14.451 Lecture Notes
Economic Growth

(and introduction to dynamic general equilibrium economies)

George-Marios Angeletos

MIT Department of Economics

Spring 2003

Contents

Preface

Chapter 1

Introduction and Growth Facts

1.1 Introduction

- In 2000, GDP per capita in the United States was \$32500 (valued at 1995 \$ prices). This high income level reflects a high standard of living.

- In contrast, standard of living is much lower in many other countries: \$9000 in Mexico, \$4000 in China, \$2500 in India, and only \$1000 in Nigeria (all figures adjusted for purchasing power parity).

- *How can countries with low level of GDP per person catch up with the high levels enjoyed by the United States or the G7?*

- Only by high growth rates sustained for long periods of time.

- *Small differences in growth rates over long periods of time can make huge differences in final outcomes.*

- US per-capita GDP grew by a factor ≈ 10 from 1870 to 2000: In 1995 prices, it was \$3300 in 1870 and \$32500 in 2000.[1] Average growth rate was $\approx 1.75\%$. If US had grown with .75% (like India, Pakistan, or the Philippines), its GDP would be only \$8700 in 1990 (i.e., $\approx 1/4$ of the actual one, similar to Mexico, less than Portugal or Greece). If US had grown with 2.75% (like Japan or Taiwan), its GDP would be \$112000 in 1990 (i.e., 3.5 times the actual one).

- At a growth rate of 1%, our children will have ≈ 1.4 our income. At a growth rate of 3%, our children will have ≈ 2.5 our income. Some East Asian countries grew by 6% over 1960-1990; this is a factor of ≈ 6 within just one generation!!!

- Once we appreciate the importance of sustained growth, the question is natural: *What can do to make growth faster?*

- Equivalently: What are the factors that explain differences in economic growth, and how can we control these factors?

- In order to prescribe policies that will promote growth, we need to understand what are the determinants of economic growth, as well as what are the effects of economic growth on social welfare. That's exactly where Growth Theory comes into picture...

[1] Let y_0 be the GDP per capital at year 0, y_T the GDP per capita at year T, and x the average annual growth rate over that period. Then, $y_T = (1+x)^T y_0$. Taking logs, we compute $\ln y_T - \ln y_0 = T \ln(1+x) \approx Tx$, or equivalenty $x \approx (\ln y_T - \ln y_0)/T$.

1.2 The World Distribution of Income Levels and Growth Rates

- As we mentioned before, in 2000 there were many countries that had much lower standards of living than the United States. This fact reflects the high cross-country dispersion in the level of income.

- **Figure 1**[2] shows the distribution of GDP per capita in 2000 across the 147 countries in the Summers and Heston dataset. The richest country was Luxembourg, with $44000 GDP per person. The United States came second, with $32500. The G7 and most of the OECD countries ranked in the top 25 positions, together with Singapore, Hong Kong, Taiwan, and Cyprus. Most African countries, on the other hand, fell in the bottom 25 of the distribution. Tanzania was the poorest country, with only $570 per person – that is, less than 2% of the income in the United States or Luxemburg!

- **Figure 2** shows the distribution of GDP per capita in 1960 across the 113 countries for which data are available. The richest country then was Switzerland, with $15000; the United States was again second, with $13000, and the poorest country was again Tanzania, with $450.

- The cross-country dispersion of income was thus as wide in 1960 as in 2000. Nevertheless, there were some important movements during this 40-year period. Argentina, Venezuela, Uruguay, Israel, and South Africa were in the top 25 in 1960, but none made it to the top 25 in 2000. On the other hand, China, Indonesia, Nepal, Pakistan, India, and Bangladesh grew fast enough to escape the bottom 25 between 1960 and 1970.

[2] Figures 1, 2 and 3 are reproduced from Barro (2003).

These large movements in the distribution of income reflects sustained differences in the rate of economic growth.

- **Figure 3** shows the distribution of the growth rates the countries experienced between 1960 and 2000. Just as there is a great dispersion in income levels, there is a great dispersion in growth rates. The mean growth rate was 1.8% per annum; that is, the world on average was twice as rich in 2000 as in 1960. The United States did slightly better than the mean. The fastest growing country was Taiwan, with a annual rate as high as 6%, which accumulates to a factor of 10 over the 40-year period. The slowest growing country was Zambia, with an negative rate at -1.8%; Zambia's residents show their income shrinking to half between 1960 and 2000.

