18.315 (Alex Postnikov, MIT, Fall 2014)

Scribed by Carl Lian

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0 Acknowledgments

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1 September 3, 2014

The topic of this class is combinatorial aspects of Schubert calculus. There's a course website, which you can find linked to the lecturer's webpage. Recommended (not required) Books: Fulton's book on Young Tableaux, Stanley, Manivel's book on Schubert polynomials.

The lecturer has been collaborating with T. Lam, who took this class a long time ago. Lam remembered nothing from the class, except something from the first lecture. Two kinds of combinatorics: extremal (Erdos), algebraic/enumerative (Stanley). In the former, you prove bounds (like $n\log(n)$), in the latter, you prove $A = B$, which suggests an interplay between algebra and combinatorics.

Things related to the topic of the class:

- Algebraic geometry and topology. $\mathrm{Gr}(k,n)$, flag manifolds.
- Representation theory, $\mathrm{GL}(n), S_n$
- Symmetric functions
- Quantum cohomology, Gromov-Witten invariants
- Total positivity

We will concentrate on the relationships to combinatorics. The only prerequisite is linear algebra. We may not cover everything, depends on how fast we go and interest.

"Main players": Young tableaux, Schur polynomials. We'll try to avoid repeating things from last year's 18.315 (symmetric functions). We will focus more generally on Schubert polynomials (generalization of Schur polynomials). Littlewood-Richardson rule, Bruhat order, matroids, recent topics (e.g. total positivity, quantum cohomology).

Let's start with an example in Schubert calculus.

Example 1.1. Find the number of lines ℓ in $\mathbb{C}^3$ that intersect 4 given generic lines L_1, L_2, L_3, L_4.

Why do we expect that this number is finite? Have 2-dimensional space of lines through the origin, then have a 2-dimensional space of affine translations of the line, so in total a 4-dimensional space of lines. Each intersection with one of the given lines is a single linear condition, so we expect finitely many solutions.

Definition 1.2. Fix $0 \le k \le n$, and a field $\mathbb{F}$ (above, $\mathbb{F} = \mathbb{C}$). Define the **Grassmannian** $\mathrm{Gr}(k, n, \mathbb{F})$ to be the "space" of k-dimensional subspaces of $\mathbb{F}^n$.

An affine line in $\mathbb{C}^3$ corresponds to a 2-dimensional subspace of $\mathbb{C}^4$, i.e. an element of $\mathrm{Gr}(2, 4)$. One way to see this is to put your $\mathbb{C}^3$ inside C^4 as a hyperplane not passing through the origin; then, to get from an affine line in $\mathbb{C}^3$ to a 2-plane in $\mathbb{C}^4$, just take the span of the points on line.

Fix $A \in \mathrm{Gr}(2, 4)$. Define $\overline{\Omega_A} = \{B \in \mathrm{Gr}(2, 4) \mid \dim(A \cap B) \ge 1\}$. This is an example of a **Schubert variety**. The example above is asking about the size of $\overline{\Omega_{L_1}} \cap \overline{\Omega_{L_2}} \cap \overline{\Omega_{L_3}} \cap \overline{\Omega_{L_4}}$, i.e. the value of an **intersection number**. This number is also the coefficient of the Schur polynomial $s_{2,2}$ in $(s_1)^4$.

Definition 1.3. A Young diagram associated to a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ with $\lambda_1 \ge \lambda_2 \ge$ is the usual thing with λ_i boxes in the i-th row and everything left-aligned. They fit into the **Young lattice** $\mathbb{Y}$, the poset of Young diagrams ordered by inclusion.

The number above is also the number of paths in $\mathbb{Y}$ from the empty partition to $(2, 2)$.

Let's now start a systematic discussion. Can think of an element of $\mathrm{Gr}(k, n)$ as a $k \times n$ matrix, where the rows are linearly independent, i.e. our matrix has maximal rank k. Let $\mathrm{Mat}^*(k, n)$ be the set of such matrices; then $\mathrm{Gr}(k, n)$ is just $\mathrm{GL}_k \setminus \mathrm{Mat}^*(k, n)$, i.e. full-rank $k \times n$ matrices up to row operations.

Exercise 1.4. $\mathrm{Gr}(k, n)$ *is* $\mathrm{GL}(n)$ *modulo matrices with all zeroes in the bottom-left* $(n-k) \times k$ *submatrix (i.e. maximal parabolic subgroup), which in turn is* $U(n)/U(k) \times U(n-k)$ *(when we work over* $\mathbb{C}$*)*

Example 1.5. $\mathrm{Gr}(1, n)$ is $1 \times n$ matrices modulo scaling, which is just **projective space** $\mathbb{P}^{n-1}$.

What's the dimension of $\mathrm{Gr}(k, n)$? Given a $k \times n$ matrix, most of the time the left-most maximal minor is non-singular, so after row operations this maximal minor becomes the identity. The number of surviving parameters is $k(n-k)$, which come from all of the other entries. Hence $\dim \mathrm{Gr}(k, n) = k(n-k)$. Also, this shows $\mathrm{Gr}(k, n) = \mathbb{F}^{k(n-k)} \cup \{\text{lower dimensional part}\}$.

Gaussian elimination and RREF give us a canonical representative for each point of the Grassmannian. Take $k = 5, n = 12$ – after row operations, get something that looks like:

$$\begin{bmatrix} 0 & 1 & * & * & 0 & * & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Because the original matrix has full rank, we should get 5 pivots. Now remove the rows with pivots. Get the $k \times (n-k)$ matrix

$$\begin{bmatrix} 0 & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * \end{bmatrix}$$

If we flip over the vertical axis, we get a Young Diagram $(6,4,4,3,1)$ of stars, that fits inside a $k \times (n-k)$ rectangle, which we denote $(6,4,4,3,1) \subset k \times (n-k)$.

Definition 1.6. A **Schubert cell** Ω_λ, with $\lambda \subset k \times (n-k)$ is the set of points of $\operatorname{Gr}(k,n)$ whose RREF has shape λ.

Theorem 1.7 (Schubert decomposition)**.** $\operatorname{Gr}(k,n) = \coprod_{\lambda \subset k\times(n-k)} \Omega_\lambda$*, with* $\Omega_\lambda \cong \mathbb{F}^{|\lambda|}$

Example 1.8. $\mathbb{P}^2 = \operatorname{Gr}(1,3) = \{[1:x:y]\} \coprod \{[0:1:x]\} \coprod \{[0:0:1]\} = \mathbb{F}^2 \coprod \mathbb{F}^1 \coprod \mathbb{F}^0$. Geometrically, take the usual plane, add a line at infinity, then add a point at infinity to the line.

The number of Schubert cells is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Application: q-binomial coefficients. $\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$, where $[n]_q = 1 + q + \cdots + q^{n-1} = (1-q^n)/(1-q)$, and $[n]_q! = [1]_q[2]_q \cdots [n]_q$. If $q=1$, get the usual things.

Theorem 1.9. $\binom{n}{k}_q$ *is a polynomial in* q *with positive integer coefficients.*

Proof. Check the q-Pascal identity,

$$\binom{n}{k}_q = q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q,$$

then use induction. ☐

More enlightening way:

Proof. Let $\mathbb{F} = \mathbb{F}_q$, where q is a prime power (there are infinitely many of these). What is $|\operatorname{Gr}(k,n)|$? This is just

$$|\operatorname{Mat}^*_{k,n}(\mathbb{F}_q)|/|\operatorname{GL}_k(\mathbb{F}_q)| = \frac{(q^n-1)(q^n-q)(q^n-q^2)\cdots(q^n-q^{k-1})}{(q^k-1)\cdots(q^k-q^{k-1})} = \binom{n}{k}_q,$$

because the action of GL on Mat is free. On the other hand,

$$|\operatorname{Gr}(k,n)| = \sum_{\lambda \subset k\times(n-k)} q^{|\lambda|},$$

so in fact we have an identity of rational functions that holds for infinitely many q, and thus for generic q. ☐

Example 1.10. $n=4, k=2$: $\binom{4}{2}_q = 1 + q + 2q^2 + q^3 + q^4$, corresponding to the Young diagrams $\emptyset, (1), (2), (1,1), (2,1), (2,2)$.

Writing $\binom{n}{k}_q = a_0 + a_1 q + \cdots + a^N q^N$, with $N = k(n-k)$, the **Gaussian coefficients** have the following properties:

1. Symmetry: $a_i = a_{N_i}$ (to see this, rotate your $k \times (n-k)$ rectangle by 180 degrees and swap λ with its complement).

2. Unimodality: $a_0 \le a_1 \le \cdots \le a_{N/2} \ge \cdots \ge a_N$ (hard – conjectured by Kelly, proven algebraically by Sylvester (1878), combinatorially by O'Hara (1990)).

Let's sketch Sylvester's proof of unimodality.

Proof. Let V be the $\mathbb{R}$-space of formal linear combinations of Young diagrams $\lambda \subset k \times (n-k)$. Then V is graded by size of Young diagram: $V_0 \oplus V_1 \oplus \cdots \oplus V_N$, where $N = k(n-k)$. Let $n_i = \dim V_i$.

Given λ, let $add(\lambda)$ denote the set of boxes that can be added to λ to get another YT, and let $remove(\lambda)$ be the same thing for removing boxes. Define the *up operator* U by

$$\lambda \mapsto \sum_{x \in add(\lambda)} \sqrt{w(x)}(\lambda + x),$$

and the *down operator* D by

$$\lambda \mapsto \sum_{y \in remove(\lambda)} \sqrt{w(y)}(\lambda - y),$$

where $w(x) = (k + c(x))(n - k - c(x))$, and $c(x)$ is the *content* of x, the row number of x minus the column number (i.e. distance from main diagonal).

Now look at the eigenvalues of the *diagonal operator* $[D, U] = DU - UD$. Possibly more details next time.

□

2 September 5, 2014

Recall the Gaussian coefficients: $a_i = |\{\lambda \subset k \times (n-k) \mid |\lambda| = i\}$. They are unimodal: $a_0 \leq a_1 \leq \cdots a_{N/2} \geq \cdots \geq a_N$, where $N = k(n-k)$. Let's finish the proof from last time.

Proof. V_i is the space of linear combinations of λ's with $|\lambda| = i$, and $V = V_0 \oplus \cdots V_N$. We have an up operator $U : \lambda \mapsto \sum_{x \in add(\lambda)} \sqrt{w(x)}(\lambda + x)$ and a down operator $D = U^* : \lambda \mapsto \sum_{y \in rem(\lambda)} \sqrt{w(y)}(\lambda - y)$, where $w(x) = (k + c(x))(n - k - c(x))$.

Now consider $H = [D, U] = DU - UD$. DU adds a box x and removes a box y with weight $\sqrt{w(x)w(y)}$; U removes a box y and adds a box x with the same weight. Unless $x = y$, the two resulting terms cancel. Thus

$$H(\lambda) = \left(\sum_{x \in add(\lambda)} w(x) - \sum_{y \in rem(\lambda)} w(y) \right) \lambda = w_\lambda \lambda$$

hence H acts diagonally by the eigenvalues w_λ above.

Lemma 2.1. $w_\lambda = k(n-k) - 2|\lambda|$.

Proof. Exercise. □

To get unimodality, need $a_i \leq a_{i+1}$ for $i < N/2$, where $a_i = \dim V_i, a_{i+1} = \dim V_{i+1}$. Let $U_i : V_i \to V_{i+1}$ be the restriction of U, and $D_i : V_i \to V_{i-1}$ be the restriction of D (similarly for other indices). By the above, $H_i = D_i U_i - U_{i-1} D_{i-1} = U_i^* U_i - U_{i-1} U_{i-1}^* = (k(n-k) - 2i)I$, where I is the identity. By linear algebra, $U_{i-1} U_{i-1}^*$ is symmetric and positive semi-definite, so $U_i^* U_i$ is positive-definite (because we are just adding H_i, which is a positive multiple of the identity). In particular, $U_i^* U_i$ is non-singular, i.e. rank a_i, so U_i, which is $a_i \times a_{i+1}$, has rank a_i. Hence $a_i \leq a_{i+1}$, completing the proof. □

Really this proof comes from representation theory: the operators U, D, H give an irreducible representation of the Lie Algebra sl_2, generated by e, f, h with $[e, f] = h, [h, e] = 2e, [h, f] = -2f$. The dimensions above are the same as the dimensions of weight spaces of the representation.

This is also related to the **Horn problem**: consider Hermitian matrices $A + B = C$, where A has real eigenvalues $\alpha_1 \leq \cdots \leq \alpha_n$, B has eigenvalues $\beta_1 \leq \cdots \leq \beta_n$, and C has eigenvalues $\gamma_1 \leq \cdots \leq \gamma_n$. What $3n$-tuples of eigenvalues can arise in this way? It turns out that there is a polyhedral cone in $\mathbb{R}^{3n}$ of allowed triples α, β, γ. This problem was open for a while, solved by Klyachko assuming the "saturation conjecture," which was later proven by Knutson-Tao. The description of the cone is recursive; it would be nice to do it non-recursively (still open).

Let's get back to the Grassmannian - we introduce **Plücker coordinates**. For simplicity, work over $\mathbb{C}$. There is an embedding $\mathrm{Gr}(k, n) \to \mathbb{CP}^{N-1}$, where $N = \binom{n}{k}$, defined as follows. Recall that a point of $\mathrm{Gr}(k, n)$ is a $k \times n$ matrix A, modulo row operations. For all $I \in \binom{[n]}{k}$, i.e. an index set $\{i_1 < \cdots < i_k\} \subset \{1, 2, \ldots, n\}$, let A_I denote the $k \times k$ submatrix of A with column set I. Let $\Delta_I(A)$ be the **maximal minor** $\det(A_I)$ (side remark: we will regard I as an *ordered* set, so that $\Delta_I(A)$ is an anti-symmetric function in the i's).

Because A has full-rank, not all of the maximal minors are zero. Moreover, row operations scale simultaneously: if $B \in \mathrm{GL}(k)$, then $\Delta_I(BA) = \det(B)\Delta_I(A)$. Hence, the $\Delta_I(A)$ form a point of $\mathbb{CP}^N - 1$, giving us the **Plücker embedding** $\varphi : \mathrm{Gr}(k, n) \to \mathbb{CP}^{N-1}$.

Lemma 2.2. *φ is injective, so the Plücker embedding is actually an embedding. (In fact, the differential $d\varphi$ is also injective.)*

Proof. Assume $\Delta_{12\cdots k} \neq 0$, so that A can be transformed via row operations into a matrix $\widetilde{A}$ whose leftmost $k \times k$ minor is the identity. Hence $\Delta_I(\widetilde{A}) = \frac{\Delta_I(A)}{\Delta_{1\cdots k}(A)}$. Now, the entry x_{ij} of $\widetilde{A}$ is $\Delta_{12\cdots(i-1)j(i+1)\cdots k}(\widetilde{A})$, so in particular we can recover $\widetilde{A}$ from the Plücker coordinates. □

There are some relations between the maximal minors:

Example 2.3. Consider $\mathrm{Gr}(2, 4)$, with the Plücker coordinates $[\Delta_{12}, \Delta_{13}, \Delta_{14}, \Delta_{23}, \Delta_{24}, \Delta_{34}]$. Then we have

$$\Delta_{13}\Delta_{24} = \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23}. \tag{2.4}$$

This can be represented by something like a skein relation:

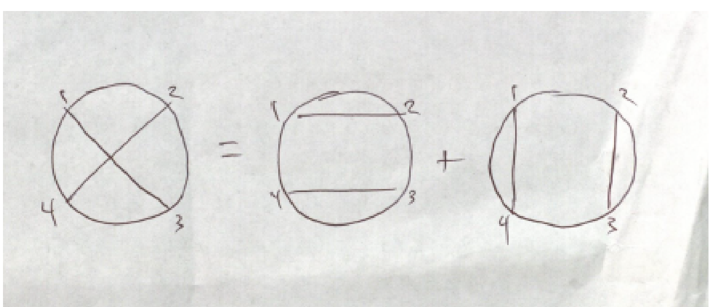

Figure 1: Plücker relation as a skein relation

To prove (2.4) in $\mathrm{Gr}(2, 4)$, it is enough to prove it for the matrix

$$\begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{bmatrix},$$

in which case we get $c(-b) = 1(ad-bc) + d(-a)$.

In fact, $\mathrm{Gr}(2,4)$ is the subvariety of $\mathbb{CP}^5$ cut out by (2.4).

In general, we have the **Plücker relations**: for all $i_1, \ldots, i_k, j_1, \ldots, j_k$ and $r = 1, \ldots, k$,

$$\Delta_{i_1 \cdots i_k} \Delta_{j_1 \cdots j_k} = \sum \Delta_{i'_1 \cdots i'_k} \Delta_{j'_1 \cdots j'_k},$$

where the indices $i'_1, \ldots, i'_k, j'_1, \ldots, j'_k$ from $i_1, \ldots, i_k, j_1, \ldots, j_k$ by switching $i_{s_1}, \ldots, i_{s_r}$ (where $s_1 < \cdots < s_r$) with $j_1, j_2, \ldots, j_k$ (it is a little cumbersome to formulate this in such a way that the signs are correct).

Example 2.5. $k = 2, n = 4, r = 1$. $\Delta_{12}\Delta_{34} = \Delta_{32}\Delta_{14} + \Delta_{13}\Delta_{24} = -\Delta_{23}\Delta_{14} + \Delta_{13}\Delta_{24}$.

Example 2.6. $r = 2, k = 3$. $\Delta_{241}\Delta_{312} = \Delta_{311}\Delta_{242} + \Delta_{341}\Delta_{212} + \Delta_{231}\Delta_{412} = \Delta_{231}\Delta_{412}$, because a repeated index means a repeated column, so zero determinant. Oops, that one was trivial.

The Plücker relations describe the image of $\mathrm{Gr}(k,n)$ in $\mathbb{CP}^{n-1}$, i.e. $\mathrm{Gr}(k,n)$ is the zero locus in projective space of the Plücker relations. In fact, all you need are the Plücker relations for $r = 1$.

Let's prove the Plücker relations (due to Sylvester).

Proof. Let $v_1, \ldots, v_k, w_1, \ldots, w_k \in \mathbb{C}^k$, and let $[v_1, \ldots, v_k]$ denote the determinant with column vectors $v_1, \ldots, v_k$ (in that order). Then the Plücker relations may be written as

$$[v_1, \ldots, v_k][w_1, \ldots, w_k] = \sum [v'_1, \ldots, v'_k][w'_1, \ldots, w'_k],$$

where as before, on the right hand side, we take all ordered subsets of the v_i of size r and swap them with $w_1, \ldots, w_r$, preserving the order, then sum.

Let $f = LHS - RHS$. First, observe that f is multilinear in $v_1, \ldots, v_k, w_1, \ldots, w_k$. We claim that f is a skew-symmetric function of the $k+1$ vectors $v_1, \ldots, v_k, w_k$, i.e. swapping two adjacent vectors flips the sign of f. It's enough to show that $f = 0$ if $v_i = v_{i+1}$ or $v_k = w_k$, i.e. the form is alternating (Editorial note: I think the terms "skew-symmetric" and "alternating" may have been reversed in class. In any case, there is no issue with characteristic 2, because alternating indeed implies skew-symmetric in all characteristics.)

If $v_i = v_{i+1}$, then $LHS = 0$. The only non-zero terms on the RHS look like $[\cdots v_i \cdots][\cdots v_{i+1} \cdots]$ or $[\cdots v_{i+1} \cdots][\cdots v_i \cdots]$ (i.e. exactly one of v_i, v_{i+1} get swapped with one of the w_i). All of pairs these terms will cancel, because they are obtained from each other by swapping two columns in the matrix on the left.

If $v_k = w_k$, assume $r < k$, or else there's nothing to check. Note that w_k doesn't get swapped, and v_k doesn't either, or else v_k, w_k are columns of the same matrix, and we get a zero term on the RHS. So the only non-vanishing terms have the form $[\cdots v_k][\cdots w_k]$. In an appropriate basis we can assume $v_k = w_k$ is the vector $(0, \ldots, 0, 1)$, in which case we just get a Plücker relation with k replaced by $k-1$ (apply induction).

It now follows that $f = 0$. □

3 September 10, 2014

Recall that we have a Plücker Embedding $\varphi : \mathrm{Gr}(k,n) \hookrightarrow \mathbb{CP}^{\binom{n}{k}-1}$, where the target points are indexed by Plücker coordinates Δ_I, $I \in \binom{[n]}{k}$.

These satisfy the **Plücker relations**: for $I, J \in \binom{[n]}{k}$, $r > 0$, $i_1, \ldots, i_r \in I$,

$$\Delta_I \Delta_J = \sum \pm \Delta_{I'} \Delta_{J'},$$

where the sum is over all $j_1, \ldots, j_r \in J$ and $I' = (I \backslash \{i_1, \ldots, i_r\}) \cup \{j_1, \ldots, j_r\}$ and $J' = (J \backslash \{j_1, \ldots, j_r\}) \cup \{i_1, \ldots, i_r\}$.

Consider the ideal $I_{kn} \subset \mathbb{C}[\Delta_I]$ generated by Plücker relations for $r = 1$, and let J_{kn} be the ideal generated by all of the Plücker relations (make the correct choice of signs, see above).

Proposition 3.1. $\varphi(\mathrm{Gr}(k, n))$ *in* $\mathbb{CP}^{\binom{n}{k}-1}$ *is the zero locus of* I_{kn}.

Proof left as an exercise. Not too hard; use induction.

Theorem 3.2 (Nullstellensatz)**.** *Let I be a (non-irrelevant) homogeneous ideal in* $\mathbb{C}[x_1, \ldots, x_N]$, $X \in \mathbb{CP}^{N-1}$ *be the zero locus of I, and J be the ideal of polynomials vanishing on X. Then* $J = \sqrt{I}$.

It can be proven that $\sqrt{I_{kn}} = J_{kn}$; need to check that J_{kn} is a radical ideal. The proof is in Fulton's book.

"Row picture" vs. "Column picture" in $\mathrm{Gr}(k, n)$. Let A be a $k \times n$ matrix of rank k. Thinking of the row space of A, this corresponds to a k-dimensional subspace of $\mathbb{C}^n$. Looking at the column space instead, a point of the Grassmanian corresponds to a collection of n vectors spanning $\mathbb{C}^k$ modulo a GL_k action.

A matroid captures the information of which sets of k vectors are linearly independent, i.e which Plücker coordinates are non-zero. That is, we have a matroid $\mathcal{M}$ corresponding to a point of $\mathrm{Gr}(k, n)$ is the set $\{I \in \binom{[n]}{k} \mid \Delta_I \neq 0\}$. Elements of $\mathcal{M}$ are called **bases**. The Plücker relations impose some conditions on such an object: namely that if the left hand side $\Delta_I \Delta_J$ is non-zero, then at least one of the terms on the right hand side $\sum \Delta'_I \Delta'_J$ must be non-zero. This motivates the following definition:

Definition 3.3. A non-empty subset $\mathcal{M}$ of $\binom{[n]}{k}$ is a **matroid of rank** k if it satisfies the **Exchange Axiom (E)**: $\forall I, J \in \mathcal{M}, \forall i \in I, \exists j \in J$ such that $(I \backslash \{i\}) \cup \{j\} \in \mathcal{M}$.

This only says that $\Delta_{I'} \neq 0$ on the right hand side of the Plücker Relation for $r = 1$. We can require that $\Delta_{J'} \neq 0$ as well:

A **Stronger Version (E')** of the Exchange Axiom: $\forall I, J \in \mathcal{M}, \forall i \in I, \exists j \in J$ such that $(I \backslash \{i\}) \cup \{j\} \in \mathcal{M}$ and $(J \backslash \{j\}) \cup \{i\} \in \mathcal{M}$.

We can require this to be true for all Plücker relations, not just $r = 1$, leading to:

Even Stronger Version (E"): $\forall I, J \in \mathcal{M}. \forall r > 0, \forall i_1, \ldots, i_r, \exists j_1, \ldots, j_r \in J$ such that $(I \backslash \{i_1, \ldots, i_r\}) \cup \{j_1, \ldots, j_r\} \in \mathcal{M}$ and $(J \backslash \{j_1, \ldots, j_r\}) \cup \{i_1, \ldots, i_r\} \in \mathcal{M}$.

Exercise 3.4. *Are (E), (E'), (E") equivalent?*

Definition 3.5. $\mathcal{M}$ is realizable (over $\mathbb{F}$) if it comes from a point of the Grassmannian $\mathrm{Gr}(k, n, \mathbb{F})$.

Example 3.6. Matroids of rank $k = 2$. The only relations between vectors you can get in 2-dimensional space come from parallel or zero vectors. Let $[n] = B_0 \coprod B_1 \coprod \cdots$, and take $\{i, j\}$ to be a base if i and j are in different blocks and are not in B_0 (B_0 corresponds to zero vectors). These turn out to be all rank 2 matroids, and all are realizable.

Becomes a mess in $k = 3$.

