Stochastic Continuous Time Games and Optimization Problems with Controlled Point Process Arrivals*

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Abstract

There is a very large literature on stochastic control of jump diffusions and a smaller literature on such games. With the exception of two long-forgotten papers this literature assumes that it is the sizes of the jumps, rather than their arrival intensities, that is controlled. The second assumption, which is more natural in many economic contexts, is typically avoided because a failed Lipschitz condition means that the classical existence and uniqueness proofs cannot be used. We here derive an asymptotic solution to the game with controlled jump intensities and show that its equilibrium is very similar to that of the game with controlled jump sizes. The paper thus makes two contributions: It supplies a way to solve some control problems and games with controlled jump intensities and it shows that the commonly used formulation with controlled jump sizes is quite defensible for at least some classes of games.

Key Words: Stochastic control, Games with jump diffusions, Point processes with evolving intensities JEL Codes: C73, C61, C02

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1 INTRODUCTION

Optimization problems with jump diffusions have found wide-spread use in economics, operations research, and related fields.[1] As far as we know, all published applications assume that it is the jump sizes, rather than the jump intensities that are controlled. At least in economics, this is an unnatural assumption in many applications: For example, when agents engage in search or R&D investments, it seems most logical to assume that higher efforts lead to faster, rather than bigger, arrivals.

A (the?) major reason for not looking at controlled arrival rates is that the resulting state dynamics do not satisfy Lipschitz conditions, such that one cannot apply the classical existence and uniqueness results. The fact that Jacod and Protter (1982) and Protter (1983) identified a set of conditions under which this class of problems can be solved made no difference. These results are simply not used: Protter himself does not cite them in his textbook on Stochastic Integration (2005) and Oeksendahl (2022) believes that the two assumptions give very similar results and thus that the choice makes little difference. The results in this paper support that belief but we also show how to find an approximate solution to games with controlled jump intensities. We formulate and solve investment games with controlled jump sizes and controlled jump intensities in Sections II and III, respectively. Comparisons and concluding comments are made in Section IV.

2 GAME WITH CONTROLLED JUMP SIZES

Two players, X and Y, compete in continuous time. The outcome at time t depends on both their stocks, the state variables $(x_t, y_t) \in R^{+2}, x_0 = y_0 = 0$. At time $T < \infty$, the game ends, and all stocks become worthless. Until then, the stocks of X and Y grow according to independent Poisson processes with intensity λ and jumps of size (u_t, v_t), respectively. The players choose the size of the jumps in their stocks as C^2 Markov controls; X's strategy is $u(x_t, y_t, t)$ and Y's is $v\,(y_t, x_t, t)$. Their stocks therefore grow

[1] The first control theory application in economics was due to Merton (1971) and Oeksendahl and Sulem (2005) is a comprehensive textbook. There are fewer game theory applications, but Wernerfelt (1988) gives conditions for existence of several equilibrium concepts, in particular Markov.

over time according to

$$x_t = \int_0^t \boldsymbol{u}\left(x_{s-}, y_{s-}, s-\right) dN_x[\lambda]ds \tag{1}$$

$$y_t = \int_0^t \boldsymbol{v}\left(y_{s-}, x_{s-}, s-\right) dN_Y[\lambda]ds \tag{2}$$

where $N_x[\lambda]$ and $N_Y[\lambda]$ are independent Poisson processes with intensity λ and the integrals sum the jumps at arrivals between 0 and t.[2] Since having explicit solutions helps us make a detailed comparison between the two formulations, we assume that X and Y maximize the second order polynomials $\int_0^T \boldsymbol{e}^{-\rho t}\left[\gamma x_t^2 - \eta y_t^2 - \theta x_t y_t + \alpha x_t - \beta y_t + \sigma - 1/2u_t^2 -\pi u_t\right] dt$ and $\int_0^T e^{-\rho t}\left[\gamma y_t^2 - \eta x_t^2 - \theta x_t y_t + \alpha y_t - \beta x_t + \sigma - 1/2v_t^2 - \pi v_t\right] dt,$ respectively. We can, for example, think of the game as describing a duopoly in which two firms invest in various assets that improve their competitive positions. To keep the expressions shorter and make the argument more transparent, we start by looking at the case in which $\alpha = \beta = \sigma = \pi = 0, (\rho, \eta, \theta) \in R^{+3}$, and $\gamma > \eta$. This means that X and Y maximize $\int_0^T \boldsymbol{e}^{-\rho t}\left[\gamma x_t^2 - \eta y_t^2 - \theta x_t y_t - 1/2u_t^2\right] dt$ and $\int_0^T \boldsymbol{e}^{-\rho t}\left[\gamma y_t^2 - \eta x_t^2 - \theta x_t y_t - 1/2v_t^2\right] dt$, respectively. (The extension to the more general case is easy and we will return to it later.)

