Special Relativity

For any pair of space-time events measured from two inertial reference frames S and S′, where S′ is moving relative to S with constant velocity v in the positive x direction:

$$\gamma \equiv \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Delta x = \gamma(\Delta x' + v\Delta t') \qquad \Delta x' = \gamma(\Delta x - v\Delta t)$$
$$\Delta y = \Delta y' \qquad \Delta y' = \Delta y$$
$$\Delta z = \Delta z' \qquad \Delta z' = \Delta z$$
$$\Delta t = \gamma(\Delta t' + \frac{v}{c^2}\Delta x') \qquad \Delta t' = \gamma(\Delta t - \frac{v}{c^2}\Delta x)$$

If $\Delta t'$ is the proper time between two events, then the time interval measured from another frame will be

$$\Delta t = \gamma \Delta t'$$

If $\Delta x'$ is the proper length of an object, then the length measured from another frame will be

$$\Delta x = \frac{\Delta x'}{\gamma}$$

For a particle moving with velocity $\vec{u}$ relative to S (and velocity $\vec{u}\,'$ relative to S′),

$$u_x = \frac{u'_x + v}{1 + \frac{v}{c^2}u'_x} \qquad u'_x = \frac{u_x - v}{1 - \frac{v}{c^2}u_x}$$
$$u_y = \frac{u'_y}{\gamma(1 + \frac{v}{c^2}u'_x)} \qquad u'_y = \frac{u_y}{\gamma(1 - \frac{v}{c^2}u_x)}$$
$$u_z = \frac{u'_z}{\gamma(1 + \frac{v}{c^2}u'_x)} \qquad u'_z = \frac{u_z}{\gamma(1 - \frac{v}{c^2}u_x)}$$

Electrostatics

$$\vec{F} = q\vec{E}$$ (electric force on a particle with charge q)

The electric field at point P due to a small element of charge dq is

$$\mathrm{d}\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{\mathrm{d}q}{r^2}\hat{r}$$

where $\vec{r}$ $(= r\hat{r})$ is the displacement vector that points from dq to P.

For any closed surface S,

$$\oint_{\mathrm{S}} \vec{E}\cdot \mathrm{d}\vec{A} = \frac{1}{\epsilon_0}Q_{\text{enclosed by S}}$$ (Gauss's Law)

Known Electric Field Strengths

$$E = \frac{Q}{4\pi\epsilon_0 r^2} \qquad \text{(point charge } Q\text{)}$$

$$E = \frac{\lambda}{2\pi\epsilon_0 r} \qquad \text{(infinite line of uniform charge density } \lambda \text{ C/m)}$$

$$E = \frac{\sigma}{2\epsilon_0} \qquad \text{(infinite sheet of uniform charge density } \sigma \text{ C/m}^2\text{)}$$

Electric Potential

$$U = qV \qquad \text{(electrostatic potential energy of a particle with charge } q\text{)}$$

$$V_b - V_a = -\int_{\vec{r}_a}^{\vec{r}_b} \vec{E} \cdot \mathrm{d}\vec{l} \qquad \text{(finding electric potential from electric field)}$$

$$\vec{E} = -\vec{\nabla} V \qquad \text{(finding electric field from electric potential)}$$

The electrostatic potential at point P due to a small element of charge dq, relative to $V(r = \infty) = 0$, is

$$\mathrm{d}V = \frac{1}{4\pi\epsilon_0} \frac{\mathrm{d}q}{r}$$

where r is the distance from dq to P.

Capacitance

$$Q = CV \qquad \text{(definition of capacitance)}$$

$$C = \epsilon_0 \frac{A}{d} \qquad \text{(for a vacuum- or air-filled parallel-plate capacitor with area } A \text{ and separation } d\text{)}$$

$$U = \frac{1}{2} CV^2 \qquad \text{(energy stored in a capacitor)}$$

Electric Field Energy

Electric fields store energy with a density (J/m^3) of

$$u = \frac{1}{2}\epsilon_0 E^2$$

Dielectrics

$$E = \frac{E_0}{K} \qquad \text{(electric field inside a dielectric material placed in an external field } E_0\text{)}$$

$$C = \epsilon \frac{A}{d} = K\epsilon_0 \frac{A}{d} \qquad \text{(for a parallel-plate capacitor filled with a dielectric material)}$$

Current

$$I = \frac{\mathrm{d}q}{\mathrm{d}t} = \int \vec{J} \cdot \mathrm{d}\vec{A} \qquad \text{(definition of current and current density)}$$

For any closed surface S,

$$I_{\text{leaving S}} = \oint_{\text{S}} \vec{J} \cdot \mathrm{d}\vec{A} = -\frac{\mathrm{d}}{\mathrm{d}t} Q_{\text{enclosed by S}} \qquad \text{(continuity equation)}$$

