Answers to Problem Set 7

1. Economic Development and the Wage Rate
(4 points each part, 24 total)

(a) and (b) The wage rate and interest rate functions are (the figure is for α>1)

$$w = \frac{\partial Y}{\partial L} = (1-\alpha)Ak^{\alpha} \quad \text{and} \quad r = \frac{\partial Y}{\partial K} = \alpha A k^{\alpha-1}$$

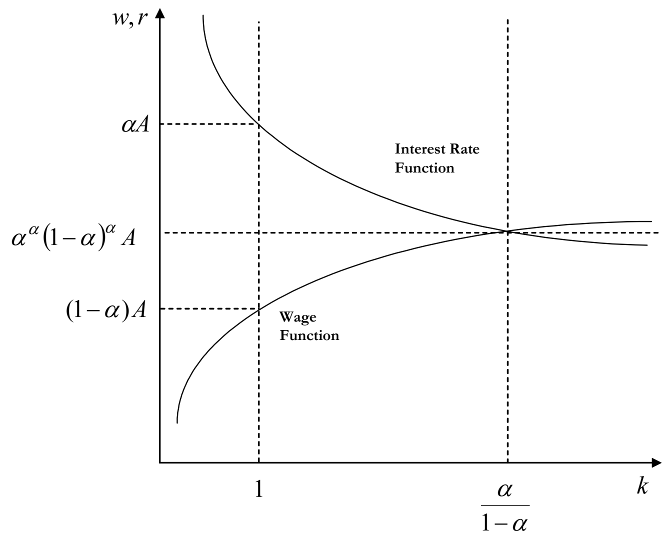

Figure 1a-b: Interest Rate and Wage as a Function of α for α>1.

(c) The convergence of a country to its steady state involves an increase in the level of capital per capita (k). The wage will increase as the level of capital per capital increases. As the capital per capita increases, the marginal productivity of each worker increases and so does his wage. When the economy arrives to the steady state level of capital per capita, capital per capita stops changing and, hence, the wage stops growing. The interest rate is the marginal return to capital so it will be falling as the economy develops and will stop falling when we arrive to the steady state.

(d) A sufficient answer for this part was to point out that even when population is growing all the time, the wage will stop increasing at the steady state and the interest rate will stop decreasing. A richer answer can be given by looking at the equations of the real wage and the interest rate at the steady state. Given a certain rate *n* of population growth, the steady state of this economy will achieved at:

$$k = \left(\frac{sA}{n+\delta}\right)^{\frac{1}{1-\alpha}}$$

the wage (w^{ss}) and the interest rate (r^{ss}) at the steady state level are given by:

$$w^{ss} = (1-\alpha)A\left(\frac{sA}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}} \quad \text{and} \quad r^{ss} = \frac{\alpha(n+\delta)}{s}$$

So that it is clear that the wage and the interest rate have a steady state level that is different from the one that is achieved when there is no population growth. The new steady state level of capital per capita is lower, the reason is that population growth puts an additional demand on the economy's capability of maintaining its capital per capita levels. The economy cannot afford to sustain the higher level of capital per capita that it had without population growth. Since capital per capita is lower, the marginal productivity of labor is lower and the marginal productivity of capital is higher. So, they are constant at the steady state but at a different level.

(e) With productivity growth there will still be a steady state for this economy, only it will not be for capital per capita but rather for capital per *productivity-weighted* capita. However the wage per person will change. Because each person in the economy will become more and more productive. On the other hand, the interest rate will stop at the steady state since the accumulation of capital will reach a point at which savings will consider productivity growth. Since productivity is growing, the returns to capital are higher, hence the steady state interest rate will be higher than without productivity growth. It was a sufficient answer to state this.