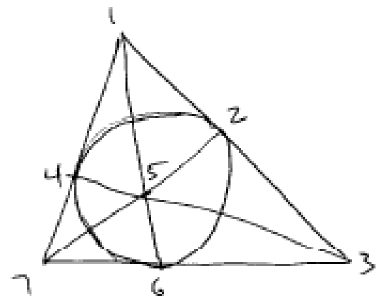

Figure 2: Fano plane

Example 3.7. Fano plane (see Figure 2). $\{i, j, k\}$ is a base if i, j, k are not on the same line (or circle). This forms a non-realizable matroid over $\mathbb{R}$ (it is realizable over $\mathbb{F}_2$ – take all non-zero vectors in $\mathbb{F}_2^3$).

Example 3.8. Pappus Theorem (see Figure 3); again $\{i, j, k\} \in \mathcal{M}$ is a base if i, j, k are not on the same line. This is a realizable matroid over $\mathbb{R}$. On the other hand, $\mathcal{M} + \{7, 8, 9\}$ is a non-realizable matroid over $\mathbb{R}$ - the fact that it is not realizable follows from Pappus.

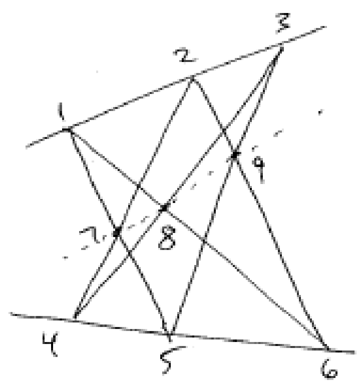

Figure 3: Pappus Theorem

Example 3.9. Desargues' Theorem (didn't bother with picture): Given ABC and $A'B'C'$ with AA', BB', CC' concurrent at P, then $X = BC \cap B'C', Y = CA \cap C'A', Z = AB \cap A'B'$ are collinear. The matroid $\mathcal{D} \subset \binom{[10]}{3}$ of all triples of points that do not lie on a line, plus $\{X, Y, Z\}$, is a non-realizable matroid.

Let's get back to Schubert cells. Recall

$$\mathrm{Gr}(k, n) = \coprod_{\lambda \subset k \times (n-k)} \Omega_\lambda,$$

with $\Lambda = \mathbb{F}^{|\lambda|}$. We can also index the cells by $I \in \binom{[n]}{k}$. We have a bijection between partitions $\lambda \subset k \times (n-k)$ and down-left lattice paths from the point $(n-k, k)$ by tracing the bottom and right edges of λ. Then, taking the set of down steps gives us a k-element subset of $[n]$.

3 definitions of Schubert cells.

Definition 3.10 (via Gaussian elimination, RREF)**.** Done last week. Note that one can extract $I \in \binom{[n]}{k}$ in bijection above by taking the column numbers of the pivots in the RREF of the point in the Grassmannian.

Consider the **Gale order** on $\binom{[n]}{k}$: given $I = \{i_1 < \cdots < i_k\}, J = \{j_1 < \cdots < j_k\}$, we have $I \leq J$ if $i_1 \leq j_1, \ldots, i_k \leq j_k$. Note that this agrees with the *reverse* of the partial order on Young diagrams by containment, using the bijection above. Also, note by row reduction that the matroid associated to a point of the Grassmannian has a unique Gale-minimal element.

Exercise 3.11. *The last sentence above is true for arbitrary matroids.*

Definition 3.12. $\Omega_I = \{A \in \mathrm{Gr}(k,n) \mid I \text{ Gale-minimal in the corresponding matroid}\}$.

It's easy to see that the first two definitions are equivalent.

Definition 3.13 (Classical)**.** Fix the complete flag of subspaces in $\mathbb{F}^n$, $0 \subset V_1 \subset \cdots \subset V_n = \mathbb{F}^n$, with $V_i = \{e_n, \ldots, e_{n-i+1}\}$ (this is the opposite convention from most of the literature). Now, for $\underline{d} = (d_1, \ldots, d_n) \in \mathbb{Z}^n_{\geq 0}$, define

$$\Gamma_{\underline{d}} = \{V \in \mathrm{Gr}(k,n) \mid \forall i, \dim(V \cap V_i) = d_i\}.$$

Need to require $0 \leq d_1 \leq \cdots \leq d_n = k$, and $d_i - d_{i-1} \geq \{0, 1\}$.

To show that the classical definition agrees with the previous one, consider an up-right lattice path from $(0,0)$ to $(n-k, k)$. Get $d_1, \ldots, d_n$ by starting with $d_0 = 0$, and adding 1 every time you go up, and 0 every time you go right. The rest is not hard using RREF. More on Schubert cells next time.

"Do you want a lot of homework, or not a lot of homework?"

4 September 12, 2014

Let's clarify why the last (classical) definition of Schubert cells agrees with the other two.

Given a full rank $k \times n$ matrix A in RREF, suppose the pivots lie in columns $i_1 \leq \cdots \leq i_k$; let I be the set of these indices. Let $u_1, \ldots, u_k$ denote the row vectors of A; the row space of A is the space of vectors of the form $\alpha_1 u_1 + \cdots + \alpha_k u_k$. Let p be such that $\alpha_p \neq 0$, but $\alpha_1, \ldots, \alpha_{p-1} = 0$. Then $u \in \langle e_{i_p}, e_{i_p+1}, \ldots, e_n \rangle$, but $u \notin \langle e_r, \ldots, e_n\}$ if $r > i_p$.

Thus $\mathrm{Rowspace}(A) \cap V_i$ is spanned by the u_j such that $i_j \in \{n, n-1, \ldots, n-i+1\}$, so $d_i = \dim \mathrm{Rowspace}(A) \cap V_i$, which is the size of $I \cap \{n, n-1, \ldots, n-i+1\}$.

Example 4.1. If $\lambda = (3,3,1) \subset 3 \times 4$ (so $n = 7, k = 3$), then $I = \{2,3,6\}$ and $\underline{d} = (0,1,1,1,2,3,3)$.

The classical definition only depends on the flag, not the coordinates.

We now define **Schubert varieties**; work over $\mathbb{C}$. $\overline{\Omega_\sigma}$ is the closure (in the usual topology of $\mathbb{C}^N$ – should also get the same thing in the Zariski topology) in $\mathrm{Gr}(k,n)$.

Theorem 4.2. *Three (clearly) equivalent formulations:*

(1) $\overline{\Omega_\sigma} = \coprod_{\mu \subseteq \lambda} \Omega_\mu$.

(2) $\overline{\Omega_\sigma} = \coprod_{K \geq I} \Gamma_K$*, where* $\geq$ *denotes the Gale order from last time.*

(3) $\overline{\Omega_\sigma} = \{A \in \mathrm{Gr}(k,n) \mid \Delta_J = 0$ *unless* $J \geq I\}$.

Proof. First show that $\overline{\Omega_\sigma} \subset \{A \in \mathrm{Gr}(k,n) \mid \Delta_J = 0 \text{ unless } J \geq I\}$; fix $A \in \overline{\Omega_\sigma}$. Then $A = \lim_{\epsilon\to 0} A(\epsilon)$ such that $A(\epsilon) \in \Gamma_I$. Then, for $\epsilon \neq 0$, $\Delta_J(A(\epsilon)) = 0$ unless $J \geq I$ in the Gale order, so the same is true for A, which is exactly what we need.

Conversely, suppose $A \in \Omega_K$ for $K \geq I$. Now let $A(\epsilon)$ be the sum of A and ϵA_I, where A_I has 1s in the positions a_{mi_m} and zeroes everywhere else. Then $A(\epsilon) \in \Gamma_I$ for all $\epsilon \neq 0$, and $A = \lim_{\epsilon \to 0} A(\epsilon)$. □

Corollary 4.3. *The following are equivalent:*

- $\overline{\Omega_I} \supseteq \overline{\Omega_J}$.
- $I \leq J$ *(in the Gale order).*
- $\lambda \supset \mu$*, where* $\lambda \leftrightarrow I$ *and* $\mu \leftrightarrow J$.

We also have a **Matroid stratification** of $\mathrm{Gr}(k,n)$. Define the matroid stratum $S_{\mathcal{M}} = \{A \in \mathrm{Gr}(k,n) \mid \Delta_I(A) \neq 0 \Leftrightarrow I \in \mathcal{M}$. Then $\mathrm{Gr}(k,n) = \coprod_{\mathcal{M}} S_{\mathcal{M}}$. This is a much finer stratification: instead of specifying the Gale-minimal non-zero Plücker coordinate, we specify all of the non-zero Plücker coordinates. The $S_{\mathcal{M}}$ are also called thin "cells" or Gelfand-Serganova "cells."

Theorem 4.4 (Mnëv Universality)**.** *$S_{\mathcal{M}}$ can be "as complicated as any algebraic variety." (In fact, this is also true in rank 3)*

Pick a permutation $w = w_1 \cdots w_n \in S_n$. Then define the permuted flag $V_\cdot^w = V_1^w \subset V_2^w \subset \cdots \subset V_n^w$, where $V_i^w = w(V_i) = \langle e_{w(n)}, e_{w(n-1)}, \ldots, e_{w(n-i+1)} \rangle$. Then, let $\Omega_\lambda^w = w(\Omega_\lambda)$, i.e. the Schubert cell defined by the flag $V_\cdot^w$.

Theorem 4.5 (Gelfand-Goresky-MacPherson-Serganova)**.** *The matroid decomposition is the common refinement of the* $n!$ *permuted Schubert decompositions* $\mathrm{Gr}(k,n) = \coprod_\lambda \Omega_\lambda^w$.

Proof. Should be obvious: the matroid strata tell you which Plücker coordinates are zero and non-zero, which is the same as knowing the Gale-minimal (equivalently, lex-minimal) Plücker coordinate for any ordering of the column vectors. □

The proof is transparent because we've fixed a bunch of coordinate vectors (the things we're permuting), but this is less clear if we only make reference to a flag without a basis.

3 definitions of matroids.

Definition 4.6. Using Exchange Axiom (already done).

Given $w \in S_n$, re-order $[n]$ by $w(1) <_w \cdots <_w w(n)$ – this induces the **permuted Gale order** $I \leq_w J$.

Theorem 4.7. *$\mathcal{M}$ is a matroid iff it has a unique minimal element under $\leq_w$ for any $w \in S_n$.*

Definition 4.8. Given $I \in \binom{[n]}{k}$, define the **Schubert matroid** (or **lattice path matroid**) $\mathcal{M}_I = \{J \in \binom{[n]}{k} \mid J \geq I\}$.

In terms of lattice paths, think of I as a lattice path, then take all lattice paths J to the southeast of I.

Theorem 4.9. *Any matroid is an intersection of some permuted Schubert matroids* $\bigcap w(\mathcal{M}_{I_w})$. *(Warning: not every intersection of matroids is a matroid! This will be clear in the* $k = 2.n = 4$ *case below.)*

Example 4.10. Taking an intersection of a Schubert matroid $\mathcal{M}_I$ and a permuted Schubert matroid $w(\mathcal{M}_J))$ with the reverse order on coordinates (i.e. $w(1) = n, w(2) = n-1, \dots$) is the same as taking all lattice paths fitting below I and above J. (These might be called Richardson matroids?)

Given any $I \in \binom{[n]}{k}$, let e_I be the 0-1 vector $\sum_{i \in I} e_i$. Then, given $\mathcal{M} = \binom{[n]}{k}$, define the polytope $P_{\mathcal{M}}$ to be the convex hull of the e_I for $I \subset \mathcal{M}$.

Example 4.11. Take $k = 2, n = 4$. The six vectors e_I with $I \in \binom{[4]}{2}$ form an octahedron; opposite vertices correspond to sets that are complements of each other. A subset of these vertices forms a matroid iff all edges of the convex hull of these vertices are already edges of the octahedron. (This is true in general.)

5 September 17, 2014

We alluded to the following last time:

Theorem 5.1 (GGMS). *Given* $\mathcal{M} \subset \binom{[n]}{k}$, $\mathcal{M}$ *is a matroid iff any edge of the convex hull* $P_{\mathcal{M}}$ *of the vectors* $e_I = \sum_{i \in I} e_i$ *looks like* $[e_I, e_J]$, *where* $J = (I \backslash \{i\}) \cup \{j\}$. *Such polytopes are called* ***matroid polytopes***.

Note that if $P_{\mathcal{M}}$ contains one pair of opposite vertices, then it should contain another pair. This can be seen from the Plücker relations, e.g. for $k = 2, n = 4$

Torus action on $\mathrm{Gr}(k, n, \mathbb{C})$. The complex torus $T = (\mathbb{C} \backslash \{0\})^n$ acts on $\mathbb{C}^n$ by rescaling: $(t_1, \dots, t_n)$ sends $(x_1, \dots, x_n)$ to $(t_1 x_1, \dots, t_n x_n)$. This gives an action on subspaces, and in particular on $\mathrm{Gr}(k, n)$. Thinking of $\mathrm{Gr}(k, n) \backslash \mathrm{Mat}^*(k, n)$, T acts by right multiplication by the diagonal matrix $(t_1, \dots, t_n)$. In terms of Plücker coordinates, $\Delta_i \mapsto (\prod_{i \in I} t_i) \Delta_I$ under the action by $(t_1, \dots, t_n)$.

What are the fixed points of this action? This is equivalent to having exactly one non-zero Plücker coordinate. Also, for any one-dimensional orbit of T, there are exactly two non-zero Δ_I, Δ_J, and furthermore $J = (I \backslash \{i\}) \cup \{j\}$.

Fix $A \in \mathrm{Gr}(k, n)$. Then the orbit $T \cdot A \in \mathrm{Gr}(k, n)$ (where $(t_1, \dots, t_n)A = A \,\mathrm{diag}(t_1, \dots, t_n)$) is some quasi-projective subvariety of $\mathbb{CP}^n$. What is its degree (more accurately, degree of its closure), i.e. the number of intersection points of X with a generic linear subspace of complimentary dimension? (The dimension of $T \cdot A$ should be equal to the dimension of the corresponding matroid polytope.)

Example 5.2. In $\mathrm{Gr}(2, 4)$, fix a (sufficiently general) point $A = (\Delta_{12} : \cdots : \Delta_{34}) \in \mathbb{CP}^5$. $(t_1, \dots, t_4)$ acts by multiplication by $(t_1 t_2, \dots, t_3, t_4)$: the orbit cuts out a 3-dimensional subvariety of $\mathbb{CP}^5$ (note that simultaneously the t_i produces the same point). Now, intersect $T \cdot A$ with 3 generic linear equations:

$$\alpha_1 \Delta_{12} t_1 t_2 + \cdots + \alpha_6 \Delta_{34} t_3 t_4 = 0$$
$$\beta_1 \Delta_{12} t_1 t_2 + \cdots + \beta_6 \Delta_{34} t_3 t_4 = 0$$
$$\gamma_1 \Delta_{12} t_1 t_2 + \cdots + \gamma_6 \Delta_{34} t_3 t_4 = 0.$$

Now apply the **Bernstein-Kushnirenko Theorem**: Given finite sets $A_1, \ldots, A_m \subset \mathbb{Z}^m$, denote $x^a = x_1^{a_1} \cdots x_m^{a_m}$ for $a \in \mathbb{Z}^m$. Also, let P_i be the convex hull of the A_i (Newton polytope). Then, consider the system of m equations in m variables $\sum_{a \in A_i} c_{i,a} x^a$ for $i = 1, 2, \ldots, m$ with generic coefficients $c_{i,a} \in \mathbb{C}$. The number of solutions to this system of equations in $(\mathbb{C}^*)^m$ is equal to the *mixed volume* of $P_1, \ldots, P_m$ (we won't define this). In the special case $P_1 = \cdots = P_m = P$, the number of solutions is $m! \cdot \mathrm{Vol}(P)$, the **normalized volume** of P.

In the case of the system above, the Newton polytope P is exactly the matroid polytope, defined by when the Plücker coordinates are zero vs. non-zero. Hence the degree of the torus orbit is equal to the the normalized volume of $P_{\mathcal{M}}$.

Consider the case $\mathcal{M} = \binom{[n]}{k}$, corresponding to the hypersimplex $\Delta_{kn} = P_{\binom{[n]}{k}} = [0,1]^n \cap \{x_1 + \cdots + x_n = k\}$ (warning: we're using Δ to use something different than before). This is the same as $\widetilde{\Delta}_{k,n-1} = [0,1]^{n-1} \cap \{k-1 \leq x_1 + \cdots + x_{n-1} \leq k\}$, via the map $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1})$.

Theorem 5.3 (Laplace)**.** *The normalized volume of Δ_{kn}, $(n-1)!\,\mathrm{Vol}_{n-1} \widetilde{\Delta}_{k,n-1}$, is equal to the **Eulerian number** $A_{k-1,n-1}$, where $A_{k,n} = [x^{k+1}](1-x)^{n+1} \sum_{r=0}^{\infty} r^n x^r$ ($A_{kn} = 0$ is $k \geq n$). Equivalently, $A_{k,n}$ is the number of permutations $w \in S_n$ with exactly k descents.*

To prove that these two definitions of Eulerian numbers are equivalent, show that they satisfy the recurrence relation $A_{k,n} = (n-k)A_{k-1,n-1} + (k+1)A_{k,n-1}$.

One way to prove Laplace's Theorem is as follows: express the volume of a section $k - 1 \leq x_1 + \cdots + x_n \leq k \cap [0,1]^n$ by as an alternating sum of sections of quadrants.

6 September 24, 2014

Lots of facts about cohomology. We're not going to prove them, and we're not even going to define cohomology.

X a topological space. To it, we associate to it a **cohomology ring** $H^*(X) = H^0(X) \oplus H^1(X) \oplus \cdots$. All coefficients are in $\mathbb{C}$, so the H^i are vector spaces over $\mathbb{C}$. There ring structure comes from the **cup product** $H^p(X) \otimes H^q(X) \to H^{p+q}(X)$. The dimensions β_i of the H^i called **Betti numbers**.

Let X be a smooth complex n-dimensional projective variety with finite cell decomposition $X = \coprod X_i$, such that $X_i \cong \mathbb{C}^{n_i}$ are locally closed subvarieties, and $\overline{X_i} - X_i$ is a union of lower-dimensional cells. Then, the cohomology X lives in even dimension, i.e. $H^* = H^0 \oplus H^2 \oplus \cdots \oplus H^{2n}$. Also, we have **Poincaré Duality**: $H^{2p}(X) \cong H_{2n-2p}(X) \cong (H^{2n-2p}(X))^*$. In particular, the Betti numbers are symmetric. Have **fundamental classes** $[\overline{X_i}] \in H^{2(n-n_i)}(X)$, which form a linear basis of $H^*(X)$.

For $\mathrm{Gr}(k,n) = \coprod_{\lambda \subset k \times (n-k)} \Omega_\lambda$, define **Schubert varieties** $X_\lambda = \overline{\Omega_\lambda}$, and the **Schubert classes** $\sigma_\lambda = [X_\lambda^\vee]$, where $\lambda^\vee$ is the complement of λ in a $k \times (n-k)$ rectangle. From the above, $\sigma_\lambda \in H^{2|\lambda|}(\mathrm{Gr}(k,n))$ form a linear basis of $H^*(\mathrm{Gr}(k,n))$.

Let Y, Y' be subvarieties of X, such that $Y \cup Y' = Z_1 \cup Z_2 \cup \cdots \cup Z_r$. Assume that $\mathrm{codim}(Y) + \mathrm{codim}(Y') = \mathrm{codim}(Z_i)$ for all i, and that Y, Y' intersect transversely. Then, $[Y][Y'] = [Z_1] + \cdots + [Z_r]$. In particular, if $\mathrm{codim}(Y) + \mathrm{codim}(Y') = \dim(X)$, then the Z_is are points, so $[Y] \cdot [Y'] = r[*]$. The integer r is called the **intersection number** of Y and Y', and is the same as the value $\langle [Y], [Y'] \rangle$ obtained from the Poincaré pairing.

Similarly, we can talk about intersection numbers of 3 subvarieties Y, Y', Y'' such that $\mathrm{codim}(Y) + \mathrm{codim}(Y') + \mathrm{codim}(Y'') = \dim(X)$ intersecting in r points, so that $\langle [Y], [Y'], [Y''] \rangle = r$. Fact:

$$[\overline{Y_i}] \cdot [\overline{Y_j'}] = \sum_k \langle [\overline{Y_i}], [\overline{Y_j}], [\overline{Y_k}] \rangle \cdot [\overline{Y_k}]^*,$$

where the $[\overline{Y_k}]^*$ are dual basis elements. So to understand H^*, we only need to understand the double and triple intersection numbers.

Let's specialize to the Grassmannian.

Theorem 6.1 (Duality). *If $|\lambda| + |\mu| = k(n-k)$, then $\langle \sigma_\lambda, \sigma_\mu \rangle = \delta_{\lambda,\mu^\vee}$.*

In other words, the basis $\{\sigma_\lambda\}$ is self-dual with respect to the Poincaré pairing.

Theorem 6.2 (Pieri's Formula). *Let σ_r be the Schubert class corresponding the partition (r) (Young diagram with one row of size r). Then*

$$\sigma_\lambda \sigma_r = \sum_\mu \sigma_\mu,$$

*where the sum is over all μ such that μ/λ is a **horizontal r-strip**, i.e. $|\mu| - |\lambda| = r$ and each column of μ/λ has at most 1 box. Equivalently, $\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots \geq \mu_k \geq \lambda_k$, and $r = (\mu_1 - \lambda_1) + \cdots + (\mu_k - \lambda_k)$.*

The problem with intersecting X_λ and X_μ is that they don't intersect tranversely: for example, if $\lambda \subset \mu$, one Schubert variety is contained in the closure of the other. Instead, look at $X_\lambda \cap g(X_\mu)$, where g is a generic element of GL_n (presumably $g(X_\mu)$ and X_μ are in the same cohomology class?). In other words, take two different Schubert decompositions of $\mathrm{Gr}(k,n)$ whose flags are in "general position" to each other (this is much harder to do for triple intersections).

Let $V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}$ and $\widetilde{V}_1 \subset \widetilde{V}_2 \subset \cdots \subset \widetilde{V}_n = \mathbb{C}$ be two flags. Let $r_{ij} = \dim(V_i \cap \widetilde{V}_j)$, and take the $n \times n$ **rank matrix** of r_ij. It turns out there are $n!$ such matrices: to write them down, consider the $n!$ rook placements on an $n \times n$ chessboard, and let r_{ij} be the number of rooks in top-left $i \times j$ sub-board. (Side-remark: this essentially gives the Bruhat order.)

If we take $V_1 \subset V_2 \subset \cdots \subset V_n$ to be the standard coordinate flag and $\widetilde{V}_1 \subset \widetilde{V}_2 \subset \cdots \subset \widetilde{V}_n$ to be the opposite coordinate flag, then the flags intersect generically (the r_ij are all the expected generic sizes). If you're in the class and are reading this, I owe you a beer, three people maximum. Explicitly, $V_i = \langle e_n, \ldots, e_{n-i+1} \rangle$ and $\widetilde{V}_i = \langle e_1, e_2, \ldots, e_i \rangle$.

Lemma 6.3. *We have:*

1. *$\Omega_\lambda \cap \widetilde{\Omega}_\mu \neq 0$ if and only if $\lambda \supseteq \mu^\vee$ (equivalently, $\mu^\vee$ and $\lambda^\vee$ don't overlap).*
2. *$\Omega_\lambda \cap \widetilde{\Omega}_{\lambda^\vee}$ is a single point and the intersection is transversal.*

Proof. $\Omega_\lambda \cap \widetilde{\Omega}_\mu \neq 0$ means that there exists a $k \times (n-k)$ matrix A such that row reduction puts the pivots in positions $i_1 < i_2 < \cdots < i_k$ (corresponding to the partition λ), and "reverse row operations," i.e. going from right to left, puts the pivots in positions $j_1 < j_2 < \cdots < j_k$ (corresponding to $\mu^\vee$), and furthermore $i_1 \leq j_1, \ldots, i_k \leq j_k$. Using the Gale order, this is equivalent to $\lambda \supset \mu^\vee$. For the converse, it turns out that after row operations A can be transformed into a matrix with left pivots $i_1, \ldots, i_k$, right pivots are $j_1, \ldots, j_k$, and zeroes to the left of all left pivots and to the right of all right pivots.

When $i_1 = j_1, \ldots, i_k = j_k$, there is only one element in the intersection: the matrix with pivots in positions $i_1 < i_2 < \cdots < i_k$ and zeroes everywhere else. □

7 September 26, 2014

We did a lot of handwaving last time, so let's summarize what we know about the cohomology ring $H^*(\mathrm{Gr}(k,n,\mathbb{C}))$.

- Commutative (usually, the cohomology ring is only commutative up to sign, but note that cohomology lives only in even dimensions), associative algebra over $\mathbb{C}$.
- Graded algebra $H^0 \oplus H^2 \oplus \cdots \oplus H^{2k(n-k)}$.
- Has a linear basis of Schubert classes $\sigma_\lambda \in H^{2|\lambda|}$
- Given $\lambda_1, \ldots, \lambda_r$, with $|\lambda_1| + \cdots + |\lambda_r| = k(n-k)$ the intersection numbers $\langle \sigma_{\lambda_1} \cdot \sigma_{\lambda_2} \cdots \sigma_{\lambda_r} \rangle$ are equal to the coefficient of $\sigma_{k(n-k)}$ in the product $\sigma_{\lambda_1} \cdot \sigma_{\lambda_2} \cdots \sigma_{\lambda_r}$; geometrically, this is the number of intersection points in generic translates of the Schubert varieties $X_{\lambda_1^\vee}, \ldots, X_{\lambda_r^\vee}$.
- (Duality theorem) $\langle \sigma_\lambda, \sigma_\mu \rangle = \delta_{\lambda, \mu^\vee}$.