- Most East Asian countries (Taiwan, Singapore, South Korea, Hong Kong, Thailand, China, and Japan), together with Bostwana (an outlier as compared to other sub-Saharan African countries), Cyprus, Romania, and Mauritus, had the most stellar growth performances; they were the "growth miracles" of our times. Some OECD countries (Ireland, Portugal, Spain, Greece, Luxemburg and Norway) also made it to the top 20 of the growth-rates chart. On the other hand, 18 out of the bottom 20 were sub-Saharan African countries. Other notable "growth disasters" were Venezuela, Chad and Iraq.

1.3 Unconditional versus Conditional Convergence

- There are important movements in the world income distribution, reflecting substantial differences in growth rates. Nonetheless, on average income and productivity differences are very persistent.

- **Figure 4**[3] graphs a country's GDP per worker in 1988 (normalized by the US level) against the same country's GDP per worker in 1960 (again normalized by the US level). Most observations close to the 45^o-line, meaning that most countries did not experienced a dramatic change in their relative position in the world income distribution. In other words, *income differences across countries are very persistent.*

- This also means that *poor countries on average do not grow faster than rich countries.* And another way to state the same fact is that unconditional convergence is zero. That is, if we ran the regression

$$\Delta \ln y_{2000-1960} = \alpha + \beta \cdot \ln y_{1960},$$

  the estimated coefficient β is zero.

- On the other hand, consider the regression

$$\Delta \ln y_{1960-90} = \alpha + \beta \cdot \ln y_{1960} + \gamma \cdot X_{1960}$$

  where X_{1960} is a set of country-specific controls, such as levels of education, fiscal and monetary policies, market competition, etc. Then, the estimated coefficient β turns to be positive (in particular, around 2% per annum). Therefore, if we look in a group of countries that share similar characteristics (as measured by X), the countries with lower intial income tend to grow faster than their rich counterparts, and therefore the poor countries tend to catch up with the rich countries in the same group. This is what we call *conditional convergence.*

- Conditional convergence is illustrated in **Figures 5 and 6**, for the group of OECD countries and the group of US states, respectively.

[3] Figure 4 is reproduced from Jones (1997).

1.4 Stylized Facts

The following are stylized facts that should guide us in the modeling of economic growth (Kaldor, Kuznets, Romer, Lucas, Barro, Mankiw-Romer-Weil, and others):

1. *In the short run, important fluctuations:* Output, employment, investment, and consumptio vary a lot across booms and recessions.

2. *In the long run, balanced growth:* Output per worker and capital per worker (Y/L and K/L) grow at roughly constant, and certainly not vanishing, rates. The capital-to-output ratio (K/Y) is nearly constant. The return to capital (r) is roughly constant, whereas the wage rate (w) grows at the same rates as output. And, the income shares of labor and capital (wL/Y and rK/Y) stay roughly constant.

3. Substantial *cross-country differences* in both income levels and growth rates.

4. Persistent differences versus conditional convergence.

5. *Formal education:* Highly correlated with high levels of income (obviously two-direction causality); together with differences in saving rates can "explain" a large fraction of the cross-country differences in output; an important predictor of high growth performance.

6. *R&D and IT:* Most powerful engines of growth (but require high skills at the first place).

7. *Government policies:* Taxation, infrastructure, inflation, law enforcement, property rights and corruption are important determinants of growth performance.

8. *Democracy:* An inverted U-shaped relation; that is, autarchies are bad for growht, and democracies are good, but too much democracy can slow down growth.

9. *Openness:* International trade and financial integration promote growth (but not necessarily if it is between the North and the South).

10. *Inequality:* The Kunzets curve, namely an inverted U-shaped relation between income inequality and GDP per capita (growth rates as well).

11. *Ferility:* High fertility rates correlated with levels of income and low rates of economic growth; and the process of development follows a Malthus curve, meaning that fertility rates initially increase and then fall as the economy develops.

12. *Financial markets and risk-sharing:* Banks, credit, stock markets, social insurance.

13. *Structural transformation:* agriculture→manifacture→services.

14. *Urbanization:* family production→organized production; small vilages→big cities; extended domestic trade.

15. Other institutional and social factors: colonial history, ethnic heterogeneity, social norms.

The theories of economic growth that we will review in this course seek to explain how all the above factors interrelate with the process of economic growth. Once we understand better the "mechanics" of economic growth, we will be able, not only to predict economic performance for given a set of fundamentals (*positive analysis*), but also to identify what government policies or socio-economic reforms can promote social welfare in the long run (*normative analysis*).