Once again a richer answer can be given by looking at the equations of the real wage and the interest rate at the steady state. First, note that the production function can be rewritten in the following way, define $A=B^{1-\alpha}$, then:

$$\tilde{y} = \left(\tilde{k}\right)^{\alpha}\text{; where } \tilde{y} = \frac{Y}{BL} \text{ and } \tilde{k} = \frac{K}{BL} = \frac{K}{A^{1/1-\alpha}L}$$

it is very clear that the sources of growth of $\tilde{k}$ are accumulation of capital in excess of productivity growth. Since there is no population growth we get:

$$\frac{\Delta\tilde{k}}{\tilde{k}} = \frac{\Delta K}{K} - \frac{a}{\alpha} \quad \text{where} \quad \frac{\Delta A}{A} = a$$

and capital will be accumulated whenever savings are greater than depreciation.

$$\Delta K = (sY - \delta K) = sK^{\alpha}(BL)^{1-\alpha} - \delta K$$

hence

$$\frac{\Delta K}{K} = s\tilde{k}^{\alpha-1} - \delta \text{ which implies that } \frac{\Delta\tilde{k}}{\tilde{k}} = s\tilde{k}^{\alpha-1} - \delta - \frac{a}{\alpha}$$

so there is a steady state for $\tilde{k}$ which is given by:

$$\tilde{k} = \left(\frac{s}{\delta + a/(1-\alpha)} \right)^{\frac{1}{1-\alpha}}$$

and hence the wage and interest rate at the steady state of this economy are given by:

$$w = (1-\alpha)B^{1-\alpha}k^{\alpha} = (1-\alpha)B^{1-2\alpha}\tilde{k}^{\alpha} = (1-\alpha)A^{\frac{1-2\alpha}{1-\alpha}}\tilde{k}^{\alpha}$$

and

$$r = \alpha B^{1-\alpha}k^{\alpha-1} = \alpha\tilde{k}^{\alpha-1}$$

it is clear that the interest rate will not change at the steady state, but the wage will change, in particular it will grow at the rate:

$$\frac{\Delta w}{w} = \frac{1-2\alpha}{1-\alpha}a$$

it is also interesting to compare the steady state levels of the interest rate in the steady state with no productivity growth (equations in part (d) with *n*=0) with this steady state (replace steady state level of $\tilde{k}$ in the interest rate equation:

With no productivity growth	With productivity growth
$r^{ss} = \frac{\alpha\delta}{s}$	$r^{ss} = \frac{\alpha\delta + \frac{\alpha a}{1-\alpha}}{s}$

It is interesting to see that the expected productivity gains from capital are reflected in the price of capital. Recall that we have discussed in class that equity is valued as the present discounted value of the dividends of the company that issued it. In this economy, every unit of capital that is invested has higher "dividends" because it is expected to increase in productivity.

(f) We could calculate an indicator of income distribution (D) as a ratio of the wage rate function to the interest rate function. In this case we have weighted each function by the population to which it is related, so that, when D=1 there is income equality. However, the owners of capital's income increases both if the interest rate increases or if capital is accumulated, so we must weight the interest rate by the capital they have. This gives:

$$D = \frac{99w}{rK} = \frac{99(1-\alpha)Ak^{\alpha}}{\alpha Ak^{\alpha-1}K} = \frac{99(1-\alpha)}{\alpha L} = \frac{(1-\alpha)}{\alpha}$$

so income distribution does not change during the converging process. Although the interest rate is falling and the wage is increasing, capitalists are accumulating capital. In the case of this production function, income distribution is fully described by parameter α.

2. Savings, Aid and Development
(5 points each part, 45 total)

(a) If the savings rate is changing, the form of the savings function will not fully reflect the form of the production function. It sill not be a scaled down version of the production function but rather a composition of two effects: the fall in productivity that is experienced when capital is accumulated and the change in the savings rate.

(b) and (c) The simplest case is the one where both models give the same steady state (Figure 2b-c).