We have **Littlewood-Richardson coefficients** $c^\nu_{\lambda\mu}$ of $H^*(\mathrm{Gr}(k,n))$:

$$\sigma_\lambda \sigma_\mu = \sum c^\nu_{\lambda\mu} \sigma_\nu.$$

These are just triple intersection numbers, because

$$\langle (\sigma_\lambda \sigma_\mu) \cdot \sigma_\nu \rangle = \langle \left(\sum_\gamma c^\gamma_{\lambda\mu} \sigma_\mu \right), \sigma_\nu \rangle = c^{\nu^\vee}_{\lambda\mu}.$$

Define $c_{\lambda\mu\nu} = c^{\nu^\vee}_{\lambda\mu}$. Because these are intersection numbers, they are non-negative integers, we want a combinatorial interpretation.

We cheated last time in proving the duality theorem, so let's do it over.

Lemma 7.1. *We have:*

1. $X_\lambda \cap \widetilde{X}_\mu \neq \emptyset$ *if and only if* $\lambda \supseteq \mu^\vee$ *(here* X_λ *is a Schubert variety and* $\widetilde{X}_\mu$ *is an opposite Schubert variety).*
2. *If* $|\lambda| + |\mu| = k(n-k)$*, then* $X_\lambda \cap \widetilde{X}_\mu \neq \emptyset$ *is a single point if* $\lambda = \mu^\vee$ *and empty otherwise.*

Proof. Let $X_I = X_\mu$, where $I \in \binom{[n]}{k}$ is the set of left steps along the border of λ (walking from the top right corner of the $k(n-k)$ rectangle to the bottom left). The first claim is equivalent to $X_I \cap \widetilde{X}_J \neq \emptyset$ iff $I \leq J^\vee$ in the Gale order, where $J^\vee = \{n+1-j_k, \ldots, n+1-j_1\}$. Given $A \in \mathrm{Gr}(k,n)$, let $\mathcal{M}_A$ be the corresponding matroid. $A \in X_I$ iff $I \leq K$ for all $K \in \mathcal{M}_A$, and $A \in \widetilde{X}_J$ iff $J^\vee \geq K$ for all $K \in \mathcal{M}_A$, so $I \leq J^\vee$. The second part now follows because if $I = J^\vee$, then $\mathcal{M}_A$ has only one element, and only one point of the Grassmannian corresponds to such a matroid. ∎

Last time we claimed that we can apply row operations to our $k \times (n-k)$ matrix A so that the left and right pivots are simultaneously in the right places (according to I and $J^\vee$). This is actually false, consider the example

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

which can't be put in the form

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

via row operations. The issue is that we are intersecting closures of Schubert cells, not the Schubert cells themselves; there will be a homework problem about this.

Theorem 7.2 (Pieri Formula)**.** *Let r be the one-part partition (r). Then*

$$\sigma_\lambda \sigma_r = \sigma_\mu \sigma_\mu,$$

where the sum on the right hand side is over all μ such that μ/λ is a horizontal strip.

The idea of the proof is to compute intersection numbers $\langle \sigma_\lambda, \sigma_{\mu^\vee}, \sigma_r \rangle$. This is not so obvious (as is the case in general with triple intersection numbers) because it's hard to write down three pairwise generically intersecting flags.

Example 7.3. What is $f_{kn} = \langle \sigma_1 \cdot \sigma_1 \cdots \sigma_1 \rangle$? We have $X_{(1)^\vee} = \{V \in \mathrm{Gr}(k,n) \mid V \cap \langle e_n, e_{n-1}, \ldots, e_{k+1} \rangle \neq 0\}$. Fix $N = k(n-k)$ generic $(n-k)$-dimensional subspaces in $\mathbb{C}^n$, $L_1, \ldots, L_N$. Then, f_{kn} is the number of k-dimensional subspaces intersecting non-trivially with all of the L_I.

Theorem 7.4. $f_{1n} = 1$ *and* $f_{2n} = C_{n-2} = \frac{1}{n-1}\binom{2(n-2)}{n-2}$.

Proof. The first part is easy. For the second part, by Pieri we have $\sigma_\lambda \sigma_1 = \sum_\mu \sigma_\mu$, where the sum over μ obtained by adding a box to λ. Thus $\sigma_\emptyset \sigma_1^n$ is

$$\sum_{\lambda : |\lambda| = m} f_\lambda \sigma_\lambda,$$

where f_λ is the number of ways to build λ one box at a time, i.e. the number of standard Young Tableaux of shape λ.

Now, f_{1n} is just $f_{(n-1)}$ and f_{2n} is just $f_{(n-2,n-2)}$, immediately implying the conclusion (for $f_{(nn)}$, there's an easy bijection to Dyck paths) □

There's a general formula for f_λ, the **hook length formula**. Namely,

$$f_\lambda = \frac{m!}{\prod_{x \in \lambda} h(x)},$$

where $m = |\lambda|$ and $h(x)$ is the number of boxes in the hook associated to a box x, i.e. the set of boxes below x in the same column or to the right of x in the same row.

Example 7.5. $k = 3, n = 6$: the number of 3-dimensional subspaces of $\mathbb{C}^6$ intersecting 9 generic 3-dimensional subspaces is $9!/(1 \cdot 2^2 \cdot 3^3 \cdot 4^2 \cdot 5)$.

The next goal is to understand the Littlewood-Richardson coefficients $c^\nu_{\lambda\mu}$. They arise in:

- $H^*(\mathrm{Gr}(k,n))$ (we've already seen this)
- products of Schur polynomials: $s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu$
- tensor products of irreps of SL_n: $V_\mu \otimes V_\mu = \oplus c^\nu_{\lambda\mu} V_\nu$
- irreps of S_n: $\mathrm{Ind}^{S_{m+n}}_{S_m \times S_n}(\pi_\lambda \otimes \pi_\mu) = \oplus_{nu} c^\nu_{\mu\nu} \pi_\nu$.

Let's see how much about symmetric functions we can cram into a third of a lecture.

Definition 7.6. The ring of **symmetric polynomials** is $\Lambda_n = \mathbb{C}[x_1, \ldots, x_n]^{S_n} = \sum \Lambda_n^d$, i.e. the ring of invariants under the usual S_n-action. It is somewhat more convenient to work in infinitely many variables, so define Λ^d to be the inverse limit of $\Lambda_1^d \leftarrow \Lambda_2^d \leftarrow \Lambda_3^d \leftarrow \cdots$, where the maps are evaluation of the last variable at zero. Explicitly, this is the ring of degree d power series invariant under any finite permutation of the variables. Then, define the ring of **symmetric functions** to be the direct sum of the Λ^d for $d = 0, 1, 2, \ldots$.

Example 7.7.

- Elementary symmetric functions $\sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}$
- Complete homogeneous symmetric functions $\sum_{i_1 \leq \cdots \leq i_k} x_{i_1} \cdots x_{i_k}$
- Power sum symmetric functions $p_k = \sum_i x_i^k$
- Monomial symmetric functions: m_λ is the symmetrization of the monomial $x_1^{\lambda_1} \cdots x_k^{\lambda_k}$. Note that $m_{(1,1\ldots,1)} = e_k$, $m_k = p_k$.

Theorem 7.8 (Fundamental Theorem)**.** *Two versions:*

1. $\Lambda_n = \mathbb{C}[e_1, \ldots, e_n] = \mathbb{C}[h_1, \ldots, h_n] = \mathbb{C}[p_1, \ldots, p_n]$ *with no relations.*
2. $\Lambda = \mathbb{C}[e_1, e_2, \ldots] = \mathbb{C}[h_1, h_2, \ldots] = \mathbb{C}[p_1, p_2, \ldots]$ *with no relations.*

We have several linear bases of Λ: $\{m_\lambda\}$ clearly form a basis. From the fundamental theorem, products of elementary symmetric functions $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_k}$ form a basis; similarly can define bases h_λ, p_λ. (If you work over $\mathbb{Z}$, the p_λ don't form a basis).

Proof of Theorem 7.8. We'll just show the first part (elementary symmetric functions generate Λ freely). The leading coefficient (using lex order) of x_λ is $x^{\lambda'}$, where λ' is the conjugate of λ. It follows that the e_λ are related to the m_λ via an upper triangular matrix, hence we get a basis. □

Definition 7.9. Given $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$, with $\gamma_1 > \cdots > \gamma_n$, define

$$a_\gamma(x_1, \ldots, x_n) = \sum_{w \in S_n} (-1)^w x_{w(1)}^{\gamma_1} \cdots x_{w(n)}^{\gamma_n},$$

where $(-1)^w$ is the sign of w. In particular, when $\delta = (n-1, n-2, \ldots, 0)$, a_δ is the Vandermonde determinant $\prod_{i<j}(x_i - x_j)$. For any λ, define the **Schur polynomial** $s_\lambda(x_1, \ldots, x_n) = a_{\lambda+\delta}/a_\delta$, and the **Schur function** $\lim_{n\to\infty} s_\lambda(x_1, \ldots, x_n)$.

These form linear bases of Λ_n and Λ, and also satisfy the Pieri Formulas:

Theorem 7.10 (Pieri Formulas)**.** *We have*

1. $s_\lambda h_r = \sum_\mu s_\mu$, *where the sum is taken over all* μ *for which* μ/λ *is a horizontal* r*-strip (note that* $h_r = s_{(r)}$*).*
2. $s_\lambda e_r = \sum_\mu s_\mu$, *where the sum is taken over all* μ *for which* μ/λ *is a vertical* r*-strip (note that* $e_r = s_{(1,\ldots,1)}$*).*

Proof. It suffices to do this for the a_λ; this is not too hard. $\square$

Easy but important fact:

Lemma 7.11. *If A is any associative $\mathbb{C}$-algebra with linear basis v_λ such that $v_\lambda v_r$ satisfying the Pieri formula, then $A \cong \Gamma$, via the map $v_\lambda \mapsto s_\lambda$*

8 October 1, 2014

Let A be the $k \times n$ Vandermonde matrix

$$\begin{bmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1^0 \\ x_2^{n-1} & x_2^{n-2} & \cdots & x_2^0 \\ \vdots & \vdots & \ddots & \vdots \\ x_k^{n-1} & x_k^{n-2} & \cdots & x_k^0 \end{bmatrix}.$$

Let $\lambda \subset k \times (n-k)$, so that $I(\lambda) = \{n-k+1-\lambda_1, n-k+2-\lambda_2, \ldots, n-\lambda_k\}$. Classically, we define the **Schubert polynomials** $s_\lambda(x_1, \ldots, x_k) = \frac{\Delta_{I(\lambda)}(A)}{\Delta_{I(\emptyset)}(A)}$. The Plücker relations give us some information:

$$\Delta_{13}\Delta_{24} = \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23} \Leftrightarrow s_{21}s_1 = s_{22}s_\emptyset + s_2 s_{11}.$$

We then define the **Schur functions** $s_\lambda(x_1, x_2, \ldots) = \lim_{k\to\infty} s_\lambda(x_1, \ldots, x_k)$. This makes sense because of the following stability condition: $s_\lambda(x_1, \ldots, x_k, 0) = s_\lambda(x_1, \ldots, x_k)$.

Recall:

Lemma 8.1. *Let $\mathcal{A}$ be a $\mathbb{C}$-algebra with linear basis v_λ satisfying the Pieri formula, i.e.*

$$v_\lambda v_r = \sum_\mu v_\mu,$$

where we sum over all μ for which μ/λ is a horizontal strip. Then $\Lambda \cong A$ via the isomorphism $i : s_\lambda \mapsto v_\lambda$.

Proof. Clearly i is an isomorphism of vector spaces; need $i(f \cdot g) = i(f) \cdot i(g)$. Pieri says $i(fh_r) = i(f)i(h_r)$ (note that $h_r = s_r$), and moreover the h_r generate Λ freely, so we are done. $\square$

Lemma 8.2. *$I_{kn} = \langle s_\lambda \mid \lambda \not\subseteq k(n-k)\rangle$ is an ideal in Λ.*

Proof. Follows from Pieri. $\square$

Exercise 8.3. $\Lambda_{kn} = \Lambda/I_{kn} = \cong \mathbb{C}[x_1, \ldots, x_k]^{S_k}/\langle h_{n-k+1}, h_{n-k+2}, \ldots, h_n\rangle$.

Corollary 8.4. *If $\mathcal{A}_{kn}$ is an algebra with basis $\langle v_\lambda \mid \lambda \subset k \times (n-k)\rangle$ satisfying Pieri, then $\mathcal{A}_{kn} \cong \Lambda_{kn}$. Hence $H^*(\mathrm{Gr}(k, n)) = \Lambda_{kn}$.*

We will now state a version of the Littlewood-Richardson rule, but we need to introduce web diagram first. Fix a horizontal line, say the x-axis. then, we associate a "left particle" to each integer n on the x-axis: this particle approaches n at a 60-degree angle from the northwest. Similarly, "right particles" associated to an integer n approaches n at a 60-degree angle from the northeast. A left and right particle then "interact as follows": at some height $h\sqrt{3}/2$, the two particles switch positions, then continue in the same direction that they were going. If the left and right particles

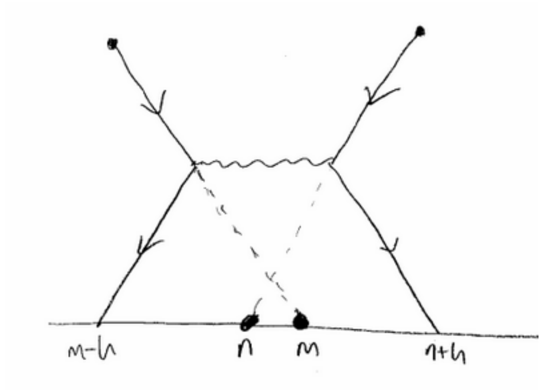

Figure 4: Interaction of left and right particle

were originally going toward the integers m, n, respectively, on the x-axis, then for some h, they land at $n+h, m-h$.

Given partitions λ, μ, consider left particles associated to $\lambda_1, \lambda_2, \ldots, \lambda_m$ and right particles associated to $\mu_1, \mu+2, \ldots, \mu_n$. Suppose every left particle interacts with every right particle exactly once, and suppose that the landing positions are $\nu_1, \nu_2, \ldots, \nu_{m+n}$. Then, the diagram obtained is called a **web diagram** with λ, μ, ν.

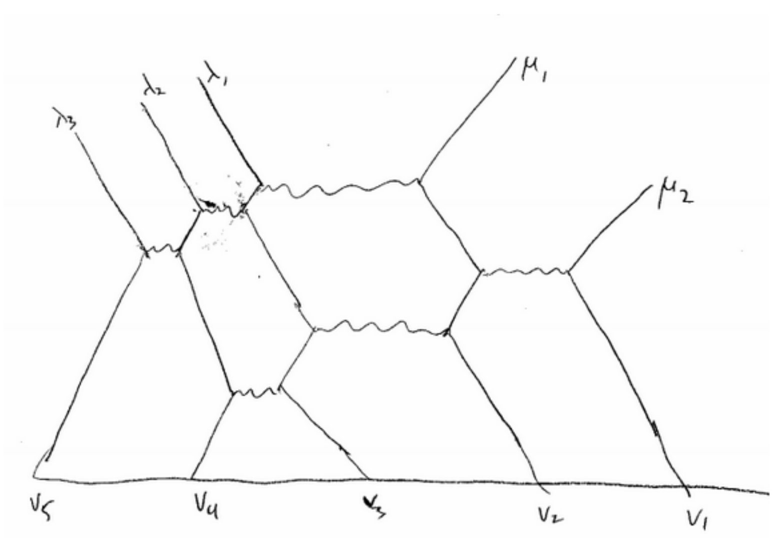

Figure 5: Web diagram (μ_1, μ_2 should be swapped above)

Theorem 8.5 (Littlewood-Richardson rule, GP version)**.** *$s_\lambda s_\mu = \sum c^\nu_{\lambda\mu} s_\nu$, where $c^\nu_{\lambda\mu}$ is the number of web diagrams with λ, μ, ν.*

Example 8.6. $m = n = 1, a \leq b$: $s_a s_b = s_{(b,a)} + s_{(b+1,a-1)} + \cdots + s_{(a+b,0)}$ (we declare $s_\lambda = 0$ if there is some negative part in λ).

Pieri rule for web diagrams. Consider $s_\lambda s_r$; say $|\lambda| = m$, the left particle corresponding to λ_i lands at μ_i, and the right particle starting at r ends at μ_{m+1}. Then $\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots \geq$

μ_{m+1}. Let ρ_i be the length of the interaction between r and λ_i. Then $\mu_{m+1} = r - \rho_1 - \cdots - \rho_m$, so $|\mu| - |\lambda| = r$, and μ/λ is a horizontal r-strip. So if we can show that the multiplication defined by web diagrams is associative, we will have established the L-R rule by Lemma [**?**]. This is not so easy (do something with local transformations of web diagrams) – we will (might?) do this at some point later.

Definition 8.7 (Combinatorial definition of Schur functions)**.** Let T be a SSYT of shape λ and weight $\beta = (\beta_1, \beta_2, \ldots)$, where β_i is the number of appearances of i in the SSYT. The **Kostka number** $K_{\lambda\beta}$ is the number of SSYTs of shape λ and weight β. Then,

$$s_\lambda = \sum_{sh(T)=\lambda} x^{wt(T)} = \sum_\beta K_{\lambda\beta} x^\beta.$$

It's not immediately obvious that s_λ is a symmetric function from this definition, but one can prove this combinatorially.

By the Pieri formula, $h_\mu = s_\emptyset h_{\mu_1} \cdots h_{\mu_\ell} = \sum_\lambda K_{\lambda\mu} s_\mu$ (using the classical definition of s_λ. The combinatorial definition says $s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$.

We have a scalar product $\langle -, - \rangle$ on Λ such that the s_λ form an orthonormal basis, i.e. $\langle s_\lambda, s\mu \rangle = \delta_{\lambda\mu}$. Then, the h_λ, m_μ are dual with respect to the scalar product; this follows from:

Theorem 8.8 (Cauchy formula)**.** *We have*

$$\prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_\lambda s_\lambda(x_1, x_2, \ldots) s_\lambda(y_1, y_2, \ldots) = \sum_\lambda h_\lambda(x_1, x_2, \ldots) m_\lambda(y_1, y_2, \ldots).$$

Here, we are using the classical definition of Schur functions.

Proof. The first and third expression are equal by expanding. We prove that the first two are equal. Let A be the $k \times \infty$ matrix (x_i^j), and B the $k \times \infty$ matrix (y_i^j). The **Cauchy-Binet formula** says

$$\det(AB^T) = \sum_{|I|=k} \Delta_I(A)\Delta_I(B).$$

Here, AB^T is the matrix $((1 - x_i y_j)^{-1})_{i,j=1}^k$, and Cauchy-Binet says its determinant is

$$\left(\prod_{i<j} (x_i - x_j)(y_i - y_j) \right) \sum_\lambda s_\lambda(x) s_\lambda(y).$$

So it suffices to show

$$\det((1 - x_i y_j)^{-1})_{i,j=1}^k = \prod_{i<j} (x_i - x_j)(y_i - y_j) \prod_{i,j=1}^k \frac{1}{1 - x_i y_j}.$$

Left as an exercise. □

We can define **Skew Schur functions** in two ways:

- (Classical) $\langle s_{\lambda/\mu}, f \rangle = \langle s_\lambda, s_\mu f \rangle$ for all $f \in \Lambda$.
- (Combinatorial) Same as for s_λ: sum over SSYT of shape λ/μ.

Ways to think about L-R coefficients:

1. $\sigma_\lambda \sigma_\mu = \sum c_{\lambda\mu}^\nu \sigma_\nu$. (Schubert classes)
2. $s_\lambda s_\mu = \sum c_{\lambda\mu}^\nu s_\lambda$. (Schur functions)
3. $s_{\lambda/\mu} = \sum c_{\mu\nu}^\lambda s_\mu$ (Skew Schur functions; immediate from classical definition).

9 October 3, 2014

Homework: prove as many non-trivial unproven statements from class as you can. Due two weeks from now. The lecturer may or may not compile a list of these at some point in the next two weeks.

Classical L-R rule.

Definition 9.1. $w = w_1 \cdots w_r \in \mathbb{Z}_{>0}^r$ is a **lattice word** if for all initial subwords $w_1 w_2 \cdots w_k$, the number of appearances of i is at least the number of appearances of j whenever $i \geq j$.

Given a SSYT T of a skew shape, define the word $w(T)$ is obtained by Hebrew reading of its entries, i.e. read from top to bottom, right to left.

Definition 9.2. A SSYT is an LR-tableau if $w(T)$ is a lattice word.

Theorem 9.3 (Classical L-R)**.** *$c_{\lambda\mu}^{\nu}$ is the number of LR-tableaux of shape ν/λ and weight μ.*

The L-R coefficients have symmetries:

1. S_3-symmetry: $c_{\lambda\mu\nu} = c_{\lambda\mu}^{\nu^\vee}$ is invariant under any permutation of $\lambda, \mu, \nu \subset k \times (n-k)$. This follows from Poincaré Duality.

2. Conjugation: $c_{\lambda\mu}^{\nu} = c_{\lambda'\mu'}^{\nu'}$. One way to see this is to use the involution $\omega : \Lambda \to \Lambda$ taking $s_\lambda \mapsto s_{\lambda'}$, which preserves the inner product on Λ. Geometrically, this follows from the fact that $\mathrm{Gr}(k, n) \cong \mathrm{Gr}(n-k, n)$

Corollary 9.4. $LR(\nu/\lambda, \mu) = LR(\nu/\mu, \lambda) = LR(\nu'/\lambda', \mu') = \cdots$.

Definition 9.5. A **picture** is a bijection φ from boxes of λ/μ to boxes of ν/γ such that:

1. If we label the boxes of λ/μ by $1, 2, \ldots, n$ in the Hebrew reading, then φ maps the labels to a SYT.

2. Same condition for φ^{-1}.

Theorem 9.6 (Zelevinsky)**.** *$\langle s_{\lambda/\mu}, s_{\nu/\gamma} \rangle$ is the number of pictures $\varphi : \lambda/\mu \to \nu/\gamma$.*

How do we get from Zelevinsky's pictures to LR-tableaux? Given a picture $\varphi : \nu/\lambda \to \mu$, if $\varphi(x)$ is in the i-th row of μ, then put i in box x of ν/λ. (Exercise: check this.)

Two partial orders on $\mathbb{Z} \times \mathbb{Z}$:

1. $x \leq_{\searrow} y$ if y is to the southeast of x.

2. $x \leq_{\swarrow} y$ if y is to the southwest of x.

Exercise 9.7. *φ is a picture iff $x \leq_{\searrow} y \Rightarrow \phi(x) \leq_{\swarrow} \phi(y)$ and $\phi(x) \leq_{\searrow} \phi(y) \Rightarrow x \leq_{\swarrow} y$.*

Definition 9.8. Let T be a SSYT of shape λ. Then, construct an inverted triangular array as follows: in the first row, write $\lambda_1, \lambda_2, \ldots$. Then, remove all instances of the largest number T, and do this again in the second row, so that the second row has one fewer number. If you're in the class and are reading this, I owe you a beer (max 3 people). The resulting array is called a **Gelfand-Tsetlin pattern**; SSYTs are in bijection with Gelfand-Tsetlin patterns where the condition on the latter is that along NE-SW rows the numbers decrease weakly and along NW-SE rows the numbers increase weakly. (If you want, fix n so that the largest entry is n and λ has n parts.)

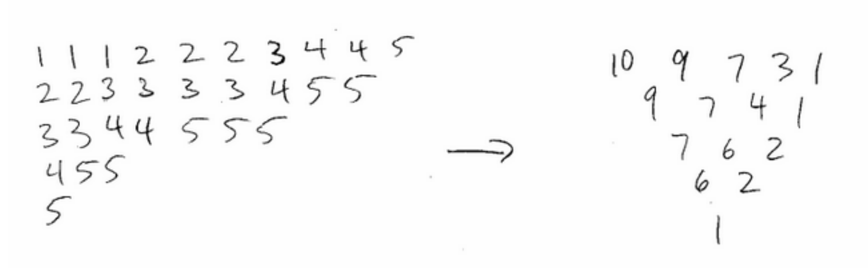

Figure 6: GT pattern example

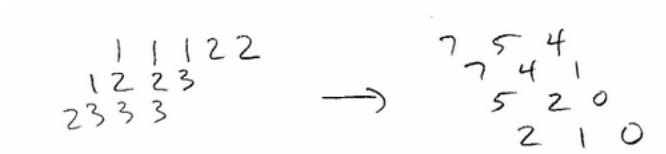

Figure 7: GT pattern example – skew shape

Example 9.9. Example of a SSYT and corresponding GT pattern – see Figure 6.

We can do a similar thing for skew shapes, and we get the same weak increasing/decreasing conditions; the shape of the Gelfand-Tsetlin pattern is a parallelogram – see Figure 7.