In a non-ideal conductor with conductivity σ (resistivity $\rho = \frac{1}{\sigma}$),

$$\vec{J} = \sigma \vec{E} = \frac{1}{\rho}\vec{E}$$

Resistance

$$V = IR \qquad \text{(definition of resistance)}$$

$$R = \rho \frac{l}{A} \qquad \text{(for a wire with length } l \text{ and cross-sectional area } A\text{)}$$

Magnetostatics

$$\vec{F} = q\vec{v} \times \vec{B} \qquad \text{(magnetic force on a particle with charge } q \text{ and velocity } \vec{v}\text{)}$$

$$\mathrm{d}\vec{F} = I\mathrm{d}\vec{l} \times \vec{B} \qquad \text{(magnetic force on a small segment } \mathrm{d}\vec{l} \text{ of wire carrying a current } I\text{)}$$

The magnetic field at point P due to a small segment of wire $\mathrm{d}\vec{l}$ carrying a current I is

$$\mathrm{d}\vec{B} = \frac{\mu_0}{4\pi} \frac{I\mathrm{d}\vec{l} \times \hat{r}}{r^2}$$

where $\vec{r}$ $(= r\hat{r})$ is the displacement vector that points from $\mathrm{d}\vec{l}$ to P.

For any closed loop C,

$$\oint_{\text{C}} \vec{B} \cdot \mathrm{d}\vec{l} = \mu_0 I_{\text{enclosed by C}} \qquad \text{(Ampere's Law)}$$

Known Magnetic Field Strengths

$$B = \frac{\mu_0 I}{2\pi r} \qquad \text{(infinite straight wire carrying current I)}$$

$$B = \mu_0 n I \qquad \text{(inside a long tightly-wrapped solenoid with } n \text{ turns per unit length)}$$

Induction

$$\mathcal{E} = \oint \vec{E} \cdot \mathrm{d}\vec{l} = \frac{-\mathrm{d}\Phi_B}{\mathrm{d}t} \qquad \text{(Faraday's Law)}$$

Mutual Inductance

$$M = \frac{\Phi_{B \text{ thru 2 due 1}}}{I_1} = \frac{\Phi_{B \text{ thru 1 due 2}}}{I_2}$$
$$\mathcal{E}_1 = -M\frac{\mathrm{d}I_2}{\mathrm{d}t}$$
$$\mathcal{E}_2 = -M\frac{\mathrm{d}I_1}{\mathrm{d}t}$$

Self Inductance

$$L = \frac{\Phi_B}{I}$$
$$\mathcal{E} = -L\frac{\mathrm{d}I}{\mathrm{d}t}$$
$$U = \frac{1}{2}LI^2 \qquad \text{(energy stored in an inductor)}$$

Magnetic Field Energy

Magnetic fields store energy with a density (J/m^3) of

$$u = \frac{1}{2}\frac{B^2}{\mu_0}$$

Circuit Elements

Type	Relationship	Series Combination	Parallel Combination
Resistor	$V = IR$	$R_{\text{eq}} = R_1 + R_2$	$\frac{1}{R_{\text{eq}}} = \frac{1}{R_1} + \frac{1}{R_2}$
Capacitor	$I = C\frac{\mathrm{d}V}{\mathrm{d}t}$	$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2}$	$C_{\text{eq}} = C_1 + C_2$
Inductor	$V = L\frac{\mathrm{d}I}{\mathrm{d}t}$	$L_{\text{eq}} = L_1 + L_2$	$\frac{1}{L_{\text{eq}}} = \frac{1}{L_1} + \frac{1}{L_2}$

Electric Power

$$P = VI \qquad \text{(power consumed by any electrical device)}$$
$$P = I^2R \qquad \text{(power dissipated as heat in a resistor)}$$
$$U = \frac{1}{2}CV^2 \qquad \text{(energy stored in a capacitor)}$$
$$U = \frac{1}{2}LI^2 \qquad \text{(energy stored in an inductor)}$$

Maxwell's Equations

Integral Form

$$\oint \vec{E} \cdot \mathrm{d}\vec{A} = \frac{Q}{\epsilon_0} \qquad \oint \vec{E} \cdot \mathrm{d}\vec{l} = \frac{-\mathrm{d}\Phi_B}{\mathrm{d}t}$$
$$\oint \vec{B} \cdot \mathrm{d}\vec{A} = 0 \qquad \oint \vec{B} \cdot \mathrm{d}\vec{l} = \mu_0 I + \mu_0\epsilon_0\frac{\mathrm{d}\Phi_E}{\mathrm{d}t}$$

Differential Form

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla}\times\vec{E} = \frac{-\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

Electromagnetic Waves

$f(x,t) = f(x-vt,0) = g(x-vt)$ (general form of a wave moving in the +x direction)

$\dfrac{\partial^2 f}{\partial x^2} = \dfrac{1}{v^2}\dfrac{\partial^2 f}{\partial t^2}$ (1-D wave equation to which $f(x,t)$ is a solution)

