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TECHNICAL NOTE

Uniqueness of Nash Equilibrium for Linear-Convex Stochastic Differential Games[1]

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Abstract. The uniqueness of Nash equilibria is shown for a class of stochastic differential games where the dynamic constraints are linear in the control variables. The result is applied to an oligopoly.

Key Words. Differential games, Nash equilibria, uniqueness, linear-convex games, stochastic games.

1. Introduction

Despite the many results available on nonzero-sum Nash differential games, we have very limited knowledge on the uniqueness of optimal strategies. In fact, quite often uncountably many equilibria exist (Basar, Ref. 1), although stochastic elements sometimes can shrink the set of equilibria considerably by rendering strategies which depend on the history of the game inoptimal (Basar and Olsder, Ref. 2, Corollary 6.4). In a deterministic setting, one can alternatively require the strategies to be functions of time and the current state only. This approach is taken by Papavassilopoulos and Cruz (Ref. 3), who derive a uniqueness theorem for a class of analytic games of this type.

In the present paper, we give a similar uniqueness result, which allows the state dynamics to be stochastic, but require them to be linear in the control variables. After deriving the result, we will apply it to a dynamic duopoly.

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2. Stochastic Differential Game

Notation. We employ the following notation:

$i=1,\ldots,n$, generic player;
$t\in[0,1]$, time;
$s_t\in\mathbb{R}^m$, value of state variable at time t;
U, a compact metric space;
A, space of measurable functions from $[0,1]\times\mathbb{R}^m$ to U;
$P_i\in A$, feedback strategy of player i;
$P\in A^n$, a set of strategies, one for each player;
$P_{it}=P_i(t,s_t)$, value of control variables of player i, for all (t,s_t);
$P_t'=(P_{it},\ldots,P_{nt})$;
B_1, space of C^2 functions from $[0,1]\times\mathbb{R}^m\times U^n$ to $\mathbb{R}^m$;
B_2, subset of B_1 in which the functions are linear in the control variables, have bounded derivatives, and are bounded at some point, say $(t,0,0)$;
$F\in B_2$;
$F(t,s_t,P_t)$, drift in s_t, for all (t,s_t,P_t);
z_t, Brownian motion in R^m;
C, space of C^2 functions from $[0,1]\times\mathbb{R}^m$ to $\mathbb{R}^m$;
$\sigma\in C$;
$\sigma(t,s_t)$, $m\times m$ matrix, which has a bounded inverse, bounded derivatives, and is bounded at $(t,0)$;
σ_{ij}, a typical element of $\sigma(t,s_t)$;
D_1, space of C^2 functions from $[0,1]\times\mathbb{R}^m\times U$ to $[0,k_1]$, $k_1\in\mathbb{R}_+$;
D_2, subset of D_1 in which the functions are convex in the control variables;
$\pi_i\in D_2$;
$\pi_i(t,s_t,P_{it})$ instantaneous, discounted payoff to player i for all (t,s_t,P_{it});
E, space of C^2 functions from $\mathbb{R}^m$ to $[0,k_2]$, $k_2\in\mathbb{R}_+$;
$\nu_i\in E$;
$\nu_i(s_1)$, discounted terminal value of s_1 to player i.

N-Player Game. We can now define the game (G) below:

$$\text{(G)}\quad \min_{P_i}\int_0^1 \pi_i(t,s_t,P_{it})\,dt+\nu_i(s_1),\qquad i=1,\ldots,n, \tag{1a}$$

$$ds_t=F(t,s_t,P_t)\,dt+\sigma(t,s_t)\,dz_t, \tag{1b}$$

$$s_0\in\mathbb{R}^m \text{ given.} \tag{1c}$$

Theorem 2.1. (G) has a Nash equilibrium.

Proof. This is a direct application of Corollary 1 in Uchida (Ref. 4). □

Noting that an equilibrium in general will depend on the initial condition, we can further get the following theorem.

Theorem 2.2. If (G) has a Nash equilibrium P^* in C^2 strategies, then that equilibrium is unique in that class of strategies.

Proof. This complex proof consists of five steps.

Step 1. Note first that the minimizing strategies are unique as functions of the arguments. That is, for any given $(t,s_t,\alpha_{it})\in[0,1]\times\mathbb{R}^m\times\mathbb{R}^n$, the equations

$$\min_{P_{it}}\alpha_{it}'F(t,s_t,P_t)+\pi_i(t,s_t,P_{it}),\qquad i=1,\ldots,n, \tag{2}$$

have at most one solution P_t in U^n.

To see this, remember that $F(\cdot)$ is linear in P_i such that (2) is additive in functions of the individual control variables, such that each player minimizes a convex function on $\mathbb{R}$, independent of the actions of other players.

Step 2. By assumption, these strategies are C^2, and we can define value functions $V_i:[0,1]\times\mathbb{R}^m\to\mathbb{R}$, $i=1,\ldots,n$, as

$$V_i(\tau,\phi)\equiv \mathop{E}_{\tau,\phi}\int_t^1 \pi_i(t,\tilde{s}_i(\tau,\phi,P^*),P_i(t,s(\tau,\phi,P_i^*)))\,dt$$
$$+\nu_i(s_1(\tau,\phi,P^*)),$$

where $\tilde{s}_t(\tau,\phi,P^*)$ is a realization of (1) starting from (τ,ϕ), with players using strategies P^*. Since all the functions defining $V_i(\cdot)$ are C^2 so is V_i.