Fix $n, \lambda = (\lambda_1, \ldots, \lambda_n), \mu = (\mu_1, \ldots, \mu_n), \nu = (\nu_1, \ldots, \nu_n)$. Let T be a LR-tableau of shape ν/λ and weight μ, corresponding to the GT pattern P. Let $A_i j$ be the number of j's in the i-th row of T. The entries of P are $b_{ij} = \lambda_i + \sum_{j' \leq j} a_{ij'}$, where the indexing is as in Figure 8

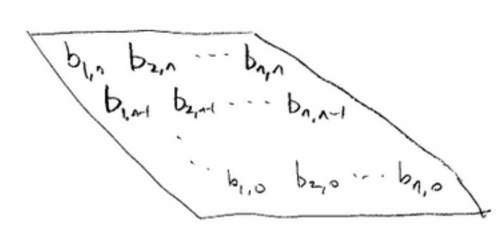

Figure 8: Indexing for parallelogramular (?) GT pattern

Note that the top row reads $\nu_1, \ldots, \nu_n$ and the bottom row reads $\lambda_1, \ldots, \lambda_n$. The word $w(T)$ has the following form: a_{11} 1's, followed be a_{22} 2's, followed by a_{21} 1's, a_{33} 3's, a_{32} 2's, a_{31} 1's, and so on. The lattice word condition is the collection of inequalities $a_{11} \geq a_{22}$, $a_{22} \geq a_{33}$, $a_{11} + a_{21} \geq a_{22} + a_{32}, \ldots$. Let $c_{ij} = \sum_{i' \leq i} a_{i'j}$, the number of j's in the first i rows of T. We have:

Lemma 9.10. *$w(T)$ is a lattice word iff (c_{ij}) is a GT-pattern (see Figure 9)*

So the LR-coefficients are the number of collections of integers a_{ij} that satisfy a bunch of inequalities corresponding to some GT-patterns; we'll write down a more symmetric way of doing this next time.

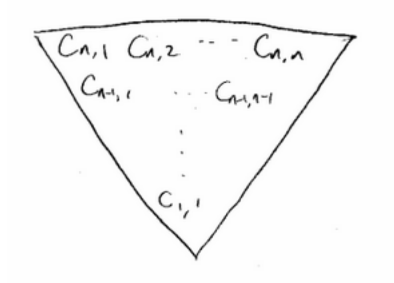

Figure 9: Indexing for triangular GT pattern

10 October 8, 2014

Recall that $c^\nu_{\lambda\mu}$ is the number of LR-tableaux T of shape ν/λ and weight μ. The conditions describe some convex polytope in $\mathbb{R}^n$; the number of lattice points inside is the L-R coefficient.

Fix k a positive integer, and λ, μ, ν partitions with k parts, $|\lambda| + |\mu| = |\nu|$. Let $\ell_i = \lambda_i - \lambda_{i+1}$, and define m_i, n_i similarly for $i = 1, 2, \ldots, k-1$. Construct the graph BZ_k (shown by example for $k = 5$ in Figure10).

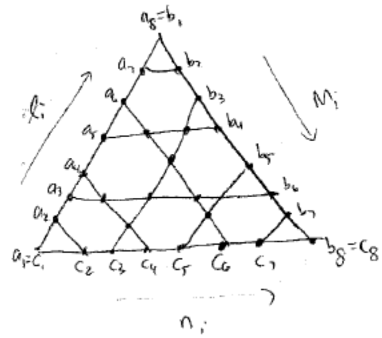

Figure 10: The graph BZ_5

Definition 10.1. A **BZ(I)-pattern** is a map $f : BZ_k \to \mathbb{Z}$ such that:

1. $f(x) \geq 0$
2. $a_1 + a_2 = \ell_1, a_3 + a_4 = \ell_2, \ldots$, $b_1 + b_2 = m_1, b_3 + b_4, \ldots$, and $c_1 + c_2 = n_1, c_3 + c_4 = n_2, \ldots$.
3. For any unit hexagon with labels a, b, c, d, e, f in clockwise order, have $a + b = d + e, b + c = e + f, c + d = f + a$ (note that one of these equations is redundant.

Theorem 10.2 (Bernstein-Zelevinsky)**.** *The number of BZ(I) patterns associated to λ, μ, ν is $c^\nu_{\lambda\mu}$.*

Letting $c_{\lambda\mu\nu} = c^{\nu^\vee}_{\lambda\mu}$, the $\mathbb{Z}/3$-symmetry $c_{\lambda\mu\nu} = c_{\mu\nu\lambda} = c_{\nu\lambda\mu}$ is clear from this picture, but the symmetry $c^\nu_{\lambda\mu} = c^\nu_{\mu\lambda}$ is not.

Let T_k be the graph shown in Figure 11 for $k = 4$ (note that the vertices of the big triangle are not vertices of T_k here). The three **tails** at a vertex v are shown as well.

Definition 10.3. A **BZ(II)-pattern** is a map $f : T_k \to \mathbb{Z}$ such that:

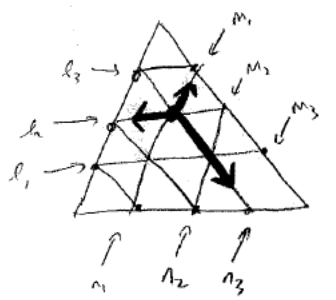

Figure 11: The graph T_4, with tail directions shown.

1. (Tail non-negativity) For any vertex $v \in T_k$, $\sum_{x \in tail(v)} f(x) \geq 0$ for each of the three tails of v
2. The sums of maximal tails in the 3 directions are ℓ_i, m_i, n_i, as in Figure 11.

Theorem 10.4. *The number of BZ(II) patterns associated to λ, μ, ν is $c^{\nu}_{\lambda\mu}$.*

Remark 10.5. From these L-R rules it is clear that the coefficients only depend on the difference vectors $(\ell_i), (m_i), (n_i)$. This can also be seen from the definitions of Schur functions.

How to get from a BZ(II) to a BZ(I)? Starting from a BZ(2), write on each edge a running tail sum in the directions indicated below. Then, take all of the numbers you get in this way and stick them into a BZ(1) triangle – then, reflect over a vertical axis. An example is shown in Figure 12.

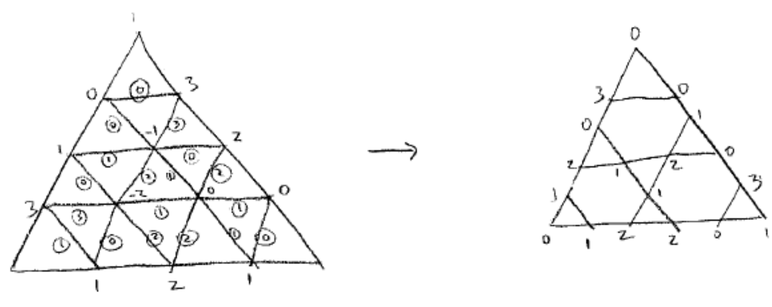

Figure 12: Bijection from BZ(II) diagram to BZ(I) diagram.

Theorem 10.6. *Using the procedure above, BZ(II) patterns for $C^{\nu}_{\lambda\mu}$ become BZ(I) patterns for $c^{\mu^\vee}_{\lambda\nu^\vee}$.*

The hexagon condition is needed to write down the inverse map.

Knutson-Tao honeycombs. Consider the plane $\{(x, y, z) \mid x+y+z = 0\}$ – we allow lines in three directions, namely lines of the form $(a, *, *), (*, b, *), (*, *, c)$. Given λ, μ, ν as above, consider the infinite rays $(\lambda_i, *, *), (*, \mu_i, *), (*, *, -\nu)$. A **honeycomb graph** looks something like the pictures in Figure 13 (we won't write down an actual definition).

Theorem 10.7 (Knutson-Tao)**.** *$c^{\nu}_{\lambda\mu}$ is the number of integer honeycombs for λ, μ, ν.*

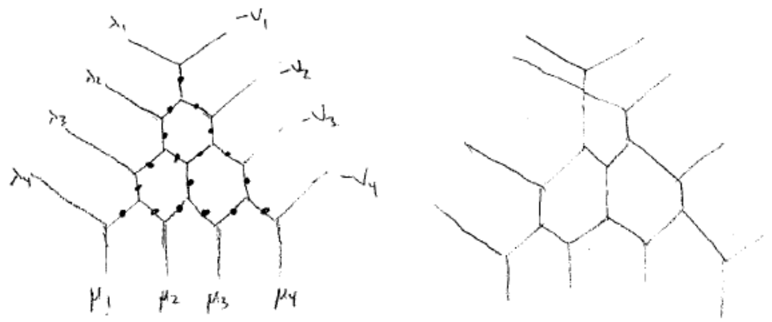

Figure 13: Honeycomb graphs: note that the one on the right has an edge of length zero.

Define $d((x,y,z),(x',y',z')) = K\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$, where K is chosen so that the vertical distance between the lines $(a,*,*)$ and $(b,*,*)$ is $|a-b|$. Then, to get from a honeycomb to a BZ(I) diagram, read off the lengths of all of the edges (some of which are zero) and stick them on the nodes of the BZ(I) in the obvious way. Note that equiangularity of the hexagons is exactly the hexagon condition.

11 October 10, 2014

Rough list of exercises for reference (a few are new).

- Lemma in the proof of unimodality of Gaussian coefficients.
- Various examples and non-examples of matroids. Equivalence of exchange axioms (regular, strong, stronger)
- Laplace Theorem on Eulerian Numbers
- Plücker Relations cut out the Grassmannian. Radical of the $r = 1$ relations gives the whole ideal. (This is essentially commutative algebra.)
- Cauchy determinant:

$$\det((1-x_iy_j)^{-1})_{i,j=1}^{k} = \prod_{i<j}(x_i-x_j)(y_i-y_j)\prod_{i,j=1}^{k}\frac{1}{1-x_iy_j}.$$

- Demazure character formula. Define

$$D_i \cdot f(x_1,\ldots,x_n) = \frac{x_i f(x_1,\ldots,x_n) - x_{i+1} f(x_1,\ldots,x_{i+1},x_i,\ldots,x_n)}{x_i - x_{i+1}}.$$

  These satisfy the relations $D_i^2 = D_i, D_iD_{i+1}D_i = D_{i+1}D_iD_{i+1}$, and $D_iD_j = D_jD_i$ if $|i-j| > 1$. Write down a product on $\binom{n}{2}$ transpositions $s_i = (i,i+1) \in S_n$ whose product is the longest element in S_n, i.e. the permutation $\sigma(1) = n, \sigma(2) = n-1, \ldots, \sigma(n) = 1$ (such products correspond to wiring diagrams). For example, when $n = 4$, take $s_1s_2s_1s_3s_2s_1$.

  Then, applying the corresponding D_i in this order (e.g. $D_1D_2D_1D_3D_2D_1$ above) to $x_1^{\lambda_1}\cdots x_n^{\lambda_n}$ yields $s_\lambda(x_1,\ldots,x_n)$.

- Knutson-Tao Saturation Theorem: if $c^{r\nu}_{r\lambda,r\mu} \neq 0$ for some r ($r\lambda$ multiply the sizes of the parts by r), then in fact $c^{\nu}_{\lambda,\mu}$. Stronger statement: if the LR-polytope $P^{\nu}_{\lambda,\mu}$ (polytope of BZ-triangles, or equivalently polytope of honeycombs) is non-empty, it contains a lattice point (including boundary). Exercise: find λ, μ, ν such that $P^{\nu}_{\lambda\mu}$ has a non-integer vertex (Hint: fill in some zeroes in a BZ triangle in such a way that the boundary conditions determine the rest of the entries, and some of them are not integers). Morally, this says that the Saturation Theorem is not trivial, because the LR-polytope may not have integer vertices despite being defined by integer equations.

12 October 15, 2014

Problem set deadline extended to next Wednesday.

Correspondence between Classical L-R rule and honeycombs (BZ-triangles).

T a LR-tableaux of shape ν/λ and weight μ. Let $a_i j$ be the number of j's in the i-th row. Then, define $b_{ij} = \lambda_i + \sum_{j' \leq j} a_{ij'}$ and $c_{ij} = \sum_{i' \leq i} a_{i'j}$. Now, the c_{ij}'s become the coordinates of the vertical edges of a honeycomb, i.e. the vertical lines $*c_{ij}*$ become vertical edges, as shown in Figure 14. The a_{ij}s become the lengths of the edges in the NE-SW direction.

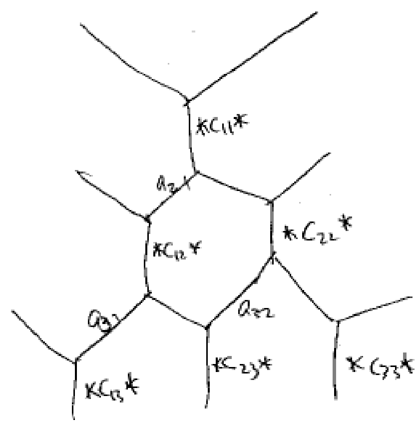

Figure 14: L-R tableaux parameters in a honeycomb.

Exercise 12.1. *Check carefully that this is all a bijection (where are the b_{ij}?). More precisely:*

- *$w(T)$ is a lattice word iff the lengths of the edges in the NW-SE direction are non-negative.*
- *T is a SSYT iff the lengths of the edges in the N-S direction are non-negative.*

Theorem 12.2 (Knutson-Tao saturation property)**.** *If $c^{r\nu}_{r\lambda,r\mu} \neq 0$, then $c^{\nu}_{\lambda\mu} \neq 0$. Equivalently, if the L-R polytope $P^{\nu}_{\lambda\mu} \neq \emptyset$, then it contains at least one integer point.*

The idea of the proof is to start with a honeycomb, possibly interior line segments not having integer coordinates, then perturb the edges until you get integer coordinates.

More general honeycombs: consider rays coming from 6 directions, as shown in Figure 15. Let a, b, c, d, ef be the number of rays coming in each direction, satisfying the hexagon condition (if we force $a = c = e = 0$, get KT honeycombs; if $a = d = 0$ we get GP web diagrams). If you allow all 6 directions, get infinitely many honeycombs. If there are 5 directions, what numbers do you get?

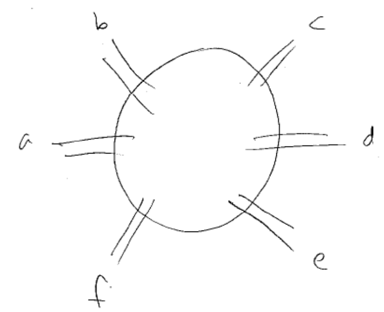

Figure 15: Generalized honeycombs?

Honeycombs live inside a certain class of web diagrams: given $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$, $\nu = (\nu_1, \nu_2, \ldots, \nu_n, 0, 0, \ldots, 0)$, the honeycomb is the right half of the web diagram, as in Figure 16.

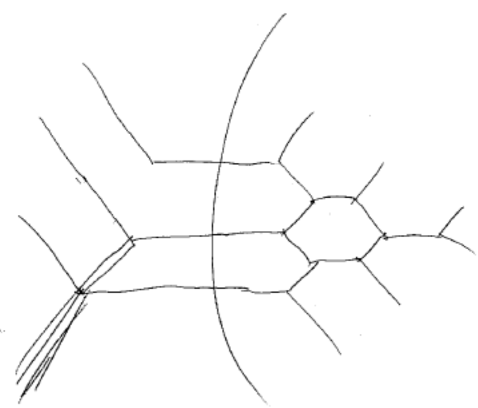

Figure 16: Honeycomb inside GP diagram.

Conversely, if you have arbitrary λ, μ, ν of $m, n, m+n$ parts, respectively, then you can add n zeroes to the end of λ and , m zeroes to the end of μ, draw the honeycomb, then identify the result with a web diagram.

Yet another L-R rule reformulation. Let V be a vector space with basis $e_0, e_1, e_2, \ldots$ (by convention, $e_{<0} = 0$). Then $R(c) : V \otimes V \to V \otimes V$ is defined by sending $e_x \otimes e_y \mapsto e_{y+c} \otimes e_{x-c}$ if $c \geq x - y$ and 0 otherwise (this is the "scattering matrix" for the particle interaction in the GP web diagrams: a left particle originally going toward x and a right particle originally going toward y that interact at height c get scattered to $y + c, x - c$, respectively).

Now given $\lambda = (\lambda_1, \ldots, \lambda_m), \mu = (\mu_1, \ldots, \mu_n)$, let $E_\lambda = e_{\lambda_m} \otimes e_{\lambda_{m-1}} \otimes \cdots e_{\lambda_1} \in V^{\otimes m}$ and define $E_\mu \in V^{\otimes n}$ similarly. Then, the operator $R_{ij}(c)$ acts by $R(c)$ on the i-th and j-th components of $V^{\otimes N}$ and by the identity on the others.

Now, draw a wiring diagram as shown by example with $m = 3, n = 2$ in Figure 17. Order the points of intersection of the wires with integers c_{ij}. Then, define the operator $R_{m,n}(c) = R_{34}(c_{34}) R_{24}(c_{24}) \cdots R_{15}(c_{15})$, where we read off the indices from left to right (we have to make some choices – note that with all of the choices the operators in question commute with each other, so it $R_{m,n}$ doesn't depend on the choices we make), and let $M_{m,n}$ be the sum of the $R_{m,n}(c)$ over all reverse plane partitions c. Then $M_{mn}(E_\lambda \otimes E_\mu) = \sum_\nu c^\nu_{\lambda,\mu} E_\nu$.

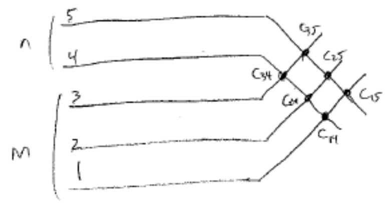

Figure 17: Wiring diagram for L-R rule.

13 October 17, 2014

Let S_n be the symmetric group, realized as the group of bijections $w : [n] \to [n]$; multiplication is by composition (gh means do h, then do g). The generators are adjacent transpositions $s_i = (i, i+1)$, and the relations are:

- $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$
- $s_i s_j = s_j s_i$ for $Ii - j| > 2$
- $s_i^2 = 1$.

A **reduced** decomposition is an expression $w = s_{i_1} \dots s_{i_r}$ of minimal possible length $\ell = \ell(w)$.

Exercise 13.1. *$\ell(w)$ is the number of inversions of w, i.e. the number of pairs (i, j) for which $i < j$ and $w(i) > w(j)$.*

To a reduced decomposition we associate a wiring diagram, shown by example.

Example 13.2. $w = s_1 s_2 s_3 s_2$, shown in Figure 18. If a wire goes from j on the left to i on the right, then $w(i) = j$.

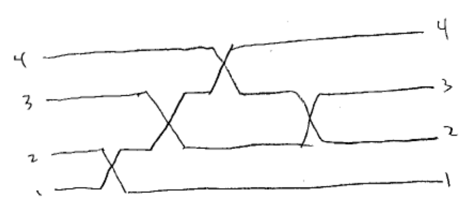

Figure 18: Wiring diagram corresponding to the reduced decomposition $w = s_1 s_2 s_3 s_2$

Because the decomposition is reduced, two wires intersect at most once; the inversions of w correspond to the points of intersection. The relations in S_n correspond to local moves, as shown in Figure 19.

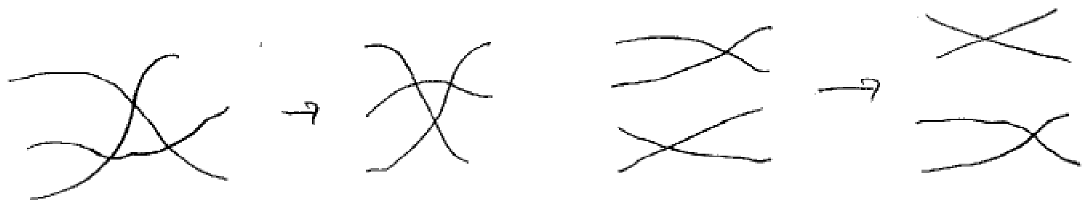

Figure 19: Local moves on wiring diagrams: $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ (3-move), $s_i s_j = s_j s_i$ (2-move)

Exercise 13.3. *Any two wiring diagrams for reduced decompositions can be obtained from each other via local moves (not obvious, because we never need to use the relation $s_i^2 = 1$.*

Let $E_i(t)$ be the $n \times n$ matrix with entries $(1, \ldots, 1, t, t^{-1}, 1, \ldots, 1)$ down the diagonal (where t is in row i), and a 1 in the $i, i+1$ entry. For $w = s_{i_1} \cdots s_{i_\ell}$, let $E(t_1, \ldots, t_\ell) = E_{i_1}(t_1) E_{i_2}(t_2) \cdots E_{i_\ell}(t_\ell)$.

Lemma 13.4. *We have the following relations corresponding to local moves:*

- *(2-move, obvious)* $E_i(a) E_j(b) = E_j(b) E_i(a)$.
- *(3-move)* $E_i(a) E_{i+1}(b) E_i(c) = E_{i+1}(a') E_i(b') E_{i+1}(c')$, *where* $(a', b', c') = ((c^{-1} + ab^{-1})^{-1}, ac, a + bc^{-1})$.

Proof. Exercise, if you want to practice multiplication of 3x3 matrices. □

Now, consider the transformation on the wiring diagram which produces a weighted bicolored directed graph, shown by example in Figure 20: this is the result of applying the shown transformation to the reduced decomposition $w = s_1 s_2 s_3 s_2$ from before.

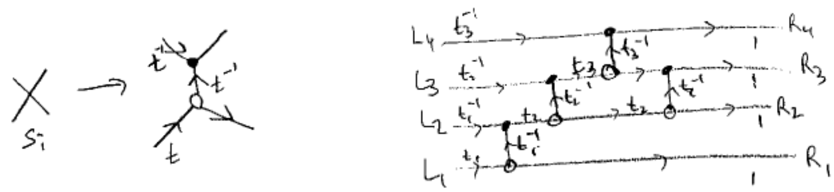

Figure 20: Weighted bicolored directed graph from wiring diagram

The (i, j) entry of $E(t_1, \ldots, t_\ell)$ is $\sum_{P: L_i \to R_j} \prod_{e \in P} w(e)$. In our example, $E(t_1, \ldots, t_\ell)$ is

$$E_1(t_1) E_2(t_2) E_3(t_3) E_2(t_4) = t_1^{-1} t_3 t_4^{-1} + t_1^{-1} t_2.$$

In fact,

Lemma 13.5 (Lindström Lemma/Gessel-Viennot Method)**.** *We have*

$$\Delta_{I,J} E(t_1, \ldots, t_\ell) = \sum_{P_1, \ldots, P_r} \prod_s w(P_s),$$

where $|I| = |J| = r$, the sum is over non-crossing paths crossing $L_i, i \in I$ to $R_j, j \in J$, and the weight of a path is the product of the weights of its edges.

You can look up the proof. Carl: Fomin-Zelevinsky's expository article on totally positive matrices proves this.

Let $D_s = \Delta_{\{w(s+1),w(s+2),\dots,w(n)\},\{s,\widehat{s+1},s+2,\dots,n\}}$, which is a positive Laurent polynomial. Starting with the digraph G, get the graph G_{n-1} by reversing all edges along the wire R_n, and inverting all of the weights. Then, get the graph G_{n-2} by reversing all edges along the wire R_{n-1} and inverting all of the weights again. Keep going...

Exercise 13.6. *$D_s = \sum_{P:R_{s+1}\to R_s} w(P)$, where the paths are taken in G_s.*

Corollary 13.7. *D_s depends only on w and not on choice of reduced decomposition.*

Tropicalize everything: replace addition with taking minimum, and multiplication by addition. For example, the 3-move is $(a,b,c)\mapsto(c^{-1}+ab^{-1}, ac, a+bc^{-1})$, and the tropical 3-move is

$$(a,b,c)\mapsto(\max(c,b-a), a+c, \min(a,b-c))$$

This corresponds to taking $t_i = q^{a_i}$ and sending $q\mapsto 0$.

14 October 24, 2014

Today we'll finish the proof of the L-R rule. Recall that if we have a reduced decomposition $w = s_{i_1}\cdots s_{i_r}$, we have a product of matrices $E(t_1,\dots,t_\ell) = e_{i_1}(t)\cdots E_{i_\ell}(t)$, where $E_i(t)$ is the $n\times n$ identity matrix except for the 2×2 block in rows and columns $i, i+1$, where it is the matrix

$$\begin{bmatrix} t & 1 \\ 0 & t^{-1}\end{bmatrix}.$$

These satisfy some relations:

Lemma 14.1. *We have the following relations corresponding to local moves:*

- *$E_i(a)E_j(b) = E_j(b)E_i(a)$.*
- *$E_i(a)E_{i+1}(b)E_i(c) = E_{i+1}(a')E_i(b')E_{i+1}(c')$, where $(a',b',c') = ((c^{-1}+ab^{-1})^{-1}, ac, a+bc^{-1})$.*

For $r = 1,2,\dots,n-1$, we have the minor $D_r = \Delta_{\{w(r+1),\dots,w(n)\},\{r,\widehat{r+1},r+2,\dots,n\}}(E(t_1,\dots,t_r))$. This corresponds to taking the wiring diagram for w, reversing the orientation of the top $n-r$ wires, and replacing all crossings with a pair of trivalent vertices (shown in Figure 21, then taking the product of the weights of paths from the $(r+1)$-st right vertex to the r-th right vertex (the weights were given last time). The value of this minor is independent of reduced decomposition (to see this, apply local moves).