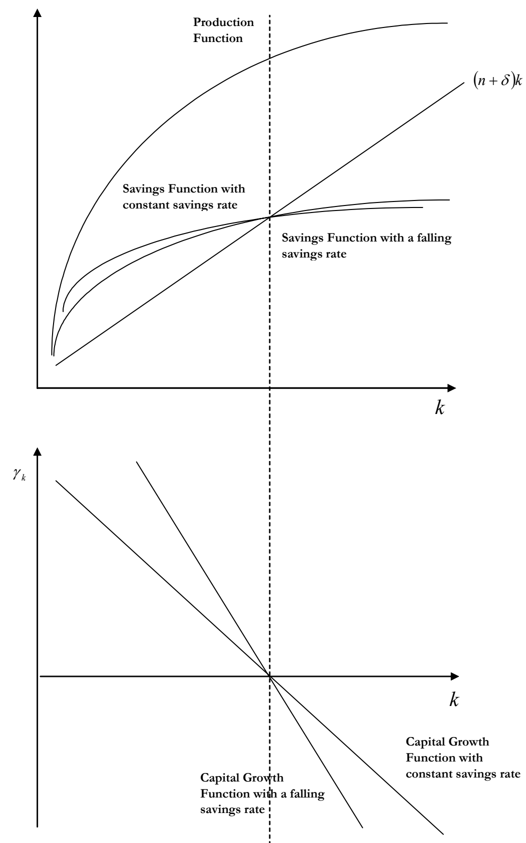

Figure 2b-c: Neo-classical Growth Model for constant savings rate and falling savings rate if both have the same steady state. For part b labels are as show, for part c switch lables of the curves.

In part b, although the steady state is the same the growth rates will be very different at different levels of capital per capita. At low levels of capital per capita the savings rate will be relatively high, so countries will start their convergence process with very high rates of growth. As they approximate the steady state there will be two effects lowering the growth rate: the old one was the fall in the marginal productivity of capital, the new one is the fall in the savings rate. In part c (swap the labels in figure 2b-c) we get the oppossite effect. We would expect countries to start of growing at lower rates and then to accelerate as they converge. They main result of convergence is not changed. What changes is the relative speed at which countries that are further away from the steady state are growing.

(d) There will be three potential steady states, the two extreme ones are stable and the central one is unstable. A middle income economy will be moved away from its unstable equilibrium by a shock. Convergence will be different for poor economies and middle income economies. We will observe convergence to the poor steady state (the poverty trap) for lower income economies and convergence to the wealthy steady state for economies with more capital per capita than the unstable equilibrium levels. We could say that convergence will be *conditional* on the initial point of an economy.

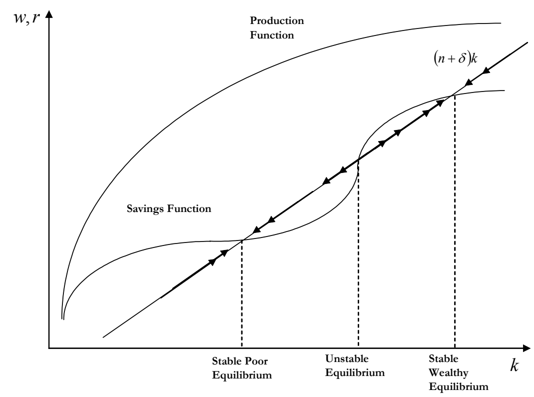

Figure 2d: Steady States for Non-Monotonic Savings Function

(e) Aid will take the form of a complement to the countries saving rate betwee k_0 and k_1.

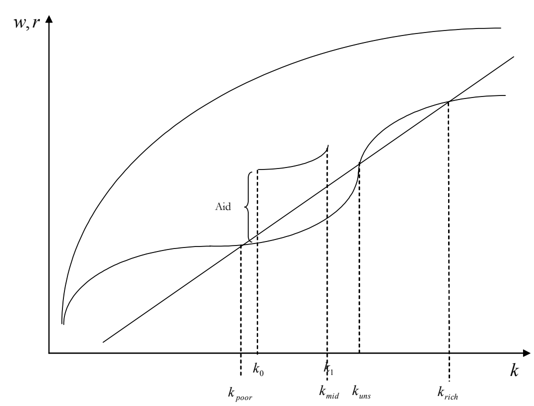

Figure 2e: Steady States for Non-Monotonic Savings Function

To make it interesting we will assume that this aid is sufficiently high to make economies between k_0 and k_1 grow. An interesting feature is that a new steady state equilibrium is created in this case (k_{mid}). This new steady state is stable but dependent on international aid.