Chapter 2

The Solow Growth Model (and looking ahead)

2.1 Centralized Dictatorial Allocations

- In this section, we start the analysis of the Solow model by pretending that there is a benevolent dictator, or social planner, that chooses the static and intertemporal allocation of resources and dictates that allocations to the households of the economy We will later show that the allocations that prevail in a decentralized competitive market environment coincide with the allocations dictated by the social planner.

2.1.1 The Economy, the Households and the Social Planner

- Time is discrete, $t \in \{0, 1, 2, ...\}$. You can think of the period as a year, as a generation, or as any other arbitrary length of time.

- The economy is an isolated island. Many households live in this island. There are no markets and production is centralized. There is a benevolent dictator, or social planner, who governs all economic and social affairs.

- There is one good, which is produced with two factors of production, capital and labor, and which can be either consumed in the same period, or invested as capital for the next period.

- Households are each endowed with one unit of labor, which they supply inelasticly to the social planner. The social planner uses the entire labor force together with the accumulated aggregate capital stock to produce the one good of the economy.

- In each period, the social planner saves a constant fraction $s \in (0,1)$ of contemporaneous output, to be added to the economy's capital stock, and distributes the remaining fraction uniformly across the households of the economy.

- In what follows, we let L_t denote the number of households (and the size of the labor force) in period t, K_t aggregate capital stock in the beginning of period t, Y_t aggregate output in period t, C_t aggregate consumption in period t, and I_t aggregate investment in period t. The corresponding lower-case variables represent per-capita measures: $k_t = K_t/L_t$, $y_t = Y_t/L_t$, $i_t = I_t/L_t$, and $c_t = C_t/L_t$.

2.1.2 Technology and Production

- The technology for producing the good is given by

$$Y_t = F(K_t, L_t) \tag{2.1}$$

where $F : \mathsf{R}_+^2 \to \mathsf{R}_+$ is a (stationary) production function. We assume that F is continuous and (although not always necessary) twice differentiable.

- We say that the technology is "*neoclassical*" if F satisfies the following properties

  1. Constant returns to scale (CRS), or linear homogeneity:

$$F(\mu K, \mu L) = \mu F(K, L), \quad \forall \mu > 0.$$

  2. Positive and diminishing marginal products:

$$F_K(K, L) > 0, \quad F_L(K, L) > 0,$$
$$F_{KK}(K, L) < 0, \quad F_{LL}(K, L) < 0.$$

  where $F_x \equiv \partial F / \partial x$ and $F_{xz} \equiv \partial^2 F / (\partial x \partial z)$ for $x, z \in \{K, L\}$.

  3. Inada conditions:

$$\begin{aligned} \lim_{K \to 0} F_K &= \lim_{L \to 0} F_L = \infty, \\ \lim_{K \to \infty} F_K &= \lim_{L \to \infty} F_L = 0. \end{aligned}$$

- By implication, F satisfies

$$Y = F(K, L) = F_K(K, L)K + F_L(K, L)L$$

or equivalently

$$1 = \varepsilon_K + \varepsilon_L$$

where

$$\varepsilon_K \equiv \frac{\partial F}{\partial K} \frac{K}{F} \quad \text{and} \quad \varepsilon_L \equiv \frac{\partial F}{\partial L} \frac{L}{F}$$

Also, F_K and F_L are homogeneous of degree zero, meaning that the marginal products depend only on the ratio K/L.

And, $F_{KL} > 0$, meaning that capital and labor are complementary.

- Technology in intensive form: Let

$$y = \frac{Y}{L} \quad \text{and} \quad k = \frac{K}{L}.$$

Then, by CRS

$$y = f(k) \tag{2.2}$$

where

$$f(k) \equiv F(k, 1).$$

By definition of f and the properties of F,

$$\begin{aligned} f(0) &= 0, \\ f'(k) &> 0 > f''(k) \\ \lim_{k\to 0} f'(k) &= \infty, \quad \lim_{k\to\infty} f'(k) = 0 \end{aligned}$$

Also,

$$\begin{aligned} F_K(K, L) &= f'(k) \\ F_L(K, L) &= f(k) - f'(k)k \end{aligned}$$