By Theorem 3.1 in Oeksendal and Sulem (2004), the players want to find C^2 value functions $W\left(x_t, y_t, t\right), W\left(y_t, x_t, t\right)$ from $R^{+3} \to R$ that satisfy the Bellman equations:[3]

$$\begin{aligned}&\text{Max}_{u(x,y,t)} \left\{e^{-\rho t}\left[\gamma x_t^2 - \eta y_t^2 - \theta x_t y_t - 1/2u_t^2\right] + \partial W\left(x_t, y_t, t\right)/\partial t + \lambda\left[W\left(x_{t-} + u_{t-}, y_{t-}, t-\right)\right.\right.\\ &\left.\left. -W\left(x_{t-}, y_{t-}, t-\right)\right] + \lambda\left[W\left(x_{t-}, y_{t-} + v_{t-}^*, t-\right) - W\left(x_{t-}, y_{t-}, t-\right)\right]\right\} = 0, W\left(x_T, y_T, T\right) = 0,\end{aligned}$$

for player X,

$$\tag{3}$$

and

$$\begin{aligned}&\text{Max}_{v(y,x,t)} \left\{e^{-\rho t}\left[\gamma y_t^2 - \eta x_t^2 - \theta x_t y_t - 1/2v_t^2\right] + \partial W\left(y_t, x_t, t\right)/\partial t + \lambda\left[W\left(y_{t-} + v_{t-}, x_{t-}, t-\right)\right.\right.\\ &\left.\left. -W\left(y_{t-}, x_{t-}, t-\right)\right] + \lambda\left[W\left(y_{t-}, x_{t-} + u_{t-}^*, t-\right) - W\left(y_{t-}, x_{t-}, t-\right)\right]\right\} = 0, W\left(y_T, x_T, T\right) = 0,\end{aligned}$$

for player Y.

$$\tag{4}$$

[2]The problem formulation suggests a filtration induced by the states (x_s, y_s, s), and we will conduct the analysis in that context. The notation reflects that we will work with the right continuous with left limits ("cadlag" from the French version) versions of all processes, x_t, y_t, u_t, and v_t., such that the jump at t is $x_t - x_{t-}$ etc. The appearance of the - subscript in the terms for jumps in both x_t and y_t does not mean that they jump at the same time but reflects that the Bellman equation looks at expected values.

[3]Since the value functions are C^2, we can use the infinitesimal generators of x_t and y_t to describe the movements of the value functions in infinitesimal time-intervals.

The first terms in (3) and (4) are the payoffs at t and the remaining terms make up a first order Taylor approximation to the dynamics of W. The (Markov) equilibrium controls u^*_{t-} and v^*_{t-} are therefore given by the first order conditions:

$$u^*_{t-}\left(x_{t-}, y_{t-}, t-\right) = e^{\rho t-}\lambda \partial W\left(x_{t-}, y_{t-,} t-\right)/\partial x \tag{5}$$

$$v^*_{t-}\left(y_{t-}, x_{t-}, t-\right) = e^{\rho t-}\lambda \partial W\left(y_{t-}, x_{t-}, t-\right)/\partial y \tag{6}$$

Since the players face symmetric problems, we will henceforth focus on X. Substitution of (5) and (6) into (3) allows us to rewrite the Bellman equation as:[4]

$$\begin{aligned}
&e^{-\rho t}\left(\gamma {x_t}^2 - \eta y_t^2 - \theta x_t y_t\right) - e^{\rho t}\lambda^2\left[\partial W\left(x_{t-}, y_{t-}, t-\right)/\partial x\right]^2/2 + \partial W\left(x_t, y_t, t\right)/\partial t + \\
&\quad + \lambda[W(x_{t-} + e^{\rho t}\lambda \partial W\left(x_{t-}, y_{t-}, t-\right)/\partial x, y_{t-}, t-\big) - W\left(x_{t-}, y_{t-}, t-\right)\big] + \\
&\lambda\left[W\left(x_{t-}, y_{t-} + e^{\rho t}\lambda \partial W\left(y_{t-}, x_{t-}, t-\right)/\partial y_, t-\right) - W\left(x_{t-}, y_{t-}, t-\right)\right] = 0, \\
&W\left(x_T, y_T, T\right) = 0
\end{aligned} \tag{7}$$

