Epistemic Foundations of Game Theory

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What is epistemic game theory?

▷ Epistemic game theory takes a descriptive and decision-theoretic approach to games.

▷ Let’s say that a play of a game is a way the game can turn out, after the decisions, but before the reveal.

▷ Here are some features of a play of a game: what the players do, what each player believes about what the others will do, whether the players are rational, whether the players believe that the others are rational, where the game is played, how many siblings the players have, which player is oldest.

▷ Some features of a play, like the first four, are relevant to evaluating the players’ actions. Others, like the last three, are not. We only care about features which are relevant to evaluating.

▷ Let $S$ be the set of all plays of a game. From $S$, we can consider interesting subsets. For example: the set of plays in which everyone believes that everyone is rational; the set of plays in which the players’ actions form a Nash equilibrium.

▷ Epistemic game theory investigates how plays picked out using epistemic concepts are related to plays picked out using game-theoretic concepts.

▷ In order to do that, we need to find some way to represent plays of a game, or, at least, those features of plays of a game which are relevant to evaluating the players’ actions. That’s what a game model does.

▷ Let $\Gamma = \langle N, \{A_i, u_i\}_{i \in N} \rangle$ be a game in strategic form.

Definition 1. An epistemic model for $\Gamma$ is a tuple $\langle W, x, \{\Pi_i\}_{i \in N}, \sigma \rangle$, where $W$ is a set of states, $x \in W$, $\Pi_i$ is player $i$’s partition, and $\sigma : W \to \prod_{i \in N} A_i$.

Definition 2. An epistemic-probability model for $\Gamma$ is a tuple $\langle W, x, \{\Pi_i, P_i\}_{i \in N}, \sigma \rangle$, where $\langle W, x, \{\Pi_i\}_{i \in N}, \sigma \rangle$ is an epistemic model, and $P_i : W \to \Delta(W)$ specifies player $i$’s partial beliefs in each world.

Remark 1. In an epistemic model $\langle W, x, \{\Pi_i\}_{i \in N}, \sigma \rangle$, $x$ represents the actual world. So the actual outcome is $\sigma(x)$. The other worlds, the partitions and the probabilities are there to represent the players’ information.
Definition 3. Let $E$ be a subset of $W$ and fix $w \in W$. We say that...

a. $i$ fully believes that $E$ at $w$ just if $p^w_i(E) = 1$.

b. $i$ knows that $E$ at $w$ just if $\Pi_i(w) \subseteq E$.

c. it’s mutual knowledge that $E$ at $w$ just if everyone knows that $E$ at $w$.

d. it’s common knowledge that $E$ at $w$ just if everyone knows that $E$ at $w$, and everyone knows that everyone knows that $E$ at $w$, and...

e. $i$ is rational at $w$ just if $i$’s action at $w$ maximizes expected utility given her (partial) beliefs at $w$.

Iterated elimination of strictly dominated strategies

We are almost in a position to state a fundamental result in epistemic game theory.

Definition 4. We say that $\sigma$ is strictly dominated by $\mu$ just if $\mu$ yields higher expected payoff than $\sigma$, no matter what the other players do.

Definition 5. We say that a strategy $\sigma$ survives iterated elimination of strictly dominated strategies if it survives the following procedure. First, delete any strictly dominated strategies in $\Gamma$, resulting in $\Gamma_1$. Then delete any strictly dominated strategies in $\Gamma_1$, resulting in $\Gamma_2$. Repeat until no strategies are strictly dominated.

Here’s the fundamental result:

Theorem 1. For any finite game in strategic form and any epistemic-probability model of that game, if the players are rational, and there is common full belief that all players are rational, then each player’s action survives iterated elimination of strictly dominated strategies.

Extensive form games

▷ Some games are sequential: players take turns to move.

▷ In some sequential games, like chess, once a player moves, it becomes common knowledge what that move was. These are perfect information games. In others, it doesn’t. Let’s focus on perfect information games.

▷ An extensive form perfect information game is a rooted tree-like structure, where leaves are labeled with the players’ payoffs, decision nodes are assigned to players, and edges at a decision node are labeled with actions.
A strategy for a player in a perfect information game specifies that player’s choice at each node assigned to her.

A strategy profile determines a path through the game tree starting from any node.

**Backward induction**

Let’s play a Centipede Game. Given any non-degenerate PI game, the backward induction strategy profile is calculated as follows. For each pre-terminal node, select the action which maximizes the payoff of the controlling player, and label that node with the resulting payoffs. Repeat until all nodes are labelled. The selected actions form the backward induction strategy profile.

Backward induction reasoning is seductive. Epistemic game theory helps clarify what’s going on.

It turns out to be a delicate matter. Backward induction illustrates the importance of representing belief revision policies in our game models.

**Common knowledge of rationality implies backward induction**

Fix a non-degenerate PI game and an epistemic model for that game.

**Definition 6.** We say that player $i$ is rational at a node $v$ in world $w$ just if, for any alternative action at $v$, for all $i$ knows, that action would not yield a higher payoff.

**Definition 7.** We say that $i$ is substantively rational in $w$ just if for all of her decision nodes $v$, $i$ is rational at $v$ in $w$.

**Theorem 2.** For any non-degenerate perfect information game and any epistemic model for $G$, if there is common knowledge of substantive rationality, then the strategy profile is the backward induction profile.

**Common knowledge of rationality does not imply backward induction**

Consider Stalnaker’s simple PI game.

Is common belief in rationality consistent with Alice playing down immediately?

See Robert Rosenthal (1981). A PI game is non-degenerate just if, for each player, different terminal nodes yield different payoffs. A node is pre-terminal just if all its daughters are labelled with payoffs.

This is a very weak concept of rationality, but that’s OK, since it just makes Aumann’s theorem stronger.
Aumann’s models do not represent the players’ belief revision policies: what they would believe were they to receive surprising information.

Belief revision policies are essential to assessing the rationality of the players.

We need to supplement our game models with a representation of the players’ belief revision policies.

How do we do that? With epistemic-plausibility models, or lexicographic probabilities, or...