Now we will observe a more complicated *conditional* convergence with economies converging to any of three stable steady states.

(f) There are two strategies, in this case to make poor and middle income economies get on a path to convergence with the richest economies. One is to effectively (getting the money to investment) lower the institutional requirements for aid (lower k_{poor}) and to increase k_{mid} up to k_{uns} (similar to what the World Bank does in most African countries). The other strategy is to make a lump-sum transfer of capital (similar to what is being tries in Afganistan), that puts the economy at a level of capital per capita that is higher than the unstable level. Both strategies have a number of difficulties in the real world that are discussed in (h).

(g) The effect is a fall of the middle income country towards the poverty trap. First because of the loss of capital per capita, and second because of the suspension of aid that puts the economy in a convergence process to poverty. This is the case of many African countries that were relatively stable places during the cold war due to aid form the Soviet Union or the United States and later fell into civil war. Aid was suspended because they lost strategic interest but also because aid agencies were not able to identify the legitimate government of each country or to stop the aid for being used for weapons purchases.

(h) In the absence of the possibility of war, there was a possibility that a country close to k_{poor} could have a positive shock (for example a local boom) and push itself institutionally to a level where it becomes eligible for aid. If war breaks out, the lower savings rate will just make it more unlikely that a country will achieve this, trapping the country in an even lower poverty trap that maintains the probability of war.

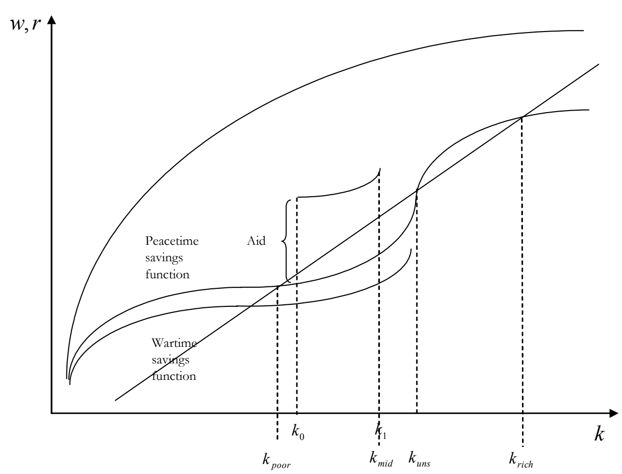

Figure 2h: War

(i) The reason for not lending to the poorest economies is that poor institutions (bad legal systems or accountability mechanisms) make it likely that aid will be diverted from its objective. Although in practice the World Bank and the United Nations still lends to these countries it finds it more attractive to help middle income countries where the effects of aid are readily observed. The effect of successful reform would be to lower the lower threshold for aid eligibility.

3. Private and Social Optima

(5 points each part, 40 total)

(a)Given a depreciation rate of δ, population growth of *n* and a savings rate for firm j of s_j. The production function can be easily expressed as:

$$y_j = k^{\alpha}{}_j A^{1-\alpha}$$

the capital accumulation function of a given firm is:

$$\Delta k_j = s_j y_j - (n+\delta)k_j$$

steady stat can be calculated as the capital per capita level where there is no more accumulation. Capital per capita, output per capita and consumption per capita in the steady state turn out to be:

$$k_j^{ss} = A\left(\frac{s_j}{n+\delta}\right)^{\frac{1}{1-\alpha}} \qquad y_j^{ss} = A\left(\frac{s_j}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}} \qquad c_j^{ss} = A(1-s_j)\left(\frac{s_j}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

(b) The golden rule for a firm is the savings rate that maximizes profits (consumption). To find the optimum we derive the consumption function and equal it to zero. First define $\phi = A(n+\delta)^{-(\alpha/(1-\alpha))}$, then:

$$\frac{\delta c_j^{ss}}{\delta s_j} = \frac{\phi}{1-\alpha}\left\{\alpha s_j^{GR\frac{2\alpha-1}{1-\alpha}} - s_j^{GR\frac{\alpha}{1-\alpha}}\right\} = 0 \quad \text{giving } s_j^{GR} = \alpha$$

(c) Steady state consumption for a given savings rate is the difference between output and the depreciation and population growth.