We guess a solution of the form:

$$\begin{aligned}
&W^g\left(x_t, y_t, t\right) \equiv e^{-\rho t}\left[a(t)x_t^2 + b(t)x_t + c(t) + d(t)y_t^2 + e(t)y_t + f(t)x_t y_t\right], \\
&a(T) = b(T) = c(T) = d(T) = e(T) = f(T) = 0
\end{aligned} \tag{8}$$

Using that the strategies are symmetric, this gives:

$$u^g_{t-} = \lambda\left[2a(t)x_{t-} + b(t) + f(t)y_{t-}\right] \tag{9}$$

$$v^g_{t-} = \lambda\left[2a(t)y_{t-} + b(t) + f(t)x_{t-}\right] \tag{10}$$

$$\begin{aligned}
&\lambda\left[W^g\left(x_{t-} + \lambda 2a(t)x_{t-} + \lambda b(t) + \lambda f(t)y_{t-}, y_{t-}, t-\right) - W^g\left(x_{t-}, y_{t-}, t-\right)\right] = \\
&e^{-\rho t}\lambda^2[a(t)\lambda + 1]\left\{4a(t)^2x_{t-}^2 + 4a(t)b(t)x_{t-} + b(t)^2 + f(t)^2y_{t-}^2 + \right. \\
&\left. 2b(t)f(t)y_{t-} + 4a(t)f(t)x_{t-}y_{t-}\right\}
\end{aligned} \tag{11}$$

$$\begin{aligned}
&\lambda\left[W^g\left(x_{t-}, y_{t-} + \lambda 2a(t)y_{t-} + \lambda b(t) + \lambda f(t)x_{t-}, t_-\right) - W^g\left(x_t, y_t, t_-\right)\right] = \\
&e^{-\rho t}\lambda^2\left\{\lambda d(t)f(t)^2x_{t-}^2 + [\lambda b(t)d(t) + e(t)]f(t)x_{t-} + b(t)[\lambda b(t)d(t) + e(t)] + \right. \\
&4a(t)d(t)[\lambda a(t) + 1]y_{t-}^2 + 2[b(t)d(t) + 2\lambda a(t)b(t)d(t) + a(t)e(t)]y_{t-} + \\
&\left. 2d(t)f(t)[2\lambda a(t) + 1]x_{t-}y_{t-}\right\}
\end{aligned} \tag{12}$$

[4]Since the value functions are C^2, we can use the infinitesimal generators of x_t and y_t to describe the movements of the value functions in infinitesimal time-intervals.

Rewriting the Bellman equation in terms of the coefficients in $W^g(x_t, y_t, t)$ gives:

$$\gamma x_t^2 - \eta y_t^2 - \theta x_t y_t - \lambda^2 \left[2a(t)x_{t-} + b(t) + f(t)y_{t-}\right]^2/2 + \left[a'(t) - \rho a(t)\right]x_t^2 + [b'(t) - \rho b(t)]x_t + [c'(t) - \rho c(t)] + [d'(t) - \rho d(t)]\, y_t^2 + [e'(t) - \rho e(t)]\, y_t + [f'(t) - \rho f(t)]\, x_t y_t + \lambda^2[a(t)\lambda + 1]\left\{4a(t)^2 x_{t-}^2 + 4a(t)b(t)x_{t-} + b(t)^2 + f(t)^2 y_{t-}^2 + 2b(t)f(t)y_{t-} + 4a(t)f(t)x_{t-}y_{t-}\right\} + \lambda^2\left\{\lambda d(t)f(t)^2 x_{t-}{}^2 + [\lambda b(t)d(t) + e(t)]f(t)x_{t-} + b(t)[\lambda b(t)d(t) + e(t)] + 4a(t)d(t)[\lambda a(t) + 1]\, y_{t-}^2 + 2[b(t)d(t) + 2\lambda a(t)b(t)d(t) + a(t)e(t)]y_{t-} + 2d(t)f(t)[2\lambda a(t) + 1]\, x_{t-1}y_{t-}\right\}.$$
$$a(T) = b(T) = c(T) = d(T) = e(T) = f(T) = 0 \tag{13}$$