- *Example: Cobb-Douglas technology*

$$F(K, L) = K^\alpha L^{1-\alpha}$$

In this case,

$$\varepsilon_K = \alpha, \quad \varepsilon_L = 1 - \alpha$$

and

$$f(k) = k^\alpha.$$

2.1.3 The Resource Constraint, and the Law of Motions for Capital and Labor

- Remember that there is a single good, which can be either consumed or invested. Of course, the sum of aggregate consumption and aggregate investment can not exceed aggregate output. That is, the social planner faces the following *resource constraint*:

$$C_t + I_t \leq Y_t \tag{2.3}$$

Equivalently, in per-capita terms:

$$c_t + i_t \leq y_t \tag{2.4}$$

- Suppose that population growth is $n \geq 0$ per period. The size of the labor force then evolves over time as follows:

$$L_t = (1+n)L_{t-1} = (1+n)^t L_0 \tag{2.5}$$

We normalize $L_0 = 1$.

- Suppose that existing capital depreciates over time at a fixed rate $\delta \in [0, 1]$. The capital stock in the beginning of next period is given by the non-depreciated part of current-period capital, plus contemporaneous investment. That is, *the law of motion for capital* is

$$K_{t+1} = (1-\delta)K_t + I_t. \tag{2.6}$$

Equivalently, in per-capita terms:

$$(1+n)k_{t+1} = (1-\delta)k_t + i_t$$

We can approximately write the above as

$$k_{t+1} \approx (1 - \delta - n)k_t + i_t \tag{2.7}$$

The sum $\delta + n$ can thus be interpreted as the "effective" depreciation rate of per-capita capital. (Remark: This approximation becomes arbitrarily good as the economy converges to its steady state. Also, it would be exact if time was continuous rather than discrete.)

2.1.4 The Dynamics of Capital and Consumption

- In most of the growth models that we will examine in this class, the key of the analysis will be to derive a dynamic system that characterizes the evolution of aggregate consumption and capital in the economy; that is, a system of difference equations in C_t and K_t (or c_t and k_t). This system is very simple in the case of the Solow model.

- Combining the law of motion for capital (2.6), the resource constraint (2.3), and the technology (2.1), we derive the difference equation for the capital stock:

$$K_{t+1} - K_t = F(K_t, L_t) - \delta K_t - C_t \tag{2.8}$$

  That is, the change in the capital stock is given by aggregate output, minus capital depreciation, minus aggregate consumption. On the other hand, aggregate consumption is, by assumption, a fixed fraction $(1 - s)$ of output:

$$C_t = (1 - s)F(K_t, L_t) \tag{2.9}$$

- Similarly, in per-capita terms, (2.6), (2.4) and (2.2) give the dynamics of capital

$$k_{t+1} - k_t = f(k_t) - (\delta + n)k_t - c_t, \tag{2.10}$$

whereas consumption is given by

$$c_t = (1-s)f(k_t). \tag{2.11}$$

- *From this point and on, we will analyze the dynamics of the economy in per capita terms only.* Translating the results to aggregate terms is a straightforward exercise.

2.1.5 The Policy Rule

- Combining (2.10) and (2.11), we derive *the fundamental equation of the Solow model*:

$$k_{t+1} - k_t = sf(k_t) - (\delta + n)k_t \tag{2.12}$$

Note that the above defines k_{t+1} as a function of k_t :

Proposition 1 *Given any initial point $k_0 > 0$, the dynamics of the dictatorial economy are given by the path $\{k_t\}_{t=0}^{\infty}$ such that*

$$k_{t+1} = G(k_t), \tag{2.13}$$

for all $t \geq 0$, where

$$G(k) \equiv sf(k) + (1-\delta-n)k.$$

Equivalently, the growth rate of capital is given by

$$\gamma_t \equiv \frac{k_{t+1} - k_t}{k_t} = \gamma(k_t), \tag{2.14}$$

where

$$\gamma(k) \equiv s\phi(k) - (\delta + n), \quad \phi(k) \equiv f(k)/k.$$

Proof. (2.13) follows from (2.12) and rearranging gives (2.14). **QED** ■

- G corresponds to what we will the *policy rule* in the Ramsey model. The dynamic evolution of the economy is concisely represented by the path $\{k_t\}_{t=0}^{\infty}$ that satisfies (2.12), or equivalently (2.13), for all $t \geq 0$, with k_0 historically given.