Tropicalization: given f a subtraction-free rational expression in some variables, $Trop(f)$ is the expression obtained by replacing $\times$ with $+$, $/$ by $-$, and $+$ by min. For example, $Trop(xy+xz^{-1}) = \min(x+y, x-z)$.

Let V be the vector space with basis $v_0, v_1, v_2, \dots$, so that $V^{\otimes n}$ has basis $v_{\alpha_1\cdots\alpha_n} = v_{\alpha_1}\otimes\cdots v_{\alpha_n}$. Define the **scattering** matrix $S_i(c): V^{\otimes n}\to V^{\otimes n}$, $c\in\mathbb{Z}$, by $v_{\alpha_1\cdots\alpha_n}\mapsto v_{\alpha_1,\dots,\alpha_{i-1},\alpha_i-c,\alpha_{i+1}+c,\alpha_{i+2},\dots,\alpha_n}$ if $\alpha_i\ge c\ge\alpha_i-\alpha_{i+1}$, and 0 otherwise.

Lemma 14.2. *$S_i(a)S_{i+1}(b)S_i(c) = S_{i+1}(a')S_i(b')S_{i+1}(c')$, where $(a',b',c') = (\max(c,b-a), a+c, \min(a,b-c))$.*

This is exactly the tropical analogue of the relation among the E_i from before.

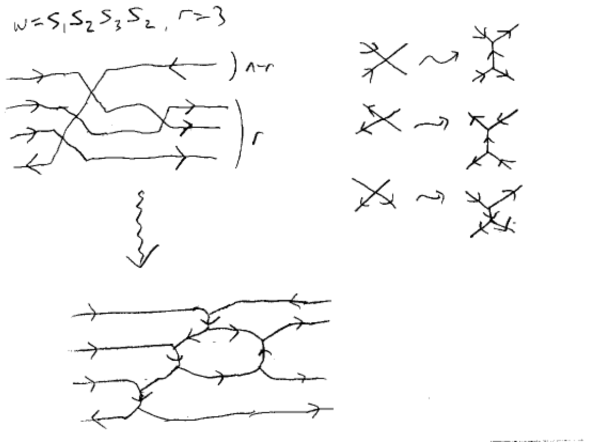

Figure 21: Transformation on a wiring diagram.

Definition 14.3. Given $w = s_{i_1} \cdots s_{i_r}$, define the operator $S_w = \sum S_{i_1}(c_1) \cdots S_{i_\ell}(c_\ell)$, where the sum is taken over all $(c_1, \ldots, c_\ell) \in \mathbb{Z}^\ell_{\geq 0}$ for which $trop(D_r) \geq 0$ for $r = 1, \ldots, n-1$.

Example 14.4. $w = 3412 = s_2 s_1 s_3 s_2$. Then, $trop(D_1) = \min(c_1 - c_3, c_2 - c_4)$, $trop(D_2) = c_4$, $trop(D_3) = \min(c_1 - c_2, c_3 - c_4)$. Thus, S_w is the sum of $S_2(c_1)S_1(c_2)S_3(c_3)S_2(c_4)$, where we sum over $c_1, c_2, c_3, c_4 \in \mathbb{Z}^4_{\geq 0}$ satisfying $c_1 \geq c_2, c_3 \geq c_4 \geq 0$.

By looking at local moves, we can show:

Theorem 14.5. *S_w depends only on w, not on its reduced decomposition.*

How is this all related to the L-R rule? Fix m, n and partitions $\lambda = (\lambda_1, \ldots, \lambda_n), \mu = (\mu_1, \ldots, \mu_n), \nu = (\nu_1, \ldots, \nu_{m+n})$. Then, let $w_{m,n}$ be the permutation $(m+1)(m+2)\cdots(m+n)12\cdots m$. Let $\lambda^R = (\lambda_m, \ldots, \lambda_1)$ (reverse order of the parts).

Theorem 14.6. *$c^\nu_{\lambda\mu}$ is the coefficient of v_{ν^R} in $S_{w_{m,n}}(v_{\lambda^R} \otimes v_{\mu^R})$.*

The latter can be represented as in Figure 22, and from here we can give a bijection to web diagrams, so to prove L-R it suffices to prove the above theorem.

Proof. Define a product on $\oplus n \geq 0 V^{\otimes n}$, by $v_\alpha \odot v_\beta = S_{w_{m,n}}(v_\alpha \otimes v_\beta)$, where α, β are arbitrary vectors taking values in non-negative integers. (Exercise: this product is zero unless $\alpha_1 \leq \alpha_2 \leq \cdots \alpha_n$ and $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_n$. We need to check that $\cdot$ satisfies the Pieri rule (this was already done for web diagrams), and associativity.

Fix $\alpha = (\alpha_1, \ldots, \alpha_m), \beta(\beta_1, \ldots, \beta_n), \gamma = (\gamma_1, \ldots, \gamma_k)$; we need to show that $(v_\alpha \odot v_\beta) \odot v_\gamma = v_\alpha \odot (v_\beta \odot v_\gamma)$. In terms of wiring diagrams, this may be represented as in Figure 23.

Note that each wiring diagram represents the same permutation $w_{m,n,k}$, and furthermore both products are equal to $S_{w_{m,n,k}}(v_\alpha \otimes v_\beta \otimes v_\gamma)$. Equivalently, if we transform one wiring diagram into another, all of the inequalities on one side transform into the other. ☐

The next topic of the course is the Flag variety (manifold) Fl_n, the spae of complete flags $0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n$. We will define a Schubert decomposition into Schubert cells labeled by permutations $w \in S_n$. The closures of Schubert cells define cohomology classes which form a basis of the cohomology ring, and the Schubert cells are ordered by the Bruhat order on S_n. We'll start with the combinatorics and then go to the geometry later.

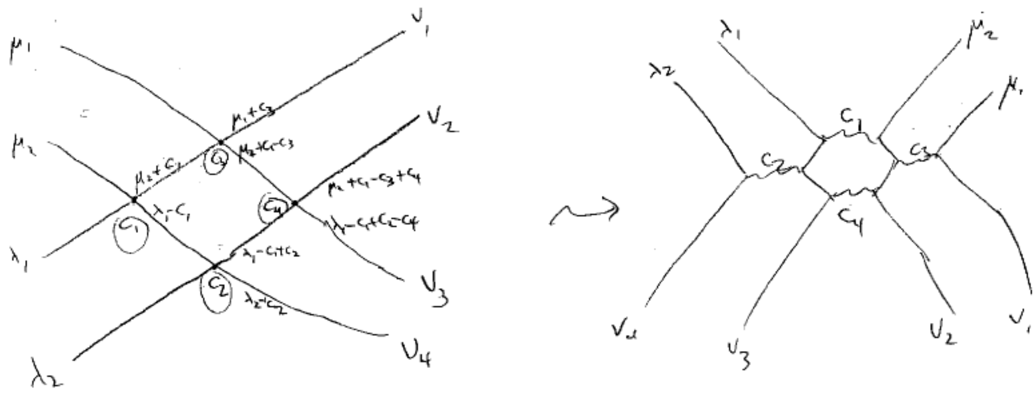

Figure 22: Wiring Diagrams and Web Diagram

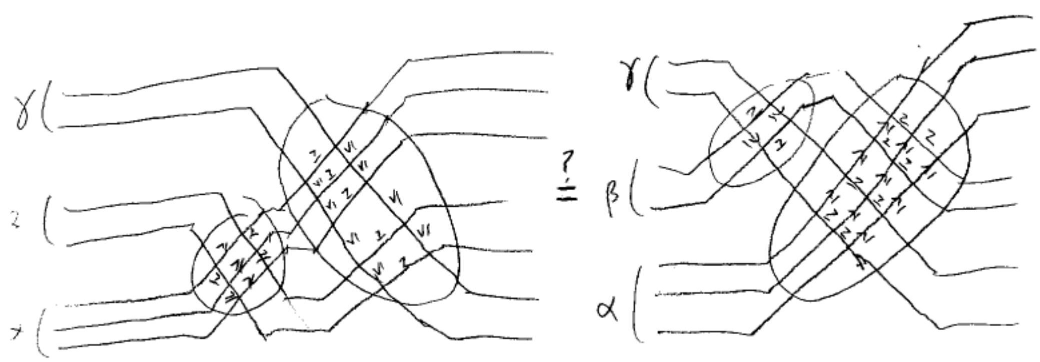

Figure 23: Associativity of $\odot$

Definition 14.7. We define the **weak Bruhat order** on S_n by declaring $u \lessdot v$ if $v = us_i$, with $s_i = (i, i+1)$, and $\ell(v) = \ell(u) + 1$. In the **strong Bruhat order**, we define $u <_s v$ the same is true, but s_i is allowed to be any transposition.

Warning: some people switch the names of the weak and strong Bruhat order.

Example 14.8. Bruhat orders on S_3: weak on the left and strong on the right in Figure 24

15 October 29, 2014

We'll define the Schubert polynomials $\mathfrak{S}_w$, where w is a permutation. Two approaches:

- "Top-to-bottom" approach via divided difference operators: Bernstein-Gelfand-Gelfand (1974), Demazure, Lascoux-Schuzenberger (1982). Define $\mathfrak{S}_w$ for the longest permutation, then go down the weak Bruhat order.

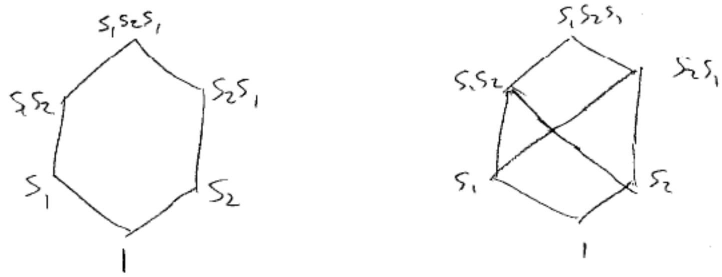

Figure 24: Weak and strong Bruhat orders

- "Bottom-to-top" approach via Monk's formula (1959), go up the strong Bruhat order.

Today, we'll do the first construction.

Definition 15.1. Define the **divided difference operators** ∂_i on $\mathbb{C}[x_1, \dots, x_n]$ by

$$\partial_i(f) = \frac{f(x_1, \dots, x_n) - f(x_1, \dots, x_{i+1}, x_i, \dots, x_n)}{x_i - x_{i+1}} = \frac{1}{x_i - x_{i+1}}(1 - s_i)(f).$$

It is clear that this is a polynomial.

Lemma 15.2. *The operators ∂_i satisfy nil-Coxeter relations:*

- $\partial_i^2 = 0$
- $\partial_i \partial_j = \partial_j \partial_i$ *if* $|i - j| \geq 2$
- $\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}$

Proof. We'll prove the first relation. Note that $\frac{1}{x_i - x_{i+1}}(1 - s_i) = (1 + s_i)\frac{1}{x_i - x_{i+1}}$. Then

$$\begin{aligned}
\partial_i^2 &= \frac{1}{x_i - x_{i+1}}(1 - s_i)\frac{1}{x_i - x_{i+1}}(1 - s_i) \\
&= \frac{1}{x_i - x_{i-1}}(1 - s_i)(1 + s_i)\frac{1}{x_i - x_{i+1}} = 0 \\
&= 0.
\end{aligned}$$

The other two relations may be checked directly. $\square$

Definition 15.3. If $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ is a reduced decomposition of $w \in S_n$, define $\partial_w = \partial_{i_1} \cdots \partial_{i_\ell}$. By the nil-Coxeter relations, this operator depends only on w (not on reduced decomposition).

We now define the Schubert polynomials $\mathfrak{S}_w(x_1, \dots, x_n)$ and Double Schubert polynomials $\mathfrak{S}_w(x_1, \dots, x_n; y_1, \dots, y_n)$ (specializing at $y_i = 0$ recovers the Schubert polynomials). First, for the longest permutation $w_0 = (n, n-1, \dots, 1)$, define

$$\mathfrak{S}_{w_0}(x; y) = \prod_{i+j \leq n, j, j \geq 1} (x_i - y_j),$$

so that $\mathfrak{S}_{w_0}(x) = x_1^{n-1} x_2^{n-2} \cdots x_n^0$. Then, if $\ell(ws_i) = \ell(w) - 1$, define $\mathfrak{S}_{ws_i}(x; y) = \partial_i \mathfrak{S}_w(x; y)$. Here, the ∂_i act only on the x_j's.

Example 15.4. $n = 3$. $\mathfrak{S}_{w_0}(x) = x_1^2 x_2$. Length 2: $\mathfrak{S}_{s_1 s_2}(x) = \partial_1(x_1^2 x_2) = x_1 x_2$ and $\mathfrak{S}_{s_2 s_1}(x)(s_1^2 s_2) = x_1^2$. Length 1: $\mathfrak{S}_{s_1}(x) = \partial_2(x_1 x_2) = x_1$ and $\mathfrak{S}_{s_2}(x) = \partial_1(x_1^2) = x_1 + x_2$. Finally, $\mathfrak{S}_1(x) = 1$.

In fact, $\mathfrak{S}_w(x) \in \mathbb{Z}_{\geq 0}[x_1, \ldots, x_n]$, and $\mathfrak{S}_w(x, -y) \in \mathbb{Z}_{\geq 0}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. This is not yet obvious.

From the perspective of AG, any choice of $\mathfrak{S}_w(x)$ will work: the polynomials you get are representatives of Schubert classes in the cohomology ring of the Flag manifold. Our particular choice of $\mathfrak{S}_w(x)$ lends itself to some nice combinatorial aspects, which we will get to next.

Definition 15.5. The **nil-Coxeter algebra** NC_n over $\mathbb{C}$ is generated by $u_1, \ldots, u_{n-1}$ with the nil-Coxeter relations as before.

This is almost the group algebra $\mathbb{C}[S_n]$, except here $u_i^2 = 0$ instead of 1. The nil-Coxeter algebra has basis indexed by permutations: if $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced decomposition, then $u_w = u_{i_1} \cdots u_{i_\ell}$. Given permutations $u, w \in S_n$, we have $u_v u_w = u_{v \cdot w}$ if $\ell(v) + \ell(w) = \ell(v + w)$ and $u_v u_w = 0$ otherwise.

(There is a closely related object, the **nil-Hecke algebra**, generated by ∂_i and x_j.)

Let $h_i(x) = 1 + x u_i \in NC_n[x]$ (so x commutes with all of the u_i). These satisfy the following relations:

- $h_i(x) h_i(y) = h_i(x + y)$
- $h_i(x) h_j(y) = h_j(y) h_i(x)$
- $h_i(x) h_{i+1}(x + y) h_i(y) = h_{i+1}(y) h_i(x + y) h_{i+1}(x)$ (Yang-Baxter relation)

Given a reduced decomposition, we produce an operator ϕ as follows: draw the wiring diagram, then multiply the operators $h_i(x_j - x_k)$ from left to right, where h_i corresponds to s_i, and $x_j - s_k$ corresponds to the transposition of the wires x_j and x_k (labeled on the right).

Example 15.6. $w = s_1 s_3 s_2 s_1 s_4 s_3$. Then, $\phi = h_1(x_2 - x_4) h_3(x_1 - x_5) h_2(x_1 - x_4) h_1(x_1 - x_2) h_4(x_3 - x_5) h_3(x_3 - x_4)$. The wiring diagram with weights is shown in Figure 25.

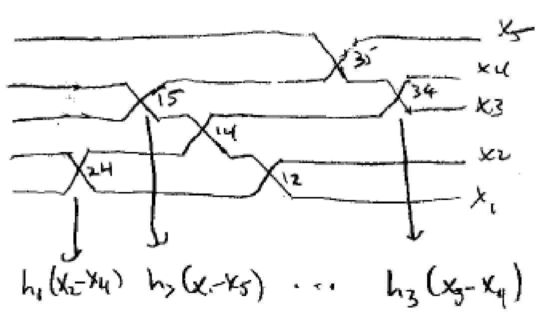

Figure 25: Operator from wiring diagram

This operator only depends on w: for example, the fact that the operator is preserved under a 3-move is just the Yang-Baxter relation.

Now, define the operator $\phi \in NC_n[x_1, \ldots, x_n; y_1, \ldots, y_n$ as follows: start with the diagram in Figure 26, then label each crossing with $x_j - y_k$, corresponding to the wires intersecting there.

Then, multiply the operators $h_i(x_j - y_k)$ from left to right, where i is the height of the crossing as before. Then $\phi_n = h_{n-1}(x_1 - y_{n-1})h_{n-2}(x_1 - y_{n-2}) \cdots h_{n-1}(x_{n-1} - y_1)$. This operator is preserved under 2-moves, but not under 3-moves. When we set $y_i = y_{n+1-i}$, this recovers the same operator from before.

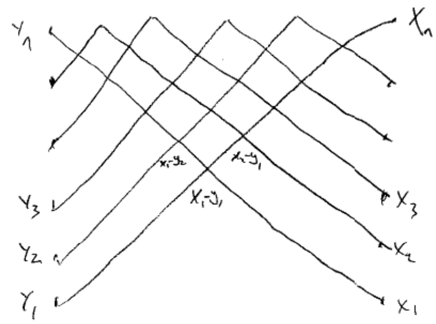

Figure 26: Operator ϕ from maximal length permutation

Theorem 15.7 (Fomin-Stanley, Billey-Jockush-Stanley). $\phi_n = \sum_{w \in S_n} \mathfrak{S}_w(x; y) u_w$.

For $n = 3$, we have $\phi_3 = h_2(x_1 - y_2)h_1(x_1 - y_1)h_2(x_2 - y_1) = (1 + (x_1 - y_2)u_2)(1 + (x_1 - y_1)u_1)(1 + (x_2 - y_1)u_2)$.

16 October 31, 2014

Recall that the nilCoxeter algebra is generated by $u_1, \ldots, u_{n-1}$ with relations $u_i^2 = 0$, $u_i u_j = u_j u_i$ for $|i - j| \geq 2$, and $u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}$. For $w = s_{i_1} \cdots s_{i_\ell}$ a reduced composition, we defined $u_w = u_{i_1} \cdots u_{i_\ell}$. We have $h_i(x) = 1 + x u_i$, satisfying the relatiosn $h_i(x)h_i(y) = h_i(x + y)$, $h_i(x)h_j(y) = h_j(y)h_i(x)$ if $|i - j| \geq 2$, and $h_i(x)h_{i+1}(x + y)h_i(y) = h_{i+1}(y)h_i(x + y)h_{i+1}(x)$ (Yang-Baxter relation).

Define (order of the terms in the product matters!)

$$\phi_n = \prod_{i=1}^{n-1} \prod_{j=n-i}^{1} h_{i+j-1}(x_i - y_j),$$

which comes from the wiring diagram of the longest permutation in Figure 26.

Theorem 16.1 (Fomin-Stanley, Billey-Jockush-Stanley). $\phi_n = \sum_{w \in S_n} \mathfrak{S}_w(x; y) u_w$.

For $n = 3$, we have

$$\begin{aligned}\phi_3 = h_2(x_1 - y_2)h_1(x_1 - y_1)h_2(x_2 - y_1) &= (1 + (x_1 - y_2)u_2)(1 + (x_1 - y_1)u_1)(1 + (x_2 - y_1)u_2) \\ &= 1 + (x_1 - y_1)u_1 + (x_1 - y_2 + x_2 - y_1)u_2 \\ &\quad + (x_1 - y_1)(x_2 - y_1)u_1 u_2 + (x_1 - y_2)(x_1 - y_1)u_2 u_1 + (x_1 - y_2)(x_1 - y_1)(x_2 - y_1)u_2 u_1 u_2.\end{aligned}$$

Proof of Theorem 16.1. Say $\phi_n = \sum_w f_w(x_1, \ldots, x_n; y_1, \ldots, y_n)$. We have $f_{w_0} = \mathfrak{S}_{w_0}$ by definition, where w_0 is the longest permutation. By induction, it suffices to show that $\partial_i(f_w) = f_{ws_i}$ if $\ell(ws_i) = \ell(w) - 1$. It suffices to prove the following lemma:

Lemma 16.2. $\partial_i(\phi_n) = \phi_n \cdot u_i$.

Indeed, this lemma would imply

$$\sum_{w\in S_n} \partial_i(f_w)u_w = \sum_{v\in S_n} f_v u_v u_i,$$

and $u_v u_i = u_{vs_i}$ if $\ell(vs_i) = \ell(v)+1$ and $u_v u_i = 0$ otherwise; then compare coefficients.

So we need to prove

$$\begin{aligned}
\frac{1}{x_i - x_{i+1}}(1-s_i)(\phi_n) &= \phi_n u_i \\
\phi_n - s_i(\phi_n) &= \phi_n(x_i - x_{i+1})u_i \\
\phi_n(1 + (x_{i+1} - x_i)u_i) &= s_i(\phi_n) \\
\phi_n \cdot h_i(x_{i+1} - x_i) &= s_i(\phi_n).
\end{aligned}$$

Example 16.3. $n = 3, i = 2$.

$$\begin{aligned}
\phi_3 \cdot h_2(x_3 - x_2) &= h_2(x_1 - y_2)h_1(x_1 - y_1)h_2(x_2 - y_1)h_2(x_3 - x_2) \\
&= h_2(x_1 - y_2)h_1(x_1 - y_1)h_2(x_3 - y_1) \\
&= s_2(\phi_3).
\end{aligned}$$

Example 16.4. $n = 3, i = 1$.

$$\begin{aligned}
\phi_3 \cdot h_1(x_2 - x_1) &= h_2(x_1 - y_2)h_1(x_1 - y_1)h_2(x_2 - y_1)h_1(x_2 - x_1) \\
&= h_2(x_1 - y_2)h_2(x_2 - x_1)h_1(x_2 - y_1)h_2(x_1 - y_1) \\
&= h_2(x_2 - y_2)h_1(x_2 - y_1)h_2(x_1 - y_1) \\
&= s_1(\phi_3).
\end{aligned}$$

Let's do the general case. We slightly change the order of the terms in ϕ_n, but enough factors commute with each other such that we get the same operator.

$$\begin{aligned}
\phi_n &= h_{n-1}(x_1 - y_{n-1})[h_{n-2}(x_1 - y_{n_2})h_{n-1}(x_2 - y_{n-2})]\cdots[h_1(x_1 - y_1)h_2(x_2 - y_1)\cdots h_{n-1}(x_{n-1} - y_1) \\
&= H^{(1)}H^{(2)}\cdots H^{(n-1)}
\end{aligned}$$

We claim that $H^{(n-1)}h_i(x_{i+1} - x_i) = s_i(H^{(n-1)})$ if $i = n-1$ and $h_{i+1}(x_{i+1} - x_i)s_i(H^{(n-1)})$ otherwise. Indeed, in the first case, the last factor in $H^{(n-1)}$ and $h_i(x_{i+1} - x_i)$ combine to get $h_{n-1}(x_n - y_1)$, and the result is exactly $s_{n-1}(H^{(n-1)})$. In the second, the $h_i(x_{i+1} - x_i)$ commutes with all of the terms going from right to left until it runs into $h_i(x_i - y_1)h_{i+1}(x_{i+1} - y_i)$. Now, apply Yang-Baxter, to get $h_{i+1}(x_{i+1} - x_1)h_i(x_{i+1} - y_1)h_{i+1}(x_i - y_1)$, then commute $h_{i+1}(x_{i+1} - x_1)$ past everything on the left; the result is $h_{i+1}(x_{i+1} - x_i)s_i(H^{(n-1)})$.

Now apply this claim repeatedly and follow your nose. □

RC-graphs/Pipe Dreams. Start with a diagram as in the left of Figure 27, and break some of the crossings in such a way that any two resulting "pipes" intersect at most one point. The weight of an intersection point is $x_i - y_j$, where x_i is the label directly below (go straight down, not along a pipe) and y_j is the label directly to the left. The weight of the diagram D is the product of the weights on the intersection points. The permutation associated to D is the permutation sending i to $\sigma(i)$, where the pipes connect x_i to $y_{\sigma(i)}$.

We have the following consequence of Theorem 16.1:

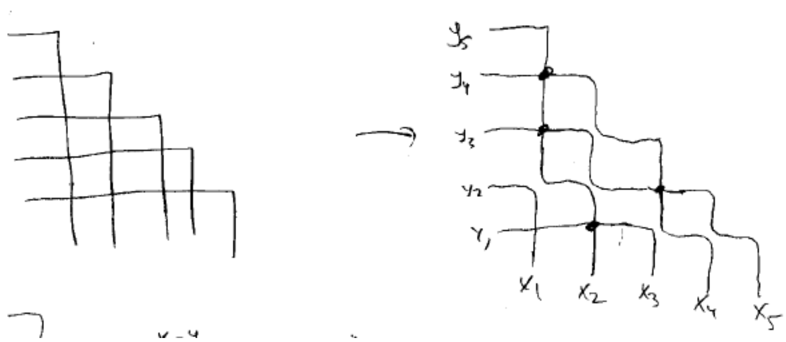

Figure 27: Pipe Dream

Theorem 16.5. $\mathfrak{S}_w(x,y) = \sum_D wt(D)$, *where the sum is taken over pipe dreams D associated to the permutation w.*

Corollary 16.6. $\mathfrak{S}_w(x,-y) \in \mathbb{Z}_{\geq 0}[x_1,\ldots,x_n;y_1,\ldots,y_n]$.