Step 3. By the Bellman principle, the V_i's therefore solve

$$\partial V_i/\partial t+\tfrac{1}{2}\sum_{j,k=1}^{m}\sigma_{ij}^2(\partial^2 V_i/\partial s_j\partial s_k)$$
$$+\min_{P_i}\left[\sum_{j=1}^{m}(\partial V_i/\partial s_j)F_j(t,s_t,P_t)+\pi_i(t,s_t,P_{it})\right]=0, \tag{3a}$$

$$V_i(1,s)=\nu_i(s_1), \tag{3b}$$

with $i=1,\ldots,n$.

Step 4. (3) has a unique solution in the class C^2 [by Theorem IV.10.1 in Ladyzenskaja *et al.* (Ref. 5)].

Step 5. So there can be only one V and this induces a unique P^*. □

Remark 2.1. Given the Markov property of the dynamic constraint, it is obvious that no other equilibrium exist in the wider class of strategies which result if we allow $P(\cdot)$ to depend on the entire history of s.

3. Example: Differentiated Duopoly

Model. A continuum of consumers are distributed evenly on [0, 1]. Each has a demand curve of the form $\alpha - 2\beta\tilde{P}_i(s)$, $(\alpha, \beta) \in \mathbb{R}^2_+$, where $\tilde{P}_i(s)$ is the effective price (price per unit quality) of firm $i = (0, 1)$ to a consumer at $s \in [0, 1]$. At any given time, all consumers in $[0, s_t]$ buy only from firm 0, whereas all consumers in $(s_t, 1]$ buy only from firm 1. The marginal buyers flow to the, for them, most attractive firm, as

$$ds_t = a(\tilde{P}_1(t, s_t) - \tilde{P}_0(t, s_t))\, dt - \sigma[s_t(1 - s_t)]^{1/2}\, dz_t, \qquad (4)$$

where $(a, \sigma) \in \mathbb{R}^2_+$ and z_t is Brownian motion.

The two firms are positioned at 0, firm 0, and 1, firm 1, and their effective prices are the products of the distance to the consumer and their nominal prices $(P_{0t}, P_{1t}) \in \mathbb{R}^2$. So, (4) takes the form

$$ds_t = a[(1 - s_t)P_{1t} - s_tP_{0t}]\, dt + \sigma[s_t(1 - s_t)]^{1/2}\, dz_t.$$

The sales of firm 0 are therefore given by

$$\int_0^s t(\alpha - 2\beta P_{0t}r)\, dr = \alpha s_t - \beta P_{0t}s_t^2,$$

whereas the sales of firm 1 are

$$\alpha(1 - s_t) - \beta P_{1t}(1 - s_t)^2.$$

The firm's strategies are functions of (t, s_t).

We will solve the game over a unit-time horizon, such that the firm's objectives are

$$\max_{P_i} \int_0^1 ((\alpha s_{it} - \beta P_{it}s_{it}^2)P_{it})\, dt + 2s_{i1} - s_{i1}^2, \qquad i = 0, 1, \qquad (5)$$

where

$$s_{0t} = s_t, \qquad s_{1t} = 1 - s_t, \qquad s_0 \in (0, 1).$$

Results. The game (4), (5) leads to value functions of the form

$$V_i(s_{it}) = A(t)s_{it}^2 + B(t)s_{it} + C(t), \qquad i = 0, 1,$$

and the price functions

$$P_{it}^* = \alpha/2\beta s_{it} - aA(t)/\beta - aB(t)/2\beta s_{it}, \qquad i = 0, 1,$$

where $A(t)$, $B(t)$, $C(t)$ solve

$$dA/dt + 3a^2A^2/\beta = 0, \qquad A(1) = -1,$$

$$dB/dt + a^2AB/\beta - 2a^2A^2/\beta = 0, \qquad B(1) = 2,$$

$$dC/dt + \sigma^2A + (\alpha^2 - 4a^2AB + a^2B^2)/(4\beta) = 0, \qquad C(1) = 0,$$

respectively. From this,

$$A(t) = -[1 + 3a^2(1 - t)/\beta]^{-1},$$

and $P_{it}^* > 0$ for all t, s_t, if α is sufficiently large.

4. Conclusions

In the present paper, the uniqueness of the feedback Nash strategies is shown for a class of stochastic differential games, where the state dynamics are linear in the control variables. Much, of course, needs to be done before the potential of this type of model can be realistically assessed. First, this is just uniqueness of Nash equilibria, although in very complex strategy spaces. If we allow more conjectural variations, the number of equilibria multiply again. Secondly, while the existence result of Uchida (Ref. 4) is reasonably general, given the type of uncertainty he postulates, it needs extension to other types of uncertainties. Our own uniqueness result is much less general and should be relatively easy to extend. Finally, there remains the problem of solving or characterizing solutions to the system (2), such that qualitative insights can be obtained.

References

1. BASAR, R., *Informationally Nonunique Equilibrium Solutions in Differential Games*, SIAM Journal on Control and Optimization, Vol. 15, pp. 636-660, 1977.
2. BASAR, T., and OLSDER, G. J., *Dynamic Noncooperative Game Theory*, Academic Press, New York, New York, 1982.
3. PAPAVASSILOPOULOS, G. P., and CRUZ, J. B., JR., *On the Uniqueness of Nash Strategies for a Class of Analytic Differential Games*, Journal of Optimization Theory and Applications, Vol. 27, pp. 309-314, 1979.

4. UCHIDA, K., *On the Existence of a Nash Equilibrium Point in N-Person, Nonzero-Sum Stochastic Differential Games*, SIAM Journal on Control and Optimization, Vol. 16, pp. 142-149, 1978.
5. LADYZENSKAJA, O. A., SOLONNIKOV, V. A., and URAL'CEVA, N. N., *Linear and Quasilinear Equations of Parabolic Type*, American Mathematical Society, Providence, Rhode Island, 1968.