2.1.6 Steady State

- A *steady state* of the economy is defined as any level k^* such that, if the economy starts with $k_0 = k^*$, then $k_t = k^*$ for all $t \geq 1$. That is, a steady state is any fixed point k^* of (2.12) or (2.13). Equivalently, a steady state is any fixed point (c^*, k^*) of the system (2.10)-(2.11).

- A trivial steady state is $c = k = 0$: There is no capital, no output, and no consumption. This would not be a steady state if $f(0) > 0$. We are interested for steady states at which capital, output and consumption are all positive and finite. We can easily show:

Proposition 2 *Suppose $\delta+n \in (0,1)$ and $s \in (0,1)$. A steady state $(c^*, k^*) \in (0,\infty)^2$ for the dictatorial economy exists and is unique. k^* and y^* increase with s and decrease with δ and n, whereas c^* is non-monotonic with s and decreases with δ and n. Finally, $y^*/k^* = (\delta+n)/s$.*

Proof. k^* is a steady state if and only if it solves

$$0 = sf(k^*) - (\delta + n)k^*,$$

Equivalently

$$\frac{y^*}{k^*} = \phi(k^*) = \frac{\delta + n}{s} \tag{2.15}$$

where

$$\phi(k) \equiv \frac{f(k)}{k}.$$

The function ϕ gives the output-to-capital ratio in the economy. The properties of f imply that ϕ is continuous (and twice differentiable), decreasing, and satisfies the Inada conditions at $k = 0$ and $k = \infty$:

$$\phi'(k) = \frac{f'(k)k - f(k)}{k^2} = -\frac{F_L}{k^2} < 0,$$

$$\phi(0) = f'(0) = \infty \quad \text{and} \quad \phi(\infty) = f'(\infty) = 0,$$

where the latter follow from L'Hospital's rule. This implies that equation (2.15) has a solution if and only if $\delta + n > 0$ and $s > 0$. and the solution unique whenever it exists. The steady state of the economy is thus unique and is given by

$$k^* = \phi^{-1}\left(\frac{\delta + n}{s}\right).$$

Since $\phi' < 0$, k^* is a decreasing function of $(\delta + n)/s$. On the other hand, consumption is given by

$$c^* = (1 - s)f(k^*).$$

It follows that c^* decreases with $\delta + n$, but s has an ambiguous effect. **QED** ■

2.1.7 Transitional Dynamics

- The above characterized the (unique) steady state of the economy. Naturally, we are interested to know whether the economy will converge to the steady state if it starts away from it. Another way to ask the same question is whether the economy will eventually return to the steady state after an exogenous shock perturbs the economy and moves away from the steady state.

- The following uses the properties of G to establish that, in the Solow model, convergence to the steady is always ensured and is monotonic:

Proposition 3 *Given any initial $k_0 \in (0, \infty)$, the dictatorial economy converges asymptotically to the steady state. The transition is monotonic. The growth rate is positive and decreases over time towards zero if $k_0 < k^*$; it is negative and increases over time towards zero if $k_0 > k^*$.*

Proof. From the properties of f, $G'(k) = sf'(k) + (1 - \delta - n) > 0$ and $G''(k) = sf''(k) < 0$. That is, G is strictly increasing and strictly concave. Moreover, $G(0) = 0$, $G'(0) = \infty$,

$G(\infty) = \infty$, $G'(\infty) = (1 - \delta - n) < 1$. By definition of k^*, $G(k) = k$ iff $k = k^*$. It follows that $G(k) > k$ for all $k < k^*$ and $G(k) < k$ for all $k > k^*$. It follows that $k_t < k_{t+1} < k^*$ whenever $k_t \in (0, k^*)$ and therefore the sequence $\{k_t\}_{t=0}^{\infty}$ is strictly increasing if $k_0 < k^*$. By monotonicity, k_t converges asymptotically to some $\hat{k} \leq k^*$. By continuity of G, $\hat{k}$ must satisfy $\hat{k} = G(\hat{k})$, that is $\hat{k}$ must be a fixed point of G. But we already proved that G has a unique fixed point, which proves that $\hat{k} = k^*$. A symmetric argument proves that, when $k_0 > k^*$, $\{k_t\}_{t=0}^{\infty}$ is stricttly decreasing and again converges asymptotically to k^*. Next, consider the growth rate of the capital stock. This is given by

$$\gamma_t \equiv \frac{k_{t+1} - k_t}{k_t} = s\phi(k_t) - (\delta + n) \equiv \gamma(k_t).$$