$W^g(x_t, y_t, t)$ solves (7) if the constant term and the coefficients on x_t^2, x_t, y_t^2, y_t and $x_t y_t$ all are zero at T. This means that the coefficients in $W^g(x_t, y_t, t)$ solve the following differential equations:

$$a'(t) = -\gamma - 2a(t)^2\lambda^2[1 + 2a(t)\lambda] - \lambda^3 d(t)f(t)^2 + \rho a(t), a(T) = 0 \tag{14}$$
$$b'(t) = \left[-4\lambda^3 a(t)^2 - 2\lambda^2 a(t) + \lambda^3 d(t)f(t) + \rho\right] b(t) + \lambda^3 e(t)f(t), b(T) = 0 \tag{15}$$
$$c'(t) = -\lambda^2 b(t)[\lambda a(t)b(t) + b(t)/2 + \lambda b(t)d(t) + e(t)] + \rho c(t), c(T) = 0 \tag{16}$$
$$d'(t) = \eta - 4\lambda^2 a(t)d(t)[a(t)\lambda + 1] - \lambda^2 f(t)^2[a(t)\lambda + 1/2] + \rho d(t), d(T) = 0 \tag{17}$$
$$e'(t) = -b(t)[2\lambda a(t) + 1][2d(t) + f(t)] + 2a(t)e(t)] + \rho e(t), e(T) = 0 \tag{18}$$
$$f'(t) = \theta - 2\lambda^2 f(t)[2a(t)\lambda + 1][a(t) + d(t)] + \rho f(t), f(T) = 0 \tag{19}$$

Since (14) - (19) is a system of ODEs with continuous right-hand sides, a unique solution exists, and we can conclude that:

Proposition 1: $W^*(x_t, y_t, t) = e^{-\rho t}\left[a(t)x_t^2 + b(t)x_t + c(t) + d(t)y_t^2 + e(t)y_t + f(t)x_t y_t\right]$, solves the Bellman equation (7), and the Markov equilibrium strategies are $u_{t-}^* = \lambda\left[2a(t)x_{t-} + b(t) + f(t)y_{t-}\right]$ and $v_{t-}^* = \lambda\left[2a(t)y_{t-} + b(t) + f(t)x_{t-}\right]$.

So the value functions are second order polynomials, and the policy functions are proportional to the arrival intensity and linear in both players' stocks.

Since Proposition 1 gives an explicit solution to the game, we can quite easily derive several interesting corollaries.

Corollary (1.1): $a(t) > 0$ *and* $f(t) < 0$ *for all* $t < T$ *and.*

Proof: From (14): First, since $a'(t) < 0$ for t close to $T, a(t) > 0$ in that neighborhood.

Second, because $a'(s) < 0$ if $a(s) = 0, a(s)$ cannot change sign and is therefore positive for all $t < T$. Similarly from (19): Since $f'(t) > 0$ for t close to $T, f(t) < 0$ in that neighborhood and because $f'(t) > 0$ if $f(t) = 0, f(t)$ cannot change sign and is therefore negative for all $t < T$.

QED

Corollary (1.2): *$E_{t=0} |x_s - y_s|$ grows with s.*

Proof: Suppose that X is ahead at time h in the sense that $x_h > y_h$. In that case (since $\lambda [2a(t)x_h + b(t) + f(t)y_h] > \lambda [2a(t)y_h + b(t) + f(t)x_h]$ when $x_h > y_h), u^*_{t-} > v^*_{t-}$ and X invests more. The players are equally likely to get the next arrival at time $h+i$, but since X will invest more, $|x_h - y_h|$ will grow more if it gets the arrival than if Y does. The expected value of $|x_{h+i} - y_{h+i}|$ is therefore larger than $|x_h - y_h|$. The same mechanism applies for all later arrivals, and if Y gets the first arrival. So while $E_{t=0} (x_s - y_s) = 0, E_{t=0} |x_s - y_s|$ is strictly positive and grows with s.

