[1+\alpha]\big)\Bigg] - \nu_{it}^i \frac{\partial g}{\partial y}\frac{\partial g}{\partial p} + \left(e^{-\rho t}TM_t MS_{it} + \nu_{it}^i \frac{\partial^2 g}{\partial p\,\partial y}\right)\dot{y} \\ + f(t) MS_{it}(1 - MS_{it})\left[\eta_{it}^j TM_t y_{jt} - \mu_{it}\frac{\dot f(t)}{f(t)}\right]. \end{aligned} \tag{VI}$$

Since the price derivative of the l.h.s. of (II) is negative, $\dot p_{it}^*(\hat p_j) > 0$, when it exists, if (VI) is positive.[6] Use that the costate variables approach their terminal values, and apply (A3) and (A4). Given these signs (VI) is indeed positive. Q.E.D.

Proof of Theorem 5. Note, first, that the price charged is the larger of p_{it}^0 and p_{it}^*, and that under the stated assumptions, $\dot p_{it}^0(\hat p_j) < 0$ and late in the life cycle $\dot p_{it}^*(\hat p_j) > 0$. We thus see that the only possible development late in the life cycle is from the pair (p_1^0, p_2^0) to (p_1^*, p_2^0) or (p_1^0, p_2^*) and then to (p_1^*, p_2^*). So, late in the life cycle we never leave the (p_1^*, p_2^*) pair.

Proof of Theorem 6. The first order condition on A_{it} is given by

$$A_{it}^{1-\lambda} = \frac{\gamma}{2\lambda} e^{\rho t} f(t)(p_{jt} - p_{it}) MS_{jt}\mu_{it}, \qquad A_{it} \geqslant 0, \qquad i = 1,2, \quad j \neq i.$$

It is trivial that $A_{it} = 0$ if $p_{jt} - p_{it} \leqslant 0$. Late in the life cycle, we can approximate μ_{it} by B_1 and find that higher market shares, ceteris paribus, lead to lower advertising. Q.E.D.

Proof of Theorem 7. The first order condition on A_{it} is given by

$$A_{it}^{1-\lambda} = \frac{\gamma}{\lambda} e^{\rho t} f(t) MS_{it}(1 - MS_{it})\mu_{it}, \qquad A_{it} \geqslant 0, \qquad i = 1,2, \quad j \neq i.$$

Late in the life cycle, when $\mu_{it} \approx B_1$, we find $A_{it} = A_{jt}$. Furthermore, advertising is biggest at $MS_{it} = 1/2$, smaller the more skewed the size distribution is. Q.E.D.

[6] I here use that, by (II), $\partial H_i/\partial p_i = 0$, and that $\partial \hat p_j/\partial MS_i = \partial \hat p_j/\partial y_i = \partial \hat p_j/\partial \epsilon_i = 0$, as has been assumed for $\hat p_j = p_j^*$.

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