Note that $\gamma(k) = 0$ iff $k = k^*$, $\gamma(k) > 0$ iff $k < k^*$, and $\gamma(k) < 0$ iff $k > k^*$. Moreover, by diminishing returns, $\gamma'(k) = s\phi'(k) < 0$. It follows that $\gamma(k_t) < \gamma(k_{t+1}) < \gamma(k^*) = 0$ whenever $k_t \in (0, k^*)$ and $\gamma(k_t) > \gamma(k_{t+1}) > \gamma(k^*) = 0$ whenever $k_t \in (k^*, \infty)$. This proves that γ_t is positive and decreases towards zero if $k_0 < k^*$ and it is negative and increases towards zero if $k_0 > k^*$. **QED** ■

- **Figure 1** depicts $G(k)$, the relation between k_t and k_{t+1}. The intersection of the graph of G with the 45■ line gives the steady-state capital stock k^*. The arrows represent the path $\{k_t\}_{t=}^{\infty}$ for a particular initial k_0.

- **Figure 2** depicts $\gamma(k)$, the relation between k_t and γ_t. The intersection of the graph of γ with the 45■ line gives the steady-state capital stock k^*. The negative slope reflects what we call "conditional convergence."

- Discuss local versus global stability: Because $\phi'(k^*) < 0$, the system is locally stable. Because ϕ is globally decreasing, the system is globally stable and transition is monotonic.

2.2 Decentralized Market Allocations

- In the previous section, we characterized the centralized allocations dictated by a social planner. We now characterize the allocations

2.2.1 Households

- Households are dynasties, living an infinite amount of time. We index households by $j \in [0, 1]$, having normalized $L_0 = 1$. The number of heads in every household grow at constant rate $n \geq 0$. Therefore, the size of the population in period t is $L_t = (1+n)^t$ and the number of persons in each household in period t is also L_t.

- We write $c_t^j, k_t^j, b_t^j, i_t^j$ for the per-head variables for household j.

- Each person in a household is endowed with one unit of labor in every period, which he supplies inelasticly in a competitive labor market for the contemporaneous wage w_t. Household j is also endowed with initial capital k_0^j. Capital in household j accumulates according to

$$(1+n)k_{t+1}^j = (1-\delta)k_t^j + i_t,$$

which we approximate by

$$k_{t+1}^j = (1-\delta-n)k_t^j + i_t. \tag{2.16}$$

Households rent the capital they own to firms in a competitive rental market for a (gross) rental rate r_t.

- The household may also hold stocks of some firms in the economy. Let π_t^j be the dividends (firm profits) that household j receive in period t. As it will become clear later on, it is without any loss of generality to assume that there is no trade of stocks.

(This is because the value of firms stocks will be zero in equilibrium and thus the value of any stock transactions will be also zero.) We thus assume that household j holds a fixed fraction α^j of the aggregate index of stocks in the economy, so that $\pi_t^j = \alpha^j \Pi_t$, where Π_t are aggregate profits. Of course, $\int \alpha^j dj = 1$.

- The household uses its income to finance either consumption or investment in new capital:

$$c_t^j + i_t^j = y_t^j.$$

Total per-head income for household j in period t is simply

$$y_t^j = w_t + r_t k_t^j + \pi_t^j. \tag{2.17}$$

Combining, we can write the budget constraint of household j in period t as

$$c_t^j + i_t^j = w_t + r_t k_t^j + \pi_t^j \tag{2.18}$$

- Finally, the consumption and investment behavior of household is a simplistic linear rule. They save fraction s and consume the rest:

$$c_t^j = (1-s) y_t^j \quad \text{and} \quad i_t^j = s y_t^i. \tag{2.19}$$

2.2.2 Firms

- There is an arbitrary number M_t of firms in period t, indexed by $m \in [0, M_t]$. Firms employ labor and rent capital in competitive labor and capital markets, have access to the same neoclassical technology, and produce a homogeneous good that they sell competitively to the households in the economy.