QED

Corollary (1.3): *If $x_0 > y_0, E_{t=0} x_s / y_s$ grows with s.*

Proof: Suppose again that X is ahead at time h. The players are equally likely to get the next arrival at time $h + i$, but the size of a player's arrival and post arrival stocks are proportional to his or her stock. So if X gets the arrival at $h + i$, we can write x_{h+i} as rx_h whereas Y's $h + i$ stock would be ry_h if it gets the first arrival. Therefore $E_{t=h} x_{h+i}/y_{h+i} = 1/2 [rx_h/y_h + x_h/(ry_h)] = 1/2 (x_h/y_h) [(r^2 + 1)/r]$ which is larger than x_h/y_h for all $r \neq 1$.

QED

Corollary (1.4): The probability that $x_s - y_s$ changes sign at $s + 1$ decreases with s.

Proof: Suppose that $x_s > y_s$. X is more likely to get the next arrival and by Corollary 1.2, the expected difference in arrival probabilities grows with $x_s - y_s$ and thus with time.

QED

3 GAME WITH CONTROLLED JUMP INTENSITIES

The model is the same as in Section II with three differences: First, the players' stocks grow in jumps of size $\varphi > 0$ that arrive according to non-homogeneous Point processes with intensities $u(x_{t-}, y_{t-}, t-)$for X and $v(y_{t-}, x_{t-}, t-)$ for Y. Second, for reasons explained below, we bound these intensities. Third, we will be looking for solutions that are unique in law.

Formally, players choose the intensities as C^2 Markov controls; X's strategy is $u(x_t, y_t, t)$ $\leq U$ and Y's is $v(y_t, x_t, t) \leq U$. The stocks therefore develop over time according to:

$$x_t = \int_0^t dM_x \left[u(x_{s-}, y_{s-}, s-)\right] ds \tag{20}$$

$$y_t = \int_0^t dM_y \left[v(y_{s-}, x_{s-}, s-)\right] ds \tag{21}$$

where $M_x[u(x_{s-}, y_{s-}, s-)]$ and$M_y[v(y_{s-}, x_{s-}, s-)]$ are Point processes with jumps of size φ and intensities $u(x_{s-}, y_{s-}, s-)$ and $v(y_{s-}, x_{s-}, s-)$, respectively. X and Y again maximize $\int_0^T e^{-\rho t}\left[\gamma x_t^2 - \eta y_t^2 - \theta x_t y_t - 1/2 u_t^2\right] dt$ and $\int_0^T \boldsymbol{e}^{-\rho t}\left[\gamma y_t^2 - \eta x_t^2 - \theta x_t y_{t-} - 1/2 v_t^2\right] dt$, respectively, and we continue to assume that $(\rho, \eta, \theta) \in R^{+3}$, and $\gamma > \eta$.

The problem no longer satisfies the conditions of Theorem 3.1 in Oeksendal and Sulem (2004) because the jump intensities $u(x_{t-}, y_{t-}, t-)$ and $v\left(y_{t-}, x_{t_-}, t-\right)$ depend on the very states they govern. Since the coefficients in (20) and (21) therefore do not satisfy Lipschitz conditions, we cannot apply classical existence and uniqueness results. Fortunately, we can rely on a more general result first obtained by Jacod and Protter (1982, Corollary 31) and Protter (1983, Corollary 3.11).[5] They show that (20) and (21) have solutions that are unique in law if there exists finite-valued increasing processes p and q such that $\int_0^t u(x_s, y_s, s)\, ds \leq p_t$ and $\int_0^t v(y_s, x_s, s) ds \leq q_t$ for all $t \geq 0$.[6] These conditions are clearly satisfied by $p_t = tU$ and $q_t = tU$ in our model. While it is unfortunate that we have to add this extra parameter, it is irrelevant for the economic intuition, and we will soon be

[5]The idea in the proof is to inductively define an increasing sequence of stopping times (jump times) $\tau_1, \tau_2, \ldots, \tau_n$, .. and piece together the entire solution from the intervals $[\tau_1, \tau_{2-})$ on which classical theory applies.

[6]Uniqueness in law means that all solutions produce the same distribution over future realizations given the same starting point and time. Classical conditions give path-wise uniqueness, a stronger property under which all solutions follow the same paths everywhere.

taking it to infinity.