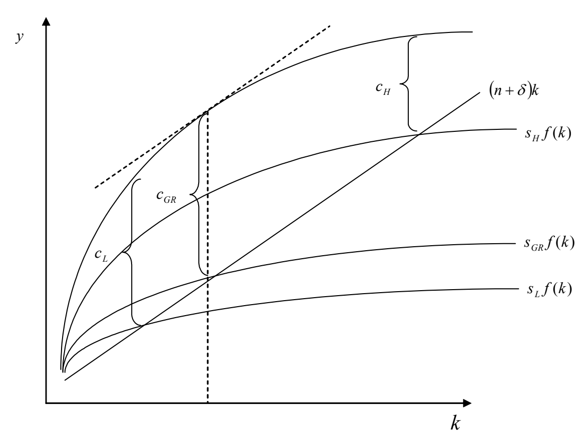

Figure 3c: The golden rule savings rate

In figure 3c we show the levels of consumptions for three savings rate. The middle one is the golden rule level rate. At this point the consumption function has zero slope, or equivalently the production function and the depreciation and population growth line have the same slope.

(d) Aggregate production in this economy (Y) will be

$$\sum_{j=1}^{M} Y_j = \sum_{j=1}^{M} K_j^{\alpha}\left(\left(\sum_{j=1}^{M} K_j\right)^{\beta} N_j\right)^{1-\alpha}$$

which, given the assumption that all firms are equal reduces to

$$Y = MY_j = MK_j^{\alpha}\left((MK_j)^{\beta} N\right)^{1-\alpha} = K_j^{\alpha+\beta(1-\alpha)} M^{1+\beta(1-\alpha)} N^{1-\alpha}$$

where K_j is the level of capital per firm. If we define national capital per capita as $K=MK_j$ then:

$$Y = K^{\alpha+\beta(1-\alpha)}(NM)^{(1-\alpha)}$$

so that

$$y = k^{\alpha+\beta(1-\alpha)} \underbrace{N^{(\beta-2)} M^{(1-\alpha)}}_{\Phi}$$

(e) If the whole economy behaved like one company the capital per capita, output per capita and consumption per capita in the steady state would turn out to be:

$$k = A\left(\frac{s\Phi}{n+\delta}\right)^{\frac{1}{(1-\alpha)(1-\beta)}} \qquad y = \Phi^{\frac{1}{(1-\alpha)(1-\beta)}}\left(\frac{s}{n+\delta}\right)^{\frac{\alpha+\beta(1-\alpha)}{(1-\alpha)(1-\beta)}} \qquad c = \Phi^{\frac{1}{(1-\alpha)(1-\beta)}}(1-s)\left(\frac{s}{n+\delta}\right)^{\frac{\alpha+\beta(1-\alpha)}{(1-\alpha)(1-\beta)}}$$

(f) First lets redefine

$$\Psi = \left(\frac{\Phi}{(n+\delta)^{\alpha+\beta(1-\alpha)}}\right)^{\frac{1}{(1-\alpha)(1-\beta)}} \quad \text{and} \quad \gamma = \frac{\alpha+\beta(1-\alpha)}{(1-\alpha)(1-\beta)}$$

then

$$c = \Psi(1-s)(s)^{\gamma}$$

the savings rate that maximizes consumption per capita is:

$$s^{GR} = \frac{\gamma}{1+\gamma} = \alpha + \beta(1-\alpha)$$

(g) The golden rule for the whole economy is greater than for each individual firm. The reason is that there is an externality generated on the whole of the economy by the investment of each firm. It would, in fact, be optimal from a social point of view to increase the savings rate of each firm.

(h) If we imposed $\beta=1$, the production function we derived in (d) would become:

$$y = \Phi k$$

which is a production function that does not have decreasing marginal returns to capital. The result is that we will have endogenous growth because every unit of capital invested in this economy will increase income without decreasing he marginal return of the next one.