We also have stability: S_n embeds into S_{n+1} in the obvious way: given $w \in S_n$, produce $\widetilde{w} \in S_{n+1}$ by acting by w in the first n letters and fixing the last one. Then, $\mathfrak{S}_w(x;y) = \mathfrak{S}_{\widetilde{w}}(x;y)$. Indeed, the only way to get a Pipe Dream associated to $\widetilde{w}$, we need to break all of the outermost crossings to get a pipe from x_{n+1} to y_{n+1}, then build a Pipe Dream associated to w underneath. The weight of the Pipe Dream of size n is the same as the one of size $n+1$.

We now define S_∞ as the injective limit of all of the finite symmetric groups: concretely this the group of permuations of $\mathbb{Z}$ that are eventually the identity. By stability, we can define $\mathfrak{S}_w(x;y)$ for any $w \in S_\infty$.

Proposition 16.7. $\mathfrak{S}_w(x;y) = \mathfrak{S}_w(x;0)$ *form a linear basis of* $\mathbb{C}[x_1,x_2,\ldots]$.

We'll prove this later.

A pipe dream for w can be reflected over the line $y=x$, yielding a pipe dream for w, and swapping the roles of the x and y. We find that $\mathfrak{S}_w(x,-y) = \mathfrak{S}_{w^{-1}}(y,-x)$.

We have Cauchy formulas: first recall the story for Schur polynomials.

- (Cauchy) $\prod_{i,j} \frac{1}{1-x_iy_j} = \sum_\lambda s_\lambda(x)s_\lambda(y)$
- (dual Cauchy) $\prod_{i,j}(1+x_iy_j) = \sum_\lambda s_\lambda(s)s_{\lambda'}(y)$

For Schubert polynomials,

$$\mathfrak{S}_{w_0}(x;-y) = \prod_{i+j\leq n, i,j\geq 1}(x_i+y_j) = \sum_{w\in S_n}\mathfrak{S}_w(x)\mathfrak{S}_{ww_0}(y).$$

More generally, we have the following theorem.

Theorem 16.8. *For any $w \in S_n$,*

$$\mathfrak{S}_w(x;-y) = \sum \mathfrak{S}_u(x)\mathfrak{S}_v(y).$$

where the sum is over all $u,v \in S_n$ where $w = v^{-1}u$ and $\ell(w) = \ell(u)+\ell(v)$.

Exercise: can you recover the usual Cauchy formulas from the Schubert polynomial version?

17 November 5, 2014

Recall:

Lemma 17.1. *$\phi_n h_i(x_{i+1} - x_i) = s_i(\phi_n)$.*

We will now prove a Cauchy formula for Schubert polynomials:

Theorem 17.2. *For any $w \in S_n$,*

$$\mathfrak{S}_w(x; -y) = \sum \mathfrak{S}_u(x)\mathfrak{S}_v(y).$$

where the sum is over all $u, v \in S_n$ where $w = v^{-1}u$ and $\ell(w) = \ell(u) + \ell(v)$.

Proof. Let $\phi(x, y) = \phi_n = \sum_{w \in S_n} \mathfrak{S}_w(x, y)u_w$. We have $\phi(x, 0) = \sum_{w \in S_n} \mathfrak{S}_w(x)u_w$ and $\phi(0, y) = \sum_{w \in S_n} \mathfrak{S}_w(0, y)u_w = \sum_{w \in S_n} \mathfrak{S}_{w^{-1}}(-y)u_w$. Because $u_v \cdot u_w = u_{vw}$ if $\ell(vw) = \ell(v) + \ell(w)$ and $u_v \cdot u_w = 0$ otherwise, the identity is equivalent to $\phi(x, y) = \phi(0, y)\phi(x, 0)$.

In fact, we have $\phi(x, 0) = h_{n-1}(x_1) \cdots h_1(x_1)h_{n-1}(x_2) \cdots h_2(x_2) \cdots h_{n-1}(x_{n-1})$, and each factor $h_i(x)$ has inverse $h_i(-x)$, so the whole product may be inverted term-by-term (order needs to be reversed): $\phi(x, 0)^{-1} = h_{n-1}(-x_{n-1}) \cdots h_2(-x_2) \cdots h_{n-1}(-x_2)h_1(-x_1) = H$, and it suffices to prove $\phi(x, y)\phi(x, 0)^{-1} = \phi(0, y)$. Denote $\psi(x_1, \ldots, x_n) = \phi(x, y)$, so that $\psi(0, \ldots, 0) = \phi(0, y)$. We need to show that $\psi(x_1, \ldots, x_n)H = \psi(0, \ldots, 0)$.

Because Schubert polynomials don't depend on the last variable x_n (by construction, or by Pipe Dreams), $\psi(x_1, \ldots, x_n) = \psi(x_1, \ldots, x_{n-1}, 0)$. Multiply on the right by the leftmost term of H, namely $h_{n-1}(0 - x_{n-1})$, which swaps the last two variables (by the lemma from before), giving $\psi(x_1, \ldots, x_{n-2}, 0, x_{n-1}) = \psi(x_1, \ldots, x_{n-2}, 0, 0)$. Next, multiply on the right by the next two terms of H, which swaps x_{n-2} with the penultimate 0, then the last 0; then replace the x_{n-2}, which is now in the last argument, with 0. Continuing in this way, we get $\psi(0, \ldots, 0)$ at the end. ∎

Recall pipe dreams from last time, and the following theorem:

Theorem 17.3. *$\mathfrak{S}_w(x, y) = \sum_D wt(D)$, where the sum is taken over pipe dreams D associated to the permutation w.*

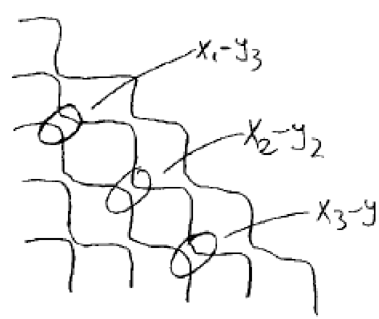

Figure 28: $\mathfrak{S}_{\text{id}}$

Note that $\mathfrak{S}_{\text{id}} = 1$, because there's only one pipe dream associated to the identity, and there are no intersections. Also, $\mathfrak{S}_{s_k}(x, y) = x_1 + \cdots + x_k - y_1 - \cdots - y_k$ – shown by picture for $k = 3$ in Figure 28. In particular, $\mathfrak{S}_{s_k}(x) = x_1 + \cdots + x_k = e_1(x_1, \ldots, x_k)$. In general, $\mathfrak{S}_w(x, y)$ is a bihomogeneous polynomial of degree $\ell(w)$ (clear from pipe dreams or divided difference operators).

Definition 17.4. Fix $1 \leq k \leq n$. $w \in S_n$ is **Grassmannian** if $w(1) < w(2) < \cdots < w(k)$ and $w(k+1) < \cdots < w(n)$, w has at most one descent, in position k.

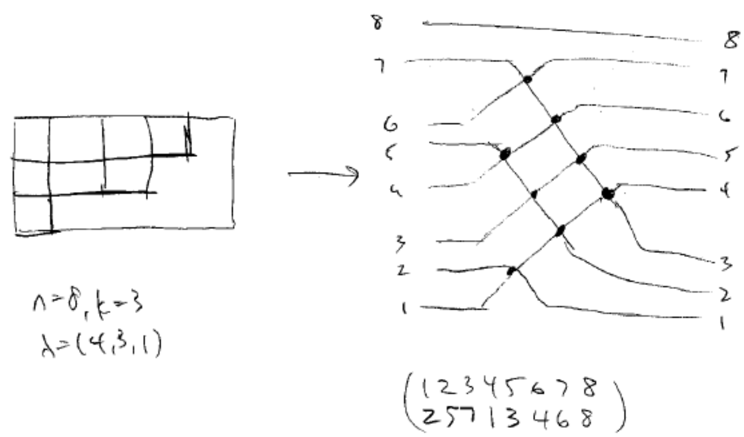

Figure 29: Young diagram to Grassmann permutation

Such permutations are in bijection with k-element subsets $I = \{w(1), \ldots, w(k)\}$, which in turn are in bijection with Young diagrams $\lambda \subset k \times (n-k)$. After some rotations, we can express this bijection as in Figure 29.

Sidenote:

Exercise 17.5. *$w \in S_n$ is* ***fully commutative*** *if any two reduced decompositions for w are obtained from each other via 2-moves (these include Grassmannian permutations). Show that w is fully commutative iff it is 321-avoiding, and that the number of such permutations is the Catalan number C_n.*

What is the Schubert polynomial associated to a Grasmannian permutation? Given $\lambda \subset k \times (n-k)$, let w_λ be the corresponding Grasmannian permutation. Then:

Theorem 17.6. $\mathfrak{S}_{w_\lambda}(x) = s_\lambda(x_1, \ldots, x_k)$.

Monomials on the left hand side correspond to pipe dreams, and monomials on the right hand side correspond to SSYT. We want a bijection between these objects. For convenience, we'll consider anti-SSYT, which means that numbers weakly decrease across rows and strongly decrease down rows. Start with the anti-SSYT as in the left of Figure 30

Then, build a pipe dream as follows: start in the third column, then go up over two crossings, corresponding to the two 3s in the first row of the anti-SSYT. Then, avoid the next crossing, move to the left, then start moving up again. The next entry in the first row is a single 1, so move up and go over one crossing, then avoid the next by moving to the left. Now start over with the second row of the anti-SSYT, and the second column of the pipe dream, and finally do the third row and the first column of the pipe dream. The resulting pipes should not cross, and the rest of the pipes can be filled in uniquely so that there are no more crossings.

The result is shown on the right of Figure 30.

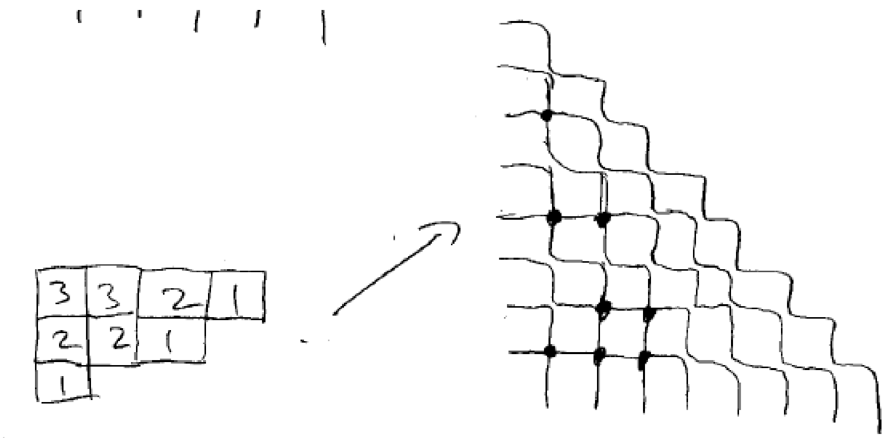

Figure 30: Pipe Dream from SSYT

18 November 7, 2014

Today all Schubert polynomials will be in one variable.

Definition 18.1. The **coinvariant algebra** C_n is $\mathbb{C}[x_1, \ldots, x_n]/I_n$, where I_n is the ideal generated by by symmetric polynomials with no constant term.

Theorem 18.2. $\dim C_n = n!$. *In fact,* $H^*(\mathrm{Fl}_n, \mathbb{C}) \cong C_n$, *and the cohomology ring has a linear basis of Schubert classes.*

Let V_n denote the span of "staircase monomials" $x^a = x_1^{a_1} \cdots x_n^{a_n}$, where $0 \le a_i \le n - i$ for all i (i.e. monomials dividing $\mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$. Clearly $\dim V_n = n!$.

Theorem 18.3. *We have:*

1. *The Schubert polynomials* $\mathfrak{S}_w$ *form a linear basis of* V_n*.*
2. *The cosets of Schubert polynomials* $\overline{\mathfrak{S}_w} = \mathfrak{S}_w + I_n$ *form a linear basis of* C_n*.*
3. *The cosets of staircase polynomials* $\overline{x^a} = x^a + I_n$ *form a linear basis of* C_n*.*

Recall that if $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced decomposition, $\partial_w = \partial_{i_1} \cdots \partial_{i_\ell}$, then $\mathfrak{S}_w = \mathfrak{S}_{w^{-1}w_0}(\mathfrak{S}_{w_0})$.

Lemma 18.4. *If* $u, w \in S_n$ *with* $\ell(u) = \ell(w)$*, then* $\partial_u(\mathfrak{S}_w) = \partial_{uw}$*.*

Proof. $\partial_u(\mathfrak{S}_w) = \partial_u(\partial_{w^{-1}w_0}\mathfrak{S}_{w_0}) = (\partial_u \partial_{w^{-1}w_0})(\mathfrak{S}_w)$. But $\partial_u \partial_{w^{-1}w_0}$ unless $\ell(u) + \ell(w^{-1}w_0) = \ell(uw^{-1}w_0)$. But $\ell(w^{-1}w_0) = \ell(w_0) - \ell(w)$, so this only happens when $uw^{-1}w_0 = w_0$, i.e. $u = w$. Now, $\partial_{w_0}(\mathfrak{S}_{w_0}) = \mathfrak{S}_{\mathrm{id}} = 1$ (clear from Pipe Dreams). □

Lemma 18.5. *The* $\mathfrak{S}_w$ *are linearly independent.*

Proof. Suppose not; because Schubert polynomials are homogeneous, we can assume $\sum_w \alpha_w \mathfrak{S}_w = 0$ where we sum over w of length ℓ. Now, for each such w, apply ∂_w to both sides; by the previous lemma, $\alpha_w = 0$. □

Proof of Theorem 18.3(1). First, note that $\mathfrak{S}_w \in V_n$; this is clear from Pipe Dreams (or from divided differences, or the Cauchy formula). They are linearly independent, and span a subspace of V_n with equal dimension, so it's the whole space. The other two statements are equivalent. $\square$

Recall the Kostka numbers $K_{\lambda\mu}$, defined by $s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$; $K_{\lambda\mu}$ is the number of SSYT of shape λ and weight μ. Analogously:

Definition 18.6. Let $K = (K_{w,a})$ be the **Kostka-Schubert** matrix, indexed by permutations w and vectors a corresponding to staircase monomials, where $\mathfrak{S}_w = \sum K_{w,a} x^a$; $K_{w,a}$ is the number of pipe dreams with permutation w and weight x^a.

Exercise 18.7. *Find some orders on w, a that makes this matrix upper triangular with 1s down the diagonal.*

Problem: how do you invert this matrix? (This was done by Egecioglu-Remmel for the Kostka matrix)

Corollary 18.8. *The polynomials $\mathfrak{S}_w$ for $w \in S_\infty$ form a linear basis for $\mathbb{C}[x_1, x_2, \ldots]$.*

To address parts (2) and (3) of the Theoremschubertspan, we need the theory of Grobner bases. Let $I \subset \mathbb{C}[x_1, \ldots, x_n]$ be a non-zero ideal. Fix a total order on monomials satisfying:

- $x^a \prec x^b \Rightarrow x^a x_i \prec x^b x_i$ for all i;
- $x^a \prec x^b$ if x^b is divisible by x^a.

For $f \in \mathbb{C}[x_1, \ldots, x_n]$, let $in(f)$ be the leading term of f (highest in the monomial order, and let $M = in(I)$ denote the monomial ideal spanned by $in(f)$, for $f \in I$. We can choose a set of minimal generators $x^{a_1}, \ldots, x^{a_N}$ of M; these can be found when $n = 2$ by mapping each monomial $x^a y^b$ to a point $(a, b) \in \mathbb{Z}^2$, then taking the corners of the lattice path bounding the discrete region formed by these points. (Similar thing for higher n).

Theorem 18.9. *There exists a unique **reduced Grobner basis** $f_1, \ldots, f_N \in I$ such that:*

- *$\langle f_1, \ldots, f_N \rangle = I$,*
- *$x^{a_i} = in(f_i)$ (and the coefficient of x^{a_i} in f_i is 1), and*
- *no monomial in f_i is divisible by x^{a_j} if $i \neq j$.*

Theorem 18.10. *For the reduced Grobner basis of I, the set of monomials $x^a \notin in(I)$ form a linear basis in $\mathbb{C}[x_1, \ldots, x_n]/I$.*

Buchberger's Algorithm: let $g_1, \ldots, g_m$ be generators of I; want to turn these into a reduced Grobner basis. For any g_i, g_j, let $g_i = \alpha x^a + \cdots, g_j = \beta x^b + \cdots$, where the terms not shown have lower order than the leading term. Let x^c be the least common multiple of x^a, x^b, and let $g_{ij} = \frac{1}{\alpha} x^{c-a} g_i - \frac{1}{\beta} x^{c-b} g_j$ (do this for all i, j). Then, "reduce," by subtracting away some terms from the g_{ij} if they are divisible by the initial monomials x^{a_i}. Repeat this until you get a Grobner basis (details omitted), which happens when all of the g_{ij} are zero.

Back to the coinvariant algebra: fix the degree lex term order, i.e. order first by total degree, then lexicographically.

Proposition 18.11 (Sturmfels)**.** *The reduced Grobner basis of I_n is given by $h_k(x_1, \ldots, n+1-k)$, where $k = 1, 2, \ldots, n$.*

Example 18.12. When $n = 3$, the Grobner basis is $h_1(x_1, x_2, x_3) = x_1 + x_2 + x_3$, $h_2(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2$, and $h_3(x_1) = x_1^3$. The minimal generators of the initial ideal M_3 are x_1^3, x_2^2, x_3, and the standard monomials are exactly the staircase monomials, so we get part (3) of the theorem.

Exercise 18.13. *Check that the ideal generated by the $h_k(x_1, \ldots, n+1-k)$ is indeed I_n.*

Helpful identity for the above:

Exercise 18.14. $h_k(x_1, \ldots, x_\ell) = \det(e_{j-i+1}(x_1, \ldots, x_{k+\ell-i}))_{i,j=1}^k$. *(This is some modification of Jacobi-Trudi.)*

19 November 12, 2014

Recall:

- $\mathfrak{S}_{w_\lambda} = s_\lambda(x_1, \ldots, x_n)$
- $\mathfrak{S}_{s_k} = x_1 + \cdots + x_k$
- $\mathfrak{S}_{s_{k-r+1}\cdots, s_k} = e_r(x_1, \ldots, x_k)$
- $\mathfrak{S}_{s_k s_{k-1}\cdots s_{k-r+1}} = h_r(x_1, \ldots, x_{k-r+1})$.

Theorem 19.1 (Chevalley-Monk formula, version 1 (S_∞))**.** *For all $w \in S_n$, $\mathfrak{S}_w\mathfrak{S}_{s_k} = \sum \mathfrak{S}_{wt_{ij}}$, where the sum is over all i, j such that $i \leq k < j$ and $\ell(wt_{ij}) = \ell(w) + 1$.*

Theorem 19.2 (Chevalley-Monk formula, version 2)**.** *Fix n, and $w \in S_n$ such that $w(n) = n$. Then $\mathfrak{S}_w\mathfrak{S}_{s_k} = \sum \mathfrak{S}_{wt_{ij}}$, where the sum is taken over all i, j such that $1 \leq i \leq k < j \leq n$ and $\ell(wt_{ij}) = \ell(w) + 1$.*

Recall from last time that the $\mathfrak{S}_w$ form a linear basis for the space V_n spanned by staircase monomials $x_1^{a_1} \cdots x_n^{a_n}$. Note now that if $\mathfrak{S}_w \in V_{n-1}$, then $\mathfrak{S}_w\mathfrak{S}_{s_k} \in V_n$.

Theorem 19.3 (Chevalley-Monk formula, version 3)**.** *For $w \in S_n$, let $\overline{\mathfrak{S}_w}$ denote the coset $\mathfrak{S}_w + I_n$, where the ideal $I_n \subset \mathbb{C}[x_1, \ldots, x_n]$ is generated by symmetric polynomials without constant term. Then, $\overline{\mathfrak{S}_w\mathfrak{S}_{s_k}} = \sum \overline{\mathfrak{S}_{wt_{ij}}}$.*

Exercise 19.4. *I_n is spanned by $\mathfrak{S}_w(x_1, \ldots, x_n, 0, \ldots)$ for $w \in S_\infty \backslash S_n$.*

Recall that $H^*(\mathrm{Gr}(k, n)) \cong \Lambda/\langle s_\lambda, \lambda \not\subset k \times (n-k)\rangle$. Analogously,

$$H^*(\mathrm{Fl}_n) \cong \mathbb{C}[x_1, \ldots, x_n]/\langle \mathfrak{S}_w(x_1, \ldots, x_n, 0, \ldots), w \in S_\infty \backslash S_n\rangle.$$

Proof of Version 2. Let $w \in S_n$ with $w(n) = n$, so that $\mathfrak{S}_w \in V_{n-1}$ and $\mathfrak{S}_w\mathfrak{S}_{s_k}$. Let $\mathfrak{S}_w\mathfrak{S}_{s_k} = \sum_u \beta_u \mathfrak{S}_u$; by looking at degrees, in order for $\beta_u \neq 0$ we need $\ell(u) = \ell(w) + \ell(s_k) = \ell(w) + 1$. Note also that $\beta_u = \partial_u(\mathfrak{S}_w\mathfrak{S}_{s_k})$.

Lemma 19.5 (Leibniz rule for ∂_u)**.** *For all $f, g \in \mathbb{C}[x_1, \ldots, x_n]$, $\partial_i(f \cdot g) = \partial_i(f)s_i(g) + f\partial_i(g)$.*

Proof. Exercise. □

Let $u = s_{i_1} \cdots s_{i_\ell}$. Then, $\beta_u = \partial_{i_1}\partial_{i_2}\cdots\partial_{i_\ell}(\mathfrak{S}_w\mathfrak{S}_{s_k})$. Applying the Leibniz rule and the fact that hitting $\mathfrak{S}_{s_k}$ (which has degree 1) with two divided difference operators gives zero, we get

$$\beta_u = \sum_{r=1}^{\ell} \partial_{i_1}\cdots\widehat{\partial_{i_r}}\cdots\partial_{i_\ell}(\mathfrak{S}_w)\partial_{i_r}(s_{i_{r+1}}\cdots s_{i_\ell}(\mathfrak{S}_{s_k}))$$

The first term is equal to 1 if and only if $s_{i_1}\cdots\widehat{s_{i_r}}\cdots s_{i_\ell} = w$ is a reduced decomposition. To see this, note that $w = u(s_{i_\ell}s_{i_{\ell-1}}\cdots s_{i_r}s_{i_{r+1}}\cdots s_{i_\ell})$ where the product of transpositions is itself a transposition t_{ij}; $i = s_{i_\ell}s_{i_{\ell-1}}\cdots s_{i_{r+1}}(i_r)$ and $j = s_{i_\ell}s_{i_{\ell-1}}\cdots s_{i_{r+1}}(i_r+1)$.

Now, let $\nu = s_{i_{r+1}}\cdots s_{i_\ell}$. Then, we have $\nu(\mathfrak{S}_{s_k}) = x_{\nu(1)} + x_{\nu(2)} + \cdots + x_{\nu(k)}$, so we find $\partial_{i_r}(\nu(\mathfrak{S}_{s_k})) = 1$ if $i \in \{1,\ldots,k\}, j \notin \{1,2,\ldots,k\}$, $\partial_{i_r}(\nu(\mathfrak{S}_{s_k})) = 1$ if $i \notin \{1,2,\ldots,k\}$ and $j \in \{1,2,\ldots,k\}$ (impossible, because $i < j$), and $\partial_{i_r}(\nu(\mathfrak{S}_{s_k})) = 0$ otherwise. From here, we get the conclusion. □

Let T_{ij} be an operator on $\mathbb{C}[x_1, x_2, \ldots]$ define by $T_{ij}(\mathfrak{S}_w) = \mathfrak{S}_{wt_{ij}}$ if $\ell(wt_{ij}) = \ell(w)+1$, and $T_{ij}(\mathfrak{S}_w) = 0$ otherwise. Then, the Chevalley-Monk formula says

$$\mathfrak{S}_{s_k}\mathfrak{S}_w = \sum_{i\leq k<j} T_{ij}(\mathfrak{S}_w).$$

Corollary 19.6. *Define the* ***Dunkl operators*** $X_k = -\sum_{i<k} T_{ik} + \sum_{j>k} T_{kj}$. *Then* $x_k\mathfrak{S}_w = X_k(\mathfrak{S}_w)$ *for any* $w \in S_\infty$.

This follows from the fact that $x_k = \mathfrak{S}_{s_k} - \mathfrak{S}_{s_{k-1}}$.

Consider the **generalized L-R coefficients** c_{uv}^w, so that $\mathfrak{S}_u\mathfrak{S}_v = \sum_w c_{uv}^w \mathfrak{S}_w$. No simple combinatorial interpretation is known, but one way to calculate them is to write $\mathfrak{S}_u$ as a product of sums of T_{ij}, then expand; each term will be give you a single Schubert polynomial.

Example 19.7. $\mathfrak{S}_{s_1s_2}\mathfrak{S}_w = x_1x_2\mathfrak{S}_w = (T_{12}+T_{13}+T_{14}+\cdots)(-T_{12}+T_{23}+T_{24}+\cdots)(\mathfrak{S}_w)$.