Depending on U and the realizations of (20) and (21), we can have up to four regions: (i) neither player is constrained, (ii) one player is constrained, (iii) the other player is constrained, and (iv) both players are constrained. We know that the players start out and end unconstrained but in between they can visit all four regions several times and the solution would need to meet value matching and smooth pasting conditions (in three dimensions) on each occasion. The obvious difficulties associated with finding an explicit solution to this equation probably explains why the case with controlled intensities is looked at so rarely.[7] We can, however, get a limiting result.

To develop some intuition for the idea, note that the complete solution to the constrained problem differs from that to the unconstrained problem because it must take into account the possibility that the system may transition to other regions. In particular, the value function in the unconstrained region must reflect the possibility that constraint later may bind for one or both players. The probability that this happens depends on U as well as the arrival realizations. If a player has a sequence of early arrivals and therefore a large stock at an early date, it will want to invest at a high level and could hit the constraint. On the other hand, if U is large and the player accumulates stocks slowly, it is less likely to ever be constrained (because the incentives to invest become smaller as t approaches T). Since the probability of realizations where the constraints on u_{t-} and v_{t-} never bind goes to l as U goes to infinity. the result follows.

Proposition 2: Suppose that all jumps are of size φ, that the arrival intensities (u_t, v_t) are bounded by $U > 0$, and that these are controlled by players X and Y, respectively. Further, define $A(t), B(t), C(t), D(t), E(t)$ and $F(t)$ as the solutions to:

$$A'(t) = -\gamma - \left[2A(t)^2 + F(t)^2\right]\varphi^2 + \rho A(t), A(T) = 0$$

$$B'(t) = -[A(t)\varphi + B(t)]\varphi^2[2A(t) + F(t)] - \varphi^2[D(t)\varphi + E(t)] + \rho B(t), B(T) = 0,$$

$$C'(t) = -\varphi^2[A(t)\varphi + B(t)]^2/2 - \varphi^2[A(t)\varphi + B(t)][D(t)\varphi + E(t)] + \rho C(t), C(T) = 0$$

$$D'(t) = \eta - \varphi^2[4A(t)D(t) + F(t)/2] + \rho D(t), D(T) = 0$$

$$E'(t) = -\varphi^2 F(t)[A(t)\varphi + 2B(t)] - 2\varphi^2[2A(t)D(t)\varphi + A(t)E(t) + B(t)D(t)] + \rho E(t), E(T) = 0$$

$$F'(t) = \theta - 2\varphi^2 F(t)[2A(t) + D(t)] + \rho F(t), F(T) = 0$$

[7] The four pre-pasting value functions are all quadratic, so some progress may still be possible.

In this case, as $U \to \infty$:

(2.0.1) The probability of $\mathrm{Max}_t \{u_{t-}, v_{t-}\} < U$ *converges to 1.*

(2.0.2) The value functions $W^*(x_t, y_t, t)$ *and* $W^*(y_t, x_t, t)$ *converge to* $e^{-\rho t}[A(t)x_t^2 + B(t)x_t + C(t) + D(t)y_t^2 + E(t)y_t + F(t)x_{t-}y_{t-}]$ *and* $e^{-\rho t}[A(t)y_t^2 + B(t)y_t + C(t) + D(t)x_t^2 + E(t)x_t + F(t)x_{t-}y_{t-}]$, *respectively.*

(2.0.3) The Markov equilibrium strategies converge to:

$u^*_{t-} = \varphi[2A(t)x_{t-} + \varphi A(t) + B(t) + F(t)y_{t-}]$ and $v^*_{t-} = \varphi[2A(t)y_{t-} + \varphi A(t) + B(t) + F(t)x_{t-}]$

Proof: The game starts in the unconstrained region. Consider a time $t < T$ and a pair of finitevalued functions from $[0, t]$ to $[0, U]$. Imagine that the latter pair of functions are arrival intensities and hold them and t constant. Then, for any pair of reals kx, ky, the probability of a pair of sequences of realized arrival times such that $x_t > k_x$ or $y_t > k_y$ is decreasing in U. This establishes (2.0.1) and that the extent to which the value functions in the constrained game depart from those in the unconstrained game goes to zero as U goes to infinity.