$f(x,t) = F_0\sin(kx-\omega t) = F_0\sin k(x-vt)$ (a sinusoidal wave moving in the +x direction)

$v = \dfrac{\omega}{k}$ (speed of propagation)

$\lambda = \dfrac{2\pi}{k}$ (wavelength)

$T = \dfrac{2\pi}{\omega}$ (period)

$\lambda f = v$ (relation between speed, wavelength, and frequency)

$c = \dfrac{1}{\sqrt{\mu_0\epsilon_0}} = 3\times 10^8$ m/s (speed of EM waves in a vacuum)

$B = \dfrac{E}{c}$ (relation between magnitudes of E and B in an EM plane wave)

Energy in EM Waves

The power density (energy transported per unit time per unit area) of an EM wave is given by

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} \qquad \text{(Poynting vector)}$$

Intensity is the time-averaged power density

$$S_{\text{avg}} = \frac{E_0 B_0}{2\mu_0} \qquad \text{(intensity of an EM plane wave)}$$

Math Corner

Solutions to Common Differential Equations

Decaying Exponential

The differential equation

$$\tau\frac{\mathrm{d}f(t)}{\mathrm{d}t} + f(t) = F_0$$

has solutions of the form

$$f(t) = F_0 + Ae^{-t/\tau}$$

where:

τ is called the time constant

A is an arbitrary constant that depends on the initial conditions

Simple Harmonic Oscillator

The differential equation

$$\frac{\mathrm{d}^2 f(t)}{\mathrm{d}t^2} + {\omega_0}^2 f(t) = 0$$

has solutions of the form

$$f(t) = A\cos(\omega_0 t + \phi)$$

where:
ω_0 is called the angular frequency
A and ϕ are arbitrary constants that depend on the initial conditions

Damped Harmonic Oscillator

The differential equation

$$\frac{\mathrm{d}^2 f(t)}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}f(t)}{\mathrm{d}t} + {\omega_0}^2 f(t) = 0$$

with $\gamma^2 < 4\omega_0^2$, has solutions of the form

$$f(t) = Ae^{-\frac{\gamma}{2}t}\cos(\omega_1 t + \phi)$$

where:
$\omega_1 = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$ is the angular frequency
A and ϕ are arbitrary constants that depend on the initial conditions

Vector Calculus

$$\vec{\nabla} = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \quad \text{(the ``del'' operator)}$$

$$\vec{\nabla} f = \hat{\imath}\frac{\partial f}{\partial x} + \hat{\jmath}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z} \quad \text{(gradient)}$$

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}F_x + \frac{\partial}{\partial y}F_y + \frac{\partial}{\partial z}F_z \quad \text{(divergence)}$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \hat{\imath}\left(\frac{\partial}{\partial y}F_z - \frac{\partial}{\partial z}F_y\right) - \hat{\jmath}\left(\frac{\partial}{\partial x}F_z - \frac{\partial}{\partial z}F_x\right) + \hat{k}\left(\frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x\right) \quad \text{(curl)}$$

$$\oint_S \vec{F}\cdot \mathrm{d}\vec{A} = \int_{\substack{\text{volume}\\ \text{bounded}\\ \text{by } S}} (\vec{\nabla}\cdot\vec{F})\,\mathrm{d}V \quad \text{(Divergence Theorem)}$$

$$\oint_C \vec{F}\cdot \mathrm{d}\vec{l} = \int_{\substack{\text{area}\\ \text{bounded}\\ \text{by } C}} (\vec{\nabla}\times\vec{F})\cdot \mathrm{d}\vec{A} \quad \text{(Curl Theorem)}$$

Vector Multiplication

$$\vec{A}\cdot\vec{B} = \left|\vec{A}\right|\left|\vec{B}\right|\cos\theta_{A,B} = AB_{\parallel \text{ to A}} = A_{\parallel \text{ to B}}B$$

$$\left|\vec{A}\times\vec{B}\right| = \left|\vec{A}\right|\left|\vec{B}\right|\sin\theta_{A,B} = AB_{\perp \text{ to A}} = A_{\perp \text{ to B}}B$$

Geometry

A sphere of radius R has volume $\frac{4}{3}\pi R^3$ and surface area $4\pi R^2$.

A cylinder of radius R and height h has volume $\pi R^2 h$ and surface area $2\pi Rh + 2\pi R^2$ (the first term is the area around the side, the second term is the area of the top and bottom).

Trigonometry

$\sin^2\theta + \cos^2\theta = 1$	$\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$	$\sin 45^\circ = \cos 45^\circ = \frac{1}{\sqrt{2}}$
$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$	$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$	$\sin 30^\circ = \cos 60^\circ = \frac{1}{2}$
$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$	$\sin(2\theta) = 2\sin\theta\cos\theta$	$\sin 60^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2}$

Quadratic Formula

If $ax^2 + bx + c = 0$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.