The problem here is that this doesn't make it obvious that the c_{uv}^w are non-negative.

The T_{ij} satisfy the following quadratic relations:

- $T_{ij}^2 = 0$.
- $T_{ij}T_{k\ell} = T_{k\ell}T_{ij}$ if i, j, k, ℓ are pairwise distinct.
- $T_{ij}T_{jk} = T_{ik}T_{ij} + T_{jk}T_{ik}$.
- $T_{jk}T_{ij} = T_{ij}T_{ik} + T_{ik}T_{jk}$.

The algebra generated by the T_{ij} with the relations above is the **Fomin-Kirillov algebra**. It is conjectured that these relations are enough to cancel all of the negative terms above, so that we get a non-negative formula for the generalized L-R coefficients.

20 November 14, 2014 (notes by Cesar Cuenca)

Recap from last time: Schubert polynomials $\mathfrak{S}_w, w \in S_\infty$, form a $\mathbb{C}$-linear basis of $\mathbb{C}[x_1, x_2, \ldots]$. We defined the operators T_{ij} on $\mathbb{C}[x_1, \ldots]$ such that $T_{ij}\mathfrak{S}_w = \mathfrak{S}_{wt_{ij}}$, if $l(wt_{ij}) = l(w)+1$, and $T_{ij}\mathfrak{S}_w = 0$ otherwise. We proved the Chevalley-Monk formula

$$\mathfrak{S}_{s_k} \cdot \mathfrak{S}_w = \sum_{i \le k < j} T_{ij} \mathfrak{S}_w.$$

We introduced the Dunkl operators $X_k = -\sum_{i<k} T_{ik} + \sum_{j>k} T_{kj}$ and derived, using that $\mathfrak{S}_{s_k} = x_1 + x_2 + \ldots + x_k$,

$$x_k \cdot \mathfrak{S}_w = X_k(\mathfrak{S}_w).$$

The operators T_{ij} satisfy the relations discovered by Sergey Fomin and Alexander Kirillov:

$$\begin{aligned} T_{ij}^2 &= 0 \\ T_{ij}T_{jk} &= T_{jk}T_{ik} + T_{ik}T_{ij} \\ T_{jk}T_{ij} &= T_{ik}T_{jk} + T_{ij}T_{ik} \\ T_{ij}T_{kl} &= T_{kl}T_{ij} \end{aligned}$$

Note. Alexander Postnikov told me after class that there is an analogous Orlik-Solomon algebra, which is simpler, and looks more *anticommutative*. Pavel Ilyich later told me that the Orlik-Solomon algebra is Koszul dual to the "Lie algebra" of the braid group.

By using the Fomin-Kirillov relations, one can then prove the following proposition.

Proposition 20.1 (Pieri Formula). *If $1 \le r \le k$, then*

$$e_r(x_1, \ldots, x_k) \cdot \mathfrak{S}_w = \sum_{i_1,\ldots,i_r,j_1,\ldots,j_r} T_{i_1j_1}T_{i_2j_2}\ldots T_{i_rj_r}(\mathfrak{S}_w),$$

where the sum is over all (2r)-tuples of integers $i_1, \ldots, i_r, j_1, \ldots, j_r \in \{1, \ldots, r\}$ such that:

1. *$i_1, \ldots, i_r \le k < j_1, \ldots, j_r$.*
2. *$i_1, \ldots, i_r$ are distinct.*
3. *$j_1 \le j_2 \le \ldots \le j_r$.*

Proof. Exercise. □

There is an analogous Pieri formula for the homogeneous symmetric polynomials instead of the elementary symmetric polynomials.

Proposition 20.2 (Pieri formula). *If $1 \le r \le k$, then*

$$h_r(x_1, \ldots, x_k) \cdot \mathfrak{S}_w = \sum_{i_1,\ldots,i_r,j_1,\ldots,j_r} T_{i_1j_1}T_{i_2j_2}\ldots T_{i_rj_r}(\mathfrak{S}_w),$$

where the sum is over all (2r)-tuples of integers $i_1, \ldots, i_r, j_1, \ldots, j_r \in \{1, \ldots, r\}$ such that:

1. *$i_1, \ldots, i_r \le k < j_1, \ldots, j_r$.*
2. *$i_1 \le i_2 \le \ldots \le i_r$.*
3. *$j_1, j_2, \ldots, j_r$ are distinct.*

We introduce an automorphism ω of the coinvariant algebra $\mathbb{C}[x_1, \ldots, x_n]/I_n$, that sends x_i to $-x_{n-i+1}$, for all i.

Remark 20.3. You should think of this as an analogue to the automorphism ω of the ring of symmetric functions Λ. Geometrically, in the case of symmetric functions, the automorphism reflects the geometric fact that $Gr(k, n) \cong Gr(n-k, n)$. In the case of Schubert polynomials, it reflects the isomorphism $Fl_n \longrightarrow Fl_n$ of the flag manifold that sends the complete flag $V_1 \subset V_2 \subset \ldots \subset V_n$ to its complementary flag $V_1^{\perp} \supset V_2^{\perp} \supset \ldots \supset V_n^{\perp}$.

Proposition 20.4. *ω sends $\mathfrak{S}_w$ to $\mathfrak{S}_{w_0ww_0}$, for any $w \in S_n$.*

Proof. For any k, we claim $\omega(\overline{\mathbb{G}}_{s_k}) = \overline{\mathbb{G}}_{s_{n-k}}$. This follows from two facts: (a) $\mathfrak{S}_{s_k} = e_1(x_1, \ldots, x_k) = x_1 + x_2 + \ldots + x_k$, and (b) $-x_n - x_{n-1} - \ldots - x_{n-k+1} = x_1 + x_2 + \ldots + x_{n-k}$ in the coinvariant algebra $C_n = \mathbb{C}[x_1, \ldots, x_n]/I_n$. Thus ω sends $\overline{\mathbb{G}}(s_k) - \overline{\mathbb{G}}(s_{k-1}) = x_k$ to $\overline{\mathbb{G}}(s_{n-k}) - \overline{\mathbb{G}}(s_{n-k+1}) = x_{n-k}$.

Thus ω sends $\overline{\mathbb{G}}_{w_0}$ to itself. It is not hard to prove (a) $\omega\partial_i = \omega\partial_{n-i}$ and (b) $s_i w_0 = w_0 s_{n-i}$. From both we can easily prove $\omega(\overline{\mathbb{G}})_w = \overline{\mathbb{G}}_{w_0 w_0}$ by reverse induction, i.e., top-down. □

Theorem 20.5. *The following are bases of C_n:*

1. $\mathfrak{S}_w$, $w \in S_n$.
2. $x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$, $0 \le a_i \le n-i$.
3. $e_{i_1, \ldots, e_{i_{n-1}}} := e_{i_1}(x_1) e_{i_2}(x_1, x_2) \ldots e_{i_{n-1}}(x_1, \ldots, x_{n-1})$, $0 \le i_r \le r$.
4. $h_{i_1, \ldots, h_{i_{n-1}}} := h_{i_1}(x_1) h_{i_2}(x_1, x_2) \ldots h_{i_{n-1}}(x_1, \ldots, x_{n-1})$, $0 \le i_r \le n-r$.

If we denote $e_i^{(k)} = e_i(x_1, \ldots, x_k)$, then the third basis is $e_{i_1, \ldots, i_m} = e_{i_1}^{(1)} \ldots e_{i_m}^{(m)}$.

Lemma 20.6. *The polynomials $e_{i_1, \ldots, i_m}$, for all $m; i_1, i_2, \ldots, i_m$, form a basis of $\mathbb{C}[x_1, x_2, \ldots]$.*

Proof. Indeed, one can write x_i as $e_1^{(i)} - e_1^{(i-1)}$. This proves that elements of the form $e_{i_1}^{(1)} e_{i_2}^{(1)} \cdots e_{i_m}^{(1)}$ span $\mathbb{C}[x_1, x_2, \ldots]$. From the straightening rule below, it follows that $e_{i_1, \ldots, e_m}$ span $\mathbb{C}[x_1, x_2, \ldots]$. We need to show they are linearly independent.

Assume there is a finite $\mathbb{C}$-linear combination

$$\sum \alpha_{i_1, \ldots, i_m} e_{i_1}^{(1)} \cdots e_{i_m}^{(m)} = 0.$$

We show that all $\alpha_{i_1, \ldots, i_m} = 0$. For any n, let $x_{n+1} = x_{n+2} = \ldots = 0$, so that we can assume the linear combination runs over $m \le n$, $0 \le i_r \le r$ for all r. Then, by reducing modulo I_n, we have

$$\sum \alpha_{i_1, \ldots, i_m} \overline{e}_{i_1}^{(1)} \cdots \overline{e}_{i_m}^{(m)} = 0.$$

The reductions $\overline{e}_{i_1}^{(1)} \cdots \overline{e}_{i_m}^{(m)}$ with $m \le n, 0 \le i_r \le r$, span C_n because $e_{i_1}^{(1)} \cdots e_{i_r}^{(r)}$ span $\mathbb{C}[x_1, x_2, \ldots]$. As $e_i^{(n)} \in I_n$ for being symmetric without constant term, then we $\overline{e}_{i_1}^{(1)} \cdots \overline{e}_{i_m}^{(m)}$ with $m < n, 0 \le i_r \le r$, span C_n. There are $n!$ of these terms and $\dim(C_n) = \dim(\mathbb{C}[x_1, \ldots, x_n]/I_n) = n!$, so it follows that $\alpha_{i_1, \ldots, i_m} = 0$, whenever $0 \le i_r \le m$ and $m < n$. But n was chosen arbitrarily, so all $\alpha_{i_1, \ldots, i_m} = 0$. □

Lemma 20.7 (Straightening Rule)**.** *We have*

$$e_i^{(k)}e_j^{(k)} = e_i^{(k+1)}e_j^{(k)} + \sum_{l\geq 1}(e_{i-l}^{(k+1)}e_{j-l}^{(k)} - e_{i-l}^{(k)}e_{j+l}^{(k+1)}).$$

Proof. Exercise. □

Now we turn to some geometry.

The flag manifold Fl_n consists of all complete flags $W = (W_1 \subset W_2 \subset \ldots \subset W_n = \mathbb{C}^n)$. There is a natural transitive action of GL_n. The stabilizer is the Borel subgroup $B = Stab_{GL_n}(Fl_n)$. Then $Fl_n = GL_n/B$, and also $Fl_n = U_n/T$, where T is the maximal torus.

If $w \in S_n$, we let $r_w(p,q) := \#\{i \leq p, w(i) \geq q\}$.

Definition 20.8. **(Schubert Cell)** $X_w^o = \{W \in Fl_n : \dim(W_p \cap V_q) = r_w(p,q) \text{ for all } p, q\}$.

21 November 19, 2014

Recall the Chevalley-Monk formula: for $w \in S_n$, $r = 1, 2, \ldots, n-1$, we have

$$\mathfrak{S}_{s_r}\mathfrak{S}_w = \sum \mathfrak{S}_{wt_{ij}} \pmod{I_n},$$

where the sum is over i, j satisfying $i \leq r < j, \ell(wt_{ij}) = \ell(w) + 1$, and I_n is the ideal generated by symmetric polynomials in n variables with no constant term. Note that $\mathfrak{S}_{s_r} = x_1 + \cdots + x_r$.

Corollary 21.1 (Chevalley)**.** *We have*

$$(y_1x_1 + \cdots + y_nx_n)\mathfrak{S}_w(x) = \sum (y_i - y_j)\mathfrak{S}_{wt_{ij}} \pmod{I_n},$$

where the sum is taken over i, j as before.

Consider the specialization $y_1 = n, y_2 = n-1, \ldots, y_n = 1$. Then, take the weight of a saturated chain in the strong Bruhat order to be the product of the $y_i - y_j$, where the edges along the chain correspond to the transpositions t_{ij}. Then,

Theorem 21.2. $\sum_{chains} w(chain) = \binom{n}{2}!$, *where we sum over all saturated chains in the strong Bruhat order.*

On the other hand, consider all reduced decompositions of the longest permutation $w_0 = s_{i_1}\cdots s_{i_\ell}$, corresponding to saturated chains in the weak Bruhat order. Then, let the weight of a chain be the product of the i_j. Then,

Theorem 21.3. $\sum_{chains} w(chain) = \binom{n}{2}!$, *where we sum over all saturated chains in the weak Bruhat order.*

For general y_i, we get a weighted sum of

$$\frac{\binom{n}{2}!}{1!2!\cdots(n-1)!}\prod_{1\leq i<j\leq n}(y_i - y_j).$$

We will prove the first formula (strong order), but the second (weak order) is an exercise.

Definition 21.4. Give $f, g \in \mathbb{C}[x_1, \ldots, x_n]$, define the D**-pairing** $\langle f, g \rangle_D$ to be the constant term of

$$f\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) g(x_1, \ldots, x_n).$$

This is a symmetric bilinear form on $\mathbb{C}[x_1, \ldots, x_n]$.

Example 21.5. $\{x_1^{a_1} \cdots x_n^{a_n}\}$ and $\left\{\frac{x_1^{a_1}}{a_1!} \cdots \frac{x_n^{a_n}}{a_n!}\right\}$ are dual bases with respect to the pairing.

Definition 21.6. Let $I \subset \mathbb{C}[x_1, \ldots, x_n]$ be a graded ideal. The space of I**-harmonic polynomials** is the space $H_I = I^\perp$. Because I is an ideal, this is in fact the space of f such that $g\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) f = 0$ for all $g \in I$ (this is a stronger condition than the constant term being zero).

H_I is the graded dual of $\mathbb{C}[x_1, \ldots, x_n]/I$.

Lemma 21.7. *Suppose $\{\overline{f_i}\}$ is a graded basis of $\mathbb{C}[x_1, \ldots, x_n]/I$ and $\{g_i\}$ is a basis of H_I. The following are equivalent:*

1. *$\{\overline{f_i}\}$ is dual to $\{g_i\}$.*
2. *$e^{x \cdot y} = \sum_i \overline{f_i}(x) g_i(y) \pmod{\widetilde{I}}$, where $x \cdot y = x_1 y_1 + \cdots + x_n y_n$, and $\widetilde{I}$ is the extension of I to $\mathbb{C}[[x_1, \ldots, x_n]]$.*

Proof. Let $C = \sum_j \overline{f_j}(x) g_j(y)$. Then, the constant term with respect to y of $\overline{f_i}\left(\frac{\partial}{\partial y_i}, \ldots, \frac{\partial}{\partial y_n}\right) C$ is $\sum_j \overline{f_j}(x) \langle \overline{f_i}, g_j \rangle_D$. The first condition is equivalent to this expression being equal to $\overline{f_i}(x)$. On the other hand, check that $C = e^{x \cdot y}$ is the only power series that satisfies this. □

Let $C_n = \mathbb{C}[x_1, \ldots, x_n]/I_n$ be the coinvariant algebra, and let $H_n = H_{I_n} = (C_n)^*$ be the space of S_n-harmonic polynomials, i.e. the space of polynomials f such that $g\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) f = 0$ for all $g \in I_n$.

The dimension of the degree k component of H_n is the same as that of the degree k component of C_n, which is the number of permutations of length k, because we have a basis of Schubert polynomials. We also have a basis of staircase monomials, which has size equal to the number of $(a_1, \ldots, a_n)$ with $0 \le a_i \le n - i$ and $\sum a_i = k$.

Definition 21.8. The **Dual Schubert polynomial** $\mathcal{D}_w(y_1, \ldots, y_n) \in \mathbb{C}[y_1, \ldots, y_n]$ that form the dual basis of H_n to the basis $\{\overline{\mathfrak{S}_w}\}$ of C_n.

Then, we have

$$\overline{e^{x \cdot y}} = \sum_{w \in S_n} \mathfrak{S}_w(x) \mathcal{D}_w(y) \pmod{I_n}.$$

Example 21.9. $\mathcal{D}_{\mathrm{id}} = 1$, $\mathcal{D}_{w_0} = \frac{\prod_{1 \le i < j \le n}(y_i - y_j)}{1!2! \cdots (n-1)!}$.

Both $\mathfrak{S}_w$ and $\mathcal{D}_w$ are stable under the embedding $S_n \hookrightarrow S_{n+1}$. We get that the $\mathcal{D}_w$, for $w \in S_\infty$, form a basis of $\mathbb{C}[y_1, y_2, \ldots]$. Then, we have the identity $e^{x \cdot y} = \sum_{w \in S_\infty} \mathfrak{S}_w(x) \mathcal{D}_w(y)$.

Theorem 21.10. *We have*

$$\mathcal{D}_w = \frac{1}{\ell(w)!} \sum_P w(P),$$

where the sum is over all paths in the Hasse diagram of the strong Bruhat order from id *to w, and the weight of a path is the product of the Chevalley multiplicities $y_i - y_j$.*

Proof. Using the formula $(x \cdot y)\mathfrak{S}_w(x) = \sum(y_i - y_j)\mathfrak{S}_{wt_{ij}}(x)$, check that $\mathcal{D}_w = \frac{1}{\ell(w)!}\sum_P w(P)$ satisfies $e^{x\cdot y} = \sum_{w\in S_n} \mathfrak{S}_w(x)\mathcal{D}_w(y)$. □

We have a product $C_n \otimes C_n \to C_n$, defined by $\mathfrak{S}_u(x)\mathfrak{S}_v(x) = \sum_w c_{uv}^w \mathfrak{S}_w(x) \pmod{I_n}$, where the c_{uv}^w are generalized L-R coefficients. We now define a coproduct $\Delta : H_n \to H_n \otimes H_n$ defined by $f(y_1, \dots, y_n) = f(y_1 + z_1, \dots, y_n + z_n)$.

Theorem 21.11. $\mathcal{D}_w(y+z) = \sum_{u,v\in S_n} c_{uv}^w \mathcal{D}_u(y)\mathcal{D}_v(z)$.

This is analogous to the statement $s_\lambda(y_1, y_2, \dots, z_1, z_2, \dots) = \sum_{\mu,\nu} c_{\mu\nu}^\lambda s_\lambda(y)s_\mu(z)$ (coproduct of Schur polynomials).

Proof.

$$\begin{aligned}\sum_{u,v,w} c_{uv}^w \overline{\mathfrak{S}_w}(x)\mathcal{D}_u(y)\mathcal{D}_v(z) &= \sum_{u,v} (\overline{\mathfrak{S}_u}(x)\mathcal{D}_u(y))(\overline{\mathfrak{S}_v}(y)\mathcal{D}_v(z)) \\ &= e^{x\cdot y}e^{x\cdot z} \\ &= e^{x\cdot(y+z)} \\ &= \sum_w \overline{\mathfrak{S}_w}(x)\mathcal{D}_w(y+z).\end{aligned}$$

Now match up coefficients. □

22 November 21, 2014

"I have almost finished making the problem set." famous last words?

Let $\mathfrak{S}_w = \sum_a K_{w,a}x^a$, so that $x^a = \sum_w K_{a,w}^{-1}\mathfrak{S}_w$.

Theorem 22.1. *For $w \in S_\infty$, we have $\mathcal{D}_w = \sum_a K_{a,w}^{-1}\frac{x^a}{a!}$, where by $\frac{x^a}{a!} = \frac{x_1^{a_1}}{a_1!}\frac{x_2^{a_2}}{a_2!}\cdots$.*

Proof. Recall that we have D-dual bases $\{x^a\}, \{x^a/a!\}$. On the other hand, the bases $\{\mathfrak{S}_w\}$ and $\{\mathcal{D}_w\}$ are also D-dual. Hence expressing $\mathcal{D}_w$ in terms of $\{x^a/a!\}$ is the same as expressing $\mathfrak{S}_w$ in terms of x^a. □

Theorem 22.2. *Let $e_a = e_{a_1}(x_1)e_{a_2}(x_1,x_2)e_{a_3}(x_1,x_2,x_3,\dots)\cdots$. Then, for $w \in S_n$, we have $\mathfrak{S}_{ww_0} = \sum_a K_{a,w}^{-1}e_{w0(\rho-a)}$, where $\rho = (n-1,\dots,1)$ and $a = (a_1,\dots,a_{n-1})$, so that $w_0(\rho - a) = (1 - a_{n-1}, 2 - a_{n-2}, \dots, n-1-a_{n-1})$.*

Proof. By Cauchy formula,

$$\sum_{w\in S_n} \mathfrak{S}_w(x)\mathfrak{S}_{ww_0}(y) = \prod_{i+j\le n}(x_i + y_j) = \sum_{k=1}^{n-1}\sum_{i=0}^{k} y_{n-k}^{k-i}e_1(x_1,\dots,x_k)$$

Exercise (?): finish the proof. □

New topic: quantum cohomology. Recall that

$$\begin{aligned}H^*(\mathrm{Gr}(k,n)) &\cong \Lambda/\langle s_\lambda \mid \lambda \not\subset k\times(n-k)\rangle \\ &\cong \Lambda/\langle e_i, h_j \mid i > k, j > n-k\rangle \\ &\cong \mathbb{C}[x_1,\dots,x_k]^{S_k}/\langle h_{n-k+1},\dots,h_n\rangle.\end{aligned}$$

The second equality says that in order to kill all of the s_λ doesn't fit inside a $k\times(n-k)$ rectangle, it is enough to kill rows and columns that don't fit inside the rectangle (this follows from Jacobi-Trudi). This ring has a linear basis of s_λ (corresponding to Schubert varieties), and we have intersection numbers $c_{\lambda\mu\nu} = \langle\sigma_\lambda\sigma_\mu\sigma_\nu\rangle$, equal to the coefficient of $S_{k\times(n-k)}$ in $s_\lambda s_\mu s_\nu$; geometrically this is the number of points in the triple-intersection of generic translates of the Schubert varieties associated to λ, μ, ν (need $|\lambda| + |\mu| + |\nu| = k(n-k)$). Moreover, L-R coefficient $c^\nu_{\lambda\mu}$ is $c_{\lambda\mu\nu^\vee}$.

We now consider the **Gromov-Witten invariants** $c^d_{\lambda\mu\nu}$, where $\lambda, \mu, \nu \subset k\times(n-k)$ and $d \geq 0$ is an integer. This counts the number of rational curves of degree d that pass through generic translates of the Schubert varieties $\Omega_\lambda, \Omega_\mu, \Omega_\nu$. In order for this number to be finite, we will need $|\lambda| + |\mu| + |\nu| = k(n-k) + dn$. When $d = 0$, we recover the usual L-R coefficients.

Write $c^{\nu,d}_{\lambda\mu} = c^d_{\lambda\mu\nu^\vee}$. Define the **quantum product**

$$\sigma_\lambda * \sigma_\mu = \sum_{\nu,d} c^{\nu,d}_{\lambda,\mu} q^d \sigma_\nu.$$

This is associative (not obvious). Define the **quantum cohomology ring** $QH^*(\mathrm{Gr}(k,n))$ to be $H^*(\mathrm{Gr}(k,n))\otimes\mathbb{C}[q]$ as a vector space, so that Schubert classes σ_λ still form a linear basis over $\mathbb{C}[q]$.

Theorem 22.3 (Bertram)**.** *$QH^*(\mathrm{Gr}(k,n)) \cong \mathbb{C}[q][x_1,\ldots,x_k]^{S_k}/J^q_{kn}$, where $J^q_{kn} = \langle h_{n-k+1},\ldots,h_{n-1},h_n+(-1)^k q\rangle$.*

If $\lambda \subset k\times(n-k)$, then σ_λ still corresponds s_λ.

How can we calculate the GW invariants $c^{\nu,d}_{\lambda\mu}$? **Rim hook algorithm**, due to Bertram, Ciocan-Fontanine, Fulton. To compute $\sigma_\lambda * \sigma_\mu$, first compute $s_\lambda s_\mu = \sum_{\nu=(\nu_1,\ldots,\nu_k)} c^\nu_{\lambda\mu} s_\nu$. The ideal is to reduce the right hand side modulo J^q_{kn} so that all of the s_ν have ν fitting into a $k\times(n-k)$ rectangle.

Given ν, remove rim hooks (ribbons) of size n from ν, i.e. strips of length n along the border of ν with at most one box in each diagonal, such that removing the strip results in a valid Young diagram. The resulting shape $\widetilde{\nu}$ is uniquely determined, and is called the n**-core** of ν. If $\widetilde{\nu} \subset k\times(n-k)$, then it turns out that

$$s_\nu \equiv \left(\prod_{r\in R}(-1)^{h(r)}\right) q^{|R|} s_{\widetilde{\nu}} \pmod{J^q_{kn}},$$

where R is the set of removed rim hooks, and $h(r)$ is the height of a rim hook. Otherwise, $s_\nu \equiv 0 \pmod{J^q_{kn}}$. Hence, the $c^{\nu,d}_{\lambda\mu}$ is an alternating sum of the usual L-R coefficients $c^\gamma_{\lambda\mu}$.

Fact: $\sigma_\lambda * \sigma_\mu \neq 0$ for any λ, μ. (Not true in the classical case.)

Define the (k,n) **cylinder**, where we identify (i,j) with $(i-k, j+n-k)$ for any (i,j). Hence a $k\times(n-k)$ rectangle has its lower left and upper right corners identified. A Young diagram fitting inside this rectangle may then be thought of as a closed loop passing through the point at which these two corners are identified. Then, translate the endpoint and draw a new closed loop; the region in between may be thought of as some skew shape.