Next, note that the Bellman equation for player X in the unconstrained game is:

$$\mathrm{Max}_{u(x,y,t)}\left\{e^{-\rho t}\left[\gamma x_t^2 - \eta y_t^2 - \theta x_t y_{t-} - 1/2u_t^2\right] + \partial W(x_t, y_t, t)/\partial t + u_{t-}[W(x_{t-} + \varphi, y_{t-}, t-) - W(x_t,\ y_{t_-}, t-)\right] + v_{t-}\left[W\left(x_{t_-}, y_{t_-} + \varphi, t-\right) - W\left(x_{t_-}, y_{t_-}, t-\right)\right] = 0 \tag{22}$$

The (Markov) equilibrium controls u^*_{t-} and v^*_{t-} are therefore given by the first order conditions:

$$u^*_{t\_}(x_{t-}, y_{t-}, t-) = e^{\rho t}[W(x_{t-} + \varphi, y_{t-}, t-) - W(x_{t-}, y_{t-}, t-)] \tag{23}$$

$$v^*_{t-}(y_{t-}, x_{t-}, t-) = e^{\rho t}[W(y_{t-} + \varphi, x_{t-}, t-) - W(y_{t-}, x_{t-}, t-)] \tag{24}$$

We can substitute (23) and (24) into (22) and rewrite the Bellman equation as:

$$e^{-\rho t}\left(\gamma x_t^2 - \eta y_t^2 - \theta x_t y_{t-}\right) + \partial W(x_{t-}, y_{t-}, t_-)/\partial t + e^{\rho t}[W(x_{t-} + \varphi, y_{t-}, t_-) - W(x_{t-}, y_{t-}, t_-)]^2/2 + e^{\rho t}[W(y_{t-} + \varphi, x_{t-}, t_-) - W(y_{t-}, x_{t-}, t_-)]\left[W\left(x_{t-}, y_{t_-} + \varphi, t_-\right) - W(x_{t-}, y_{t-}, t_-)\right] = 0 \tag{25}$$

We again guess a solution of the form:

$$W^g(x_t, y_t, t) \equiv e^{-\rho t}\left[A(t)x_t^2 + B(t)x_t + C(t) + D(t)y_t^2 + E(t)y_t + F(t)x_{t-}y_{t-}\right]. \quad (26)$$

Substituting (26) into (25) gives:

$$\begin{aligned}
&\left\{A'(t) - \rho A(t) + \gamma + \left[2A(t)^2 + F(t)^2\right]\varphi^2\right\} x_t^2 + \{B'(t) - \rho B(t) + [A(t)\varphi + B(t)] \\
&\varphi^2[2A(t) + F(t)] - \varphi^2[D(t)\varphi + E(t)]\big\} x_t + \big\{C'(t) - \rho C(t) + \varphi^2[A(t)\varphi + B(t)]^2/2 - \\
&\varphi^2[A(t)\varphi + B(t)][D(t)\varphi + E(t)]\big\} + \left\{D'(t) - \eta - \rho D(t) + \varphi^2[4A(t)D(t) + F(t)/2]\right\} y_t^2 + \\
&\left\{E'(t) - \rho E(t) + \varphi^2 F(t)[A(t)\varphi + 2B(t)] + 2\varphi^2[2A(t)D(t)\varphi + A(t)E(t) + B(t)D(t)]\right\} y_t + \\
&\left\{F'(t) - \rho F(t) - \theta + 2\varphi^2 F(t)[2A(t) + D(t)]\right\} x_t y_t = 0
\end{aligned} \quad (27)$$

Our guess $W^g(x_t, y_t, t)$ therefore solves (27) if the coefficients in $W^g(x_t, y_t, t)$, solve the following differential equations:

$$A'(t) = -\gamma - \left[2A(t)^2 + F(t)^2\right]\varphi^2 + \rho A(t), A(T) = 0 \quad (28)$$

$$B'(t) = -[A(t)\varphi + B(t)]\varphi^2[2A(t) + F(t)] - \varphi^2[D(t)\varphi + E(t)] + \rho B(t), B(T) = 0 \quad (29)$$

$$C'(t) = -\varphi^2[A(t)\varphi + B(t)]^2/2 - \varphi^2[A(t)\varphi + B(t)][D(t)\varphi + E(t)] + \rho C(t), C(T) = 0 \quad (30)$$

$$D'(t) = \eta - \varphi^2[4A(t)D(t) + F(t)/2] + \rho D(t), D(T) = 0 \quad (31)$$

$$\begin{aligned}E'(t) = &-\varphi^2 F(t)\left[A(t)\varphi + 2B(t)\right] - 2\varphi^2[2A(t)D(t)\varphi + A(t)E(t) + B(t)D(t)] + \\ &\rho E(t), E(T) = 0\end{aligned} \quad (32)$$

$$F'(t) = \theta - 2\varphi^2 F(t)[2A(t) + D(t)] + \rho F(t), F(T) = 0 \quad (33)$$

Since (28) – (33) have continuous right-hand sides, a solution exists and (2.0.2) and (2.0.3) follows.

QED

So just as in the game with controlled jump sizes, the value functions with controlled arrival intensities are second order polynomials, and the policy functions are linear in both players' stocks. Finally, except for the quadratic $A(t)\varphi^2$, the policy functions are proportional to the jump size which therefore play a role very similar to that played by the arrival intensity in the game with controlled jump sizes. Between the jump size and the arrival intensity, the uncontrolled magnitude plays more or less the same role in both models.

It is easy to establish that the game with controlled jump intensities behaves "like" the game with controlled jump sizes in the sense that Corollaries 2.1 – 2.4 below are analogues of Corollaries 1.1 – 1.4.

Corollary (2.1): *As $U \to \infty$, the probabilities that $a(t) > 0$ and $f(t) < 0$ both converge to 1.*

Corollary (2.2): *As $U \to \infty$, if $x_0 > y_0$, the probability that $E_{t=0} x_s / y_s$ grows with s converges to 1.*

Corollary (2.3): *As $U \to \infty$, the probability that $E_{t=0} |x_s - y_s|$ grows with s, converges to 1.*

Corollary (2.4): *As $U \to \infty$, the probability that $x_s - y_s$ changes sign at $s+1$, decreases with s.*

Proof: By arguments identical to those used to prove Corollaries 1.1 – 1.4

QED

In a natural analogue of the model presented in Section II, we obtain a limiting result according to which the value - and policy functions are very similar to what we found there. By Corollaries 2 – 4, the two games share other appealing properties as well. The analysis in this Section shows that we can solve dynamic optimization problems and games with controlled jump intensities if we are willing to accept results that depend on theoretical bounds on the control variables and solutions that are unique in law only. However, it also suggests that the qualitative properties of the solution are relatively close to those obtained in similar models with controlled jump sizes.

4 DISCUSSION

There are some theoretical limits on the acceptability of the solution to the second formulation, but we do not see them as particularly important. It is hard to think of an economic model in which a solution would be disqualified because it only is unique in law. There are certainly cases in which it is natural to assume that the arrivals cannot be too close in time ("at most one per day"?), but the two formulations do not differ in that respect; only the arrival intensities are at stake. Can there be cases in which it is important to bound the arrival intensities? We cannot think of an example, but the answer may be less clear than that about the nature of uniqueness.

We would like to close with four observations: First, as discussed at the start of Section II, we claim that the result hold for general second order polynomial objective functions of the form $\int_0^T \boldsymbol{e}^{-\rho t} \left[\gamma x_t^2 - \eta y_t^2 - \theta x_t y_t + \alpha x_t - \beta y_t + \sigma - 1/2 u_t^2 - \pi u_t\right] dt$. To see this, start by writing out the analogue of (7). This only differs because the first term (the payoff function) is longer and because $u_{t-}^* \left(x_{t-}, y_{t-}, t-\right) = e^{\rho t -} \lambda \partial W \left(x_{t-}, y_{t-}, t-\right) / \partial x - \pi$, where the π is new. You then guess a solution of the form $W^g \left(x_t, y_t, t\right) \equiv e^{-\rho t} \left[a(t) x_t^2 + b(t) x_t + c(t) + d(t) y_t^2 + e(t) y_t + f(t) x_t y_t\right]$, which is the same as (8). By going through each term in the Bellman equation you can then see that it also is a second order polynomial and therefore can be solved by $W^g \left(x_t, y_t, t\right)$. Second, while it is a limitation that the results only have been established for objective functions that are second order polynomials, such functions can be thought of as Taylor approximations to a richer set. In many models this approximation should be good enough to capture economically important effects. Third, subject to the constraint that objective functions have to be second order polynomials, it is not important that the games be symmetric. We chose that case because Corollaries 2 – 4 are uninteresting in other settings. Fourth, to help us think about the limitations of the results, it would be interesting to identify an economic situation in which the two formulations yield conflicting or at least different intuitions about what is going on.

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