From here, we get **Cylindrical Schur functions**, and from here we can pull out GW invariants.

23 November 26, 2014

Recall $HQ^*(\mathrm{Gr}(k,n)) \cong \mathbb{C}[q][x_1,\ldots,x_k]^{S_k}/J^q_{k,n}$, where $J^q_{k,n} = \langle h_{n-k+1},\ldots,h_{n-1},h_n+(-1)^k q\rangle$. Schubert classes correspond to Schur functions s_λ, $\lambda \subset k\times(n-k)$. To compute $\sigma_\lambda * \sigma_\mu$, compute $s_\lambda s_\mu \pmod{J^q_{kn}}$ via Rim Hook Algorithm.

The usual isomorphism $\mathrm{Gr}(k,n) \cong \mathrm{Gr}(n-k,n)$ induces an involution $\omega : QH^*(\mathrm{Gr}(k,n)) \to QH^*(\mathrm{Gr}(n-k,n))$ taking $\sigma_\lambda \mapsto \sigma_{\lambda'}$, where $\lambda \subset k \times (n-k)$. Warning: quantum cohomology is not functorial in the same way that usual cohomology is.

More symmetric description of HQ^*: $HQ^*(\mathrm{Gr}(k,n)) \cong \mathbb{C}[q, e_1, \ldots, e_k, h_1, \ldots, h_{n-k}]/I^q_{kn}$, where I^q_{kn} is generated by coefficients in the t-expansion of

$$(1 + te_1 + te_2 + \cdots + t^k e_k)(1 - h_1 t + h_2 t - \cdots + (-1)^{n-k} t^{n-k} h_{n-k}) - (1 + (-1)^{n-} q t^n)$$

Using this description, it is now clear that $\omega : e_i \leftrightarrow h_i$. Exercise: the two descriptions of $HQ^*(\mathrm{Gr}(k,n))$ are isomorphic.

Cylindric and toric tableaux. Let $Cyl_{kn} = \mathbb{R}^2/(-k, n-k)\mathbb{Z}$. Consider a lattice path from $(k,0)$ to $(0, n-k)$, defining a shape μ. Then, consider a lattice path from $(k,0)$ to $(0,n-k)$ cutting out the shape λ, then shifted by d in each direction, so that this is a lattice path from $(k+d, d)$ to $(d, n-k+d)$. These are both closed loops on the cylinder. Then, a cylindrical tableau is a filling of the squares in between these two closed loops that is weakly increasing across rows and strictly increasing down columns. We denote the shape of the shape $\lambda/d/\mu$.

Example 23.1. See Figure 31: here $k = 6, n = 16, \lambda = (9,7,6,2,2,0), \mu = (9,9,7,3,3,1), d = 2$. The weight of this filling is $(3,9,4,6,3,2,2)$ (note that the half-boxes at both ends are identified with each other).

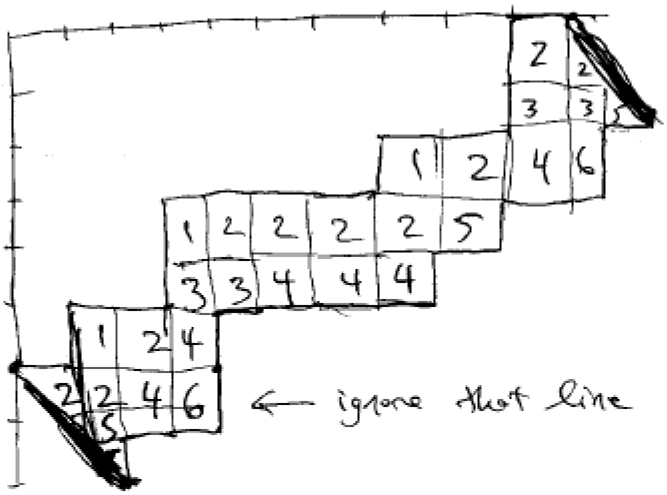

Figure 31: Example of Cylindric Tableau

Define the **cylindric Kostka numbers** $K^\beta_{\lambda/d/\mu}$ to be the number of SSYT of shape $\lambda/d/\mu$ and of weight β, and the **cylindric Schur functions** $s_{\lambda/d/\mu}(x_1, x_2, \ldots) = \sum_\beta K^\beta_{\lambda/d/\mu} x^\beta$.

Lemma 23.2. *$s_{\lambda/d/\mu}$ is symmetric.*

This can be proven using essentially the same argument as with usual Schur functions.

Definition 23.3. The **toric Schur polynomial** $s_{\lambda/d,\mu}(x_1, \ldots, x_k)$ is $s_{\lambda/d/\mu}(x_1, \ldots, x_k, 0, 0, \ldots)$.

If we define $Torus_{kn} = \mathbb{R}^2/k\mathbb{Z} \times (n-k)\mathbb{Z}$ and define toric shapes in a similar way to cylindric shapes, the name should become clear.

Recall that the GW invariants give the structure constants for multiplication in the quantum cohomology ring:

$$\sigma_\lambda * \sigma_\mu = \sum_{\nu \subset k\times(n-k)} c_{\lambda\mu}^{\nu,d} q^d \sigma_\nu.$$

Theorem 23.4. *We have*

$$s_{\lambda/d/\mu}(x_1,\ldots,x_k) = \sum_{\nu \subset k\times(n-k)} c_{\mu\nu}^{\lambda,d} s_\nu(x_1,\ldots,x_d).$$

Note that when $d=0$, $\lambda/0/\mu = \lambda/\mu$, and we recover the usual LR coefficients. Also, note from the geometric interpretation of $c_{\mu\nu}^{\lambda,d}$ that $s_{\lambda/d/\mu}(x_1,\ldots,x_k)$ is Schur-positive, but it turns out this is false for the cylindric Schur function $s_{\lambda/d/\mu}(x_1,\ldots,x_k,\ldots)$. (when we specialize, the negative terms go away).

Corollary 23.5. *We have*

$$s_{\mu^\vee/d/\lambda}(x_1,\ldots,x_k) = \sum_{\nu \subset k\times(n-k)} c_{\lambda\mu}^{\nu,d} s_{\nu^\vee}(x_1,\ldots,x_d).$$

In $H^*(\mathrm{Gr}(k,n))$, we have that $\sigma_\lambda\sigma_\mu \neq 0$ iff $\mu^\vee/\lambda$ is a valid skew shape, i.e. the paths coresponding to $\lambda,\mu^\vee$ don't intersect. In $QH^*(\mathrm{Gr}(k,n))$, $\sigma_\lambda * \sigma_\mu$ is never zero, because if the paths corresponding to $\lambda,\mu^\vee$ overlap, we can shift $\mu^\vee$ by d until they do not (this corresponds to a nonzero q^d coefficient in the quantum product). In fact, for any $\lambda,\mu \subset k\times(n-k)$ there are two non-negative integers $d_{min} \leq d_{max}$ such that q^d appears in the product $\sigma_\lambda * \sigma_\mu$ iff $d \subset [d_{min}, d_{max}]$.

d_{min} is the maximal length diagonal in the overlap region of λ and $\mu^\vee$. d_{max} is the opposite.

Exercise 23.6. $d_{min} \leq d_{max}$

To prove Theorem 23.4, first prove a quantum Pieri formula (geometrically), then use this to recover the comultiplication structure.

24 December 3, 2014

Quantum cohomology of Fl_n. As a linear space, $QH^*(\mathrm{Fl}_n) = H^*(\mathrm{Fl}_n) \otimes \mathbb{C}[q_1,\ldots,q_{n-1}]$. The Schubert classes σ_w, $w \in S_n$ form a $\mathbb{C}[q_1,\ldots,q_{n-1}]$-linear basis of $QH^*(\mathrm{Fl}_n)$.

Define the quantum product

$$\sigma_u * \sigma_v = \sum_{w\in S_n} \langle \sigma_u, \sigma_v, \sigma_w\rangle_d q^d \sigma_{w_0 w}$$

where the sum is over $d = (d_1,\ldots,d_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1}$. The $\langle \sigma_u,\sigma_v,\sigma_w\rangle_d$ is the GW invariant counting the number of rational curves of multidegree d that intersect (generic translates of) the Schubert varieties X_u, X_v, X_w. We would like combinatorial formula for this number (seems out of reach – not known for the Grassmannian).

Recall:

Theorem 24.1 (Borel)**.** $H^*(\mathrm{Fl}_n) \cong \mathbb{C}[x_1,\ldots,x_n]/\langle e_1^{(n)},\ldots,e_n^{(n)}\rangle$, *where* $e_i^{(n)} = e_i(x_1,\ldots,x_n)$.

We now have:

Theorem 24.2 (Quantum Borel; Givental-Kim)**.** *$QH^*(\mathrm{Fl}_n) \cong \mathbb{C}[q_1, \ldots, q_{n-1}][x_1, \ldots, x_n]/\langle E_1^{(n)}, \ldots, E_n^{(n)} \rangle$, where the quantum elementary polynomials $E_i^{(n)} = E_i(x_1, \ldots, x_n, q_1, \ldots, q_{n-1})$ are defined by the expansion $\det(I + \lambda A_n) = \sum_{i=0} E_i^{(n)} \lambda^i$, and*

$$A_n = \begin{bmatrix} x_1 & q_1 & & & \\ -1 & x_2 & q_2 & & \\ & \ddots & \ddots & \ddots & \\ & & & & q_{n-1} \\ & & & -1 & x_n \end{bmatrix}.$$

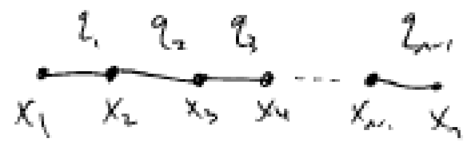

Figure 32: Weighted graph to be covered by monomers and dimers

There is also a monomer-dimer formula for $E_i^{(n)}$: E_i is the weighted sum over all disjoint systems of monomers and dimers covering n nodes in the graph shown in Figure 32. (A monomer covers one node and a dimer covers two adjacent nodes and the edge connecting them; the weight of a covering is the product of the weights of vertices associated to the monomers and edges associated to the dimers.)

We have the following recurrence:

$$E_i^{(n)} = E_i^{(n-1)} + E_{i-1}^{(n-1)} x_n + E_{i-2}^{(n-2)} q_{n-1}.$$

Schubert classes $\mathfrak{S}_w$ are represented by quantum Schubert polynomials $\mathfrak{S}_w^q(x_1, \ldots, x_n, q_1, \ldots, q_{n-1}) \in \mathbb{C}[x, q]/I_n^q$.

Proposition 24.3. *The following are bases of $\mathbb{C}[x_1, \ldots, x_n]/I_n$:*

1. $\{x_1^{a_1} \cdots x_{n-1}^{a_{n-1}}, 0 \leq a_i \leq n - i\}$.
2. $\{\mathfrak{S}_w, w \in S_n\}$.
3. *Standard elementary monomials* $e_{i_1 \cdots i_{n-1}} = e_{i_1}(x_1) e_{i_2}(x_1, x_2) \cdots e_{i_{n-1}}(x_1, \ldots, x_{n-1})$.

We can thus write $\mathfrak{S}_w = \sum \alpha_{i_1 \cdots i_{n-1}} e_{i_1 \cdots i_{n-1}}$. Now, "quantize": define $\mathfrak{S}_w^q = \sum \alpha_{i_1 \cdots i_{n-1}} E_{i_1 \cdots i_{n-1}}$, where $E_{i_1 \cdots i_{n-1}} = E_{i_1}^{(1)} E_{i_2}^{(2)} \cdots E_{i_{n-1}}^{(n-1)}$.

Theorem 24.4 (Fomin-Gelfand-Postnikov)**.** *The $\mathfrak{S}_w^q$ represent $\mathfrak{S}_w$ in $\mathbb{C}[x, q]/I_n^q$.*

As a result, we get the following algorithm for computing the quantum product: evaluate

$$\mathfrak{S}_u^q \mathfrak{S}_v^q = \sum_{w,d} C_{u,v}^{w,d} q^d \mathfrak{S}_w^q \pmod{I_n^q};$$

the $C_{u,v}^{w,d}$ are the GW-invariants.

Recall: $\mathfrak{S}_w = \sum_a K_{w,a} x^a$, where $K_{w,a}$ is the number of pipe dreams for w of weight x^a. Hence

1. $x^a = \sum_w K^{-1}_{a,w}\mathfrak{S}_w, w \in S_\infty, a \in \mathbb{Z}^\infty_{\geq 0}$.

2. $\mathcal{D}_w = \sum_a K^{-1}_{a,w} \frac{x_1^{a_1}}{a_1!}\frac{x_2^{a_2}}{a_2!}\cdots$.

3. $\mathfrak{S}_{ww_0} = \sum_{(a_1,\ldots,a_{n-1}),0\leq a_i\leq n-i} K^{-1}_{a,w} e_{1-a_{n-1},2-a_{n-2},\ldots,n-1-a_1}$.

$$\partial_k(e^{(1)}_{i_1}e^{(2)}_{i_2}\cdots e^{(n-1)}_{i_{n-1}}) = e^{(1)}_{i_1}e^{(k-1)}_{i_{k-1}}e^{(k-1)}_{i_k-1}e^{(k+1)}_{i_{k+1}}\cdots e^{(n-1)}_{i_{n-1}}.$$

To get rid of the repeated upper index, apply the straightening rule:

$$e^{(k-1)}_i e^{(k-1)}_j = \sum_{\ell\geq 0} e^{(k-1)}_{i+\ell}e^{(k)}_{j-\ell} - \sum_{m\geq 1} e^{(k-1)}_{j-m}e^{(k)}_{i+m}.$$

Example 24.5. Take $n = 4$. We can now calculate the Schubert polynomials in terms of the elementary polynomials, by applying divided difference operators and going down the weak Bruhat order. We have $\mathfrak{S}_{4321} = e_{123}$, then $\mathfrak{S}_{3421} = \partial_1(e_{123}) = e_{023}$, and

$$\begin{aligned}\mathfrak{S}_{3412} &= \partial_3(e_{0234})\\ &= \partial_3(e^{(2)}_2 e^{(3)}_3)\\ &= e^{(2)}_2 e^{(2)}_2\\ &= e_{022} - e_{013}.\end{aligned}$$

We can quantize this: so in the above example,

$$\begin{aligned}\mathfrak{S}^q_{3412} &= E_{022} - E_{013}\\ &= (x_1x_2 + q_1)(x_1x_2 + x_1x_3 + x_2x_3 + q_1 + q_2) - (x_1 + x + 2)(x_1x_2x_3 + q_1x_3 + q_2x_1)\\ &= x_1^2x_2^2 + 2q_1x_1x_2 + q_1^2 + q_1q_2 - q_2x_1^2\end{aligned}$$

Note that the first term (the only one without any q-terms) is $\mathfrak{S}_{3412}$.

Axiomatic approach for $\mathfrak{S}^q_w$.

A1. Homogeneity: $\mathfrak{S}^q_w$ is homogeneous of degree $\ell(w)$, where we make $\deg(x_i) = 1, \deg(q_i) = 2$.

A2. Classical limit: $\mathfrak{S}^q_w(x_1,\ldots,x_n,0,\ldots,0) = \mathfrak{S}_w$.

A3. Positivity: the structure constants of $\mathbb{C}[x,q]/I^q_n$ in the basis $\mathfrak{S}^q_w$ are polynomials in the q_i with non-negative integer coefficients.

A4. If $w = s_{k-i+1}s_{k-i+2}\cdots s_k$, then $\mathfrak{S}^q_w = E^{(k)}_i$.

Theorem 24.6 (Fomin-Gelfand-Postnikov)**.** *The axioms above uniquely define the $\mathfrak{S}^q_w$ (they define exactly the polynomials we describe earlier).*

Conjecture 24.7. *In fact, this is true for just the first three axioms.*

Theorem 24.8 (Quantum Monk Formula)**.**

$$\mathfrak{S}^q_{s_r}\mathfrak{S}^q_w = \sum_{a\leq k<b,\ell(wt_{ab})=\ell(w)+1}\mathfrak{S}^q_{wt_{ab}} + \sum_{a\leq k<b,\ell(wt_{ab})=\ell(w)-\ell(t_{ab})} q_{ab}\mathfrak{S}_{wt_{ab}} \pmod{I^q_n},$$

where $q_{ab} = q_aq_{a+1}\cdots q_{b+1}$. Note that $\mathfrak{S}^q_{s_r} = x_1 + \cdots + x_r$.

We can express this in terms of quantum Bruhat operators on $\mathbb{C}[S_n]$: T_{ab} sends w to wt_{ab} if $\ell(wt_{ab}) = \ell(w) + 1$, to $q_{ab}wt_{ab}$ if $\ell(wt_{ab}) = \ell(w) - \ell(t_{ab})$, and 0 otherwise. Then, multiplication by $\mathfrak{S}^q_{s_k}$ is the same as acting by $\sum_{a\leq k<b} T_{ab}$.

25 December 10, 2014

"I'll finish grading [the problem sets] at some point."

Recall that we have two ways of getting L-R coefficients:

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu$$
$$s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\nu} s_\nu.$$

We also have generalized L-R coefficients, defined by

$$\mathfrak{S}_u \mathfrak{S}_v = \sum c^w_{uv} \mathfrak{S}_w.$$

We now define **skew Schubert polynomials** $\mathfrak{S}_{u,w}$, where $u \le w$ in the strong Bruhat order, satisfying

$$\mathfrak{S}_{u,w} = \sum_v c^w_{u,w_0 v} \mathfrak{S}_v.$$

In fact, we will define $\mathfrak{S}^q_{u,v}$ for any permutations u, v.

Recall the quantum Bruhat operators T_{ij} on $QH^*(\mathrm{Fl}_n)$ sending σ_w to $\sigma_w t_{ij}$ if $\ell(wt_{ij}) = \ell(w)+1$, $q_i q_{i+1} \cdots q - j - 1 \sigma_w t_{ij}$ if $\ell(wt_{ij}) = \ell(w) - \ell(t_{ij})$, and 0 otherwise. Then, the quantum Monk formula says that

$$(x_1 + \cdots + x_k) * \sigma_w = \sigma_{s_k} * \sigma_w = \sum_{i \le k < j} T_{ij}(\sigma_w).$$

From here, we get the quantum Pieri formula

$$\sigma_{s_{k-r+1} s_{k-r+2} \cdots s_k} * \sigma_w = \sum T_{a_1,b_1} T_{a_2,b_2} \cdots T_{a_r,b_r}(\sigma_w),$$

where the sum is over the $a_1, \ldots, a_r, b_1, \ldots, b_r$ satisfying

(1) $a_1, \ldots, a_r \le k < b_1, \ldots, b_r$,

(2) a_i are distinct, and

(3) $b_1 \le b_2 \le \cdots \le b_r$.

Define the $\mathbb{C}$-linear involution ω on QH^* by $\omega(q_1^{d_1} \cdots q_{n-1}^{d_{n-1}} \sigma_w) = q_{n-1}^{d_1} \cdots q_1^{d_{n-1}} \sigma_{w_0 w w_0}$. Applying ω to the quantum Pieri formula yields

$$\sigma_{s_k s_{k+1} \cdots s_{k+r-1}} * \sigma_w = \sum T_{a_1,b_1} T_{a_2,b_2} \cdots T_{a_r,b_r}(\sigma_w),$$

where the sum is over the $a_1, \ldots, a_r, b_1, \ldots, b_r$ satisfying

(1') $a_1, \ldots, a_r \le k < b_1, \ldots, b_r$,

(2') $a_1 \le a_2 \le \cdots \le a_r$, and

(3') b_i are distinct.

Define the Pieri operator

$$\mathcal{H}_r^{(k)}(\sigma_w) = \sum_{(1'),(2'),(3')} T_{a_1,b_1}\cdots T_{a_r,b_r}(\sigma_w),$$

and

$$\mathcal{H}(y) = \sum_{\beta_1,\ldots,\beta_{n-1}, 0\le\beta_i\le n-1} y^{\rho-\beta}\mathcal{H}_{\beta_1}^{(1)}\cdots\mathcal{H}_{\beta_{n-1}}^{(n-1)},$$

where $\rho = (n-1, n-2, \ldots, 1)$.

Finally, define the quantum skew Schubert polynomial $\mathfrak{S}_{u,v}^q \in \mathbb{Z}[y_1,\ldots,y_{n-1},q_1,\ldots,q_{n-1}]$ by

$$\mathcal{H}(y)\cdot\sigma_u \mapsto \sum_v \mathfrak{S}_{u,v}^q(y,q)\sigma_v.$$

Let's reformulate this combinatorially. Define the **quantum Bruhat graph** as follows: the vertices are permutations; there is a directed edge $w \mapsto wt_{ij}$ of weight 1 if $\ell(wt_{ij}) = \ell(w)+1$, and a directed edge $w \mapsto wt_{ij}$ of weight $q_i q_{i+1}\cdots q_{j+1}$ if $\ell(wt_{ij}) = \ell(w) - \ell(t_{ij})$. Then, $\mathfrak{S}_{u,v}^q$ may be computed as follows: for $\beta \le \rho$, the coefficient of $y^{\rho-\beta}$ in $\mathfrak{S}_{u,v}^q$ is the weighted sum over directed β-admissable paths from u to v in the quantum Bruhat graph, i.e. one with labels $a_1b_1, a_2b_2, \ldots, a_{\beta_1}b_{\beta_1}, a_1'b_1', \ldots, a_{\beta_2}'b_{\beta_2}', \ldots$ (in this order; the label ij corresponds to the transposition t_{ij}), where the a's and $b's$ satisfy the conditions from $H_{\beta_1}^{(1)}, H_{\beta_2}^{(2)}, \ldots$.

Theorem 25.1.

$$\mathfrak{S}_{u,w_0v}^q(y,q) = \sum_{w,d}\langle\sigma_u,\sigma_v,\sigma_{w_0w}\rangle_d q^d\mathfrak{S}_w(y).$$

In particular, $\mathfrak{S}_{w,w_0}^q = \mathfrak{S}_{1,w_0w}^q = \mathfrak{S}_w$.

Example 25.2. Consider $u = s_1s_2, v = s_2s_1$ when $n=3$. The only $\beta = (\beta_1,\beta_2)$ for which we get a β-admissiable path is $(1,1)$; the only β-admissable path is $u \mapsto w_0 \mapsto v$, with edge labels $12, 23$. Its weight is q_2, so we get $\mathfrak{S}_{u,v}^q = y_1q_2$.

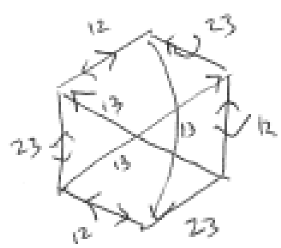

Figure 33: Quantum Bruhat graph, $n=3$

Proof of Theorem 25.1. Recall that we have the Cauchy formula

$$\prod_{i+j\le n}(x_i+y_j) = \sum_{w\in S_n}\mathfrak{S}_{ww_0}(x)\mathfrak{S}_w(y).$$

The left hand side is

$$\prod_{k=1}^{n-1}\sum_{i=0}^{k} y_{n-k}^{k-i}e_i(x_1,\ldots,x_k) = \sum_\beta y^{\rho-\beta}e_{\beta_1}^{(1)}\cdots e_{\beta_{n-1}}^{(n-1)} \in H^*(\mathrm{Fl}_n).$$

Quantizing,

$$\sum_\beta y^{\rho-\beta} E^{(1)}_{\beta_1} E^{(2)}_{\beta_2} \cdots E^{(n-1)}_{\beta_{n-1}} = \sum_w \mathfrak{S}^q_{ww_0}(x)\mathfrak{S}_w(y).$$

Apply ω:

$$\sum_\beta y^{\rho-\beta} H^{(1)}_{\beta_1} H^{(2)}_{\beta_2} \cdots H^{(n-1)}_{\beta_{n-1}} = \sum_w \mathfrak{S}^q_{w_0 w}(x)\mathfrak{S}_w(y).$$

Now, think of each side as an operator on quantum Schubert polynomials, and check that they act in the same way. □

More on the quantum Bruhat graph. Let σ be the cycle the n-cycle $(12\cdots n)$.

Theorem 25.3. *The unweighted quantum Bruhat graph is symmetric under rotations $w \mapsto cw$.*

Recall that the GW-invariants satisfy

$$c_{u,v,w}(q) = \sum_d q^d \langle \sigma_u, \sigma_v, \sigma_w \rangle_d$$

Corollary 25.4. *We have the following symmetry: $c_{u,v,w} = q_i q_{i+1} \cdots q_{j-1} c_{u,\sigma^{-1}v,\sigma w}$.*

First, rephrase the strong Bruhat order in terms of permutation matrices. Given w and its permutation matrix P_w, we can swap i and j to move up in the strong Bruhat order iff the corresponding 1's in P_w are in NW/SE relative to each other, and there are no 1's in the rectangle defined by these two entries. We can swap i, j to move down iff the corresponding 1's are NE/SW relative to each other, and there are no 1's above or below the rectangle defined by these two entries. Applying the cycle ω corresponds to moving the bottom row of P_w to the top, and we see that everything is preserved, hence we get the cyclic symmetry of the quantum Bruhat graph.