- Let K_t^m and L_t^m denote the amount of capital and labor that firm m employs in period t. Then, the profits of that firm in period t are given by

$$\Pi_t^m = F(K_t^m, L_t^m) - r_t K_t^m - w_t L_t^m.$$

- The firms seeks to maximize profits. The FOCs for an interior solution require

$$F_K(K_t^m, L_t^m) = r_t. \quad (2.20)$$

$$F_L(K_t^m, L_t^m) = w_t. \quad (2.21)$$

- Remember that the marginal products are homogenous of degree zero; that is, they depend only on the capital-labor ratio. In particular, F_K is a decreasing function of K_t^m/L_t^m and F_L is an increasing function of K_t^m/L_t^m. Each of the above conditions thus pins down a unique capital-labor ratio K_t^m/L_t^m. For an interior solution to the firms' problem to exist, it must be that r_t and w_t are consistent, that is, they imply the same K_t^m/L_t^m. This is the case if and only if there is some $X_t \in (0, \infty)$ such that

$$r_t = f'(X_t) \quad (2.22)$$

$$w_t = f(X_t) - f'(X_t)X_t \quad (2.23)$$

where $f(k) \equiv F(k, 1)$; this follows from the properties $F_K(K, L) = f'(K/L)$ and $F_L(K, L) = f(K/L) - f'(K/L) \cdot (K/L)$, which we established earlier.

- If (**??**)-(**??**) are satisfied, the FOCs reduce to $K_t^m/L_t^m = X_t$, or

$$K_t^m = X_t L_t^m. \quad (2.24)$$

That is, the FOCs pin down the capital labor ratio for each firm (K_t^m/L_t^m), but not the size of the firm (L_t^m). Moreover, because all firms have access to the same technology, they use exactly the same capital-labor ratio.

- Besides, (**??**) and (**??**) imply

$$r_t X_t + w_t = f(X_t). \tag{2.25}$$

It follows that

$$r_t K_t^m + w_t L_t^m = (r_t X_t + w_t) L_t^m = f(X_t) L_t^m = F(K_t^m, L_t^m),$$

and therefore

$$\Pi_t^m = L_t^m [f(X_t) - r_t X_t - w_t] = 0. \tag{2.26}$$

That is, when (**??**)-(**??**) are satisfied, the maximal profits that any firm makes are exactly zero, and these profits are attained for any firm size as long as the capital-labor ratio is optimal If instead (**??**)-(**??**) were violated, then either $r_t X_t + w_t < f(X_t)$, in which case the firm could make infinite profits, or $r_t X_t + w_t > f(X_t)$, in which case operating a firm of any positive size would entail strictly negative profits.

2.2.3 Market Clearing

- The *capital market* clears if and only if

$$\int_0^{M_t} K_t^m dm = \int_0^1 (1+n)^t k_t^j dj$$

Equivalently,

$$\int_0^{M_t} K_t^m dm = K_t \tag{2.27}$$

where $K_t \equiv \int_0^{L_t} k_t^j dj$ is the aggregate capital stock in the economy.

- The *labor market*, on the other hand, clears if and only if

$$\int_0^{M_t} L_t^m dm = \int_0^1 (1+n)^t dj$$

Equivalently,

$$\int_0^{M_t} L_t^m dm = L_t \tag{2.28}$$

where L_t is the size of the labor force in the economy.

2.2.4 General Equilibrium: Definition

- The definition of a *general equilibrium* is more meaningful when households optimize their behavior (maximize utility) rather than being automata (mechanically save a constant fraction of income). Nonetheless, it is always important to have clear in mind what is the definition of equilibrium in any model. For the decentralized version of the Solow model, we let:

Definition 4 *An equilibrium of the economy is an allocation* $\{(k_t^j, c_t^j, i_t^j)_{j\in[0,1]}, (K_t^m, L_t^m)_{m\in[0,M_t]}\}_{t=0}^{\infty}$, *a distribution of profits* $\{(\pi_t^j)_{j\in[0,1]}\}$, *and a price path* $\{r_t, w_t\}_{t=0}^{\infty}$ *such that*

(i) Given $\{r_t, w_t\}_{t=0}^{\infty}$ *and* $\{\pi_t^j\}_{t=0}^{\infty}$, *the path* $\{k_t^j, c_t^j, i_t^j\}$ *is consistent with the behavior of household* j, *for every* j.

(ii) (K_t^m, L_t^m) *maximizes firm profits, for every* m *and* t.

(iii) The capital and labor markets clear in every period

(iv) Aggregate dividends equal aggregate profits.

2.2.5 General Equilibrium: Existence, Uniqueness, and Characterization

- In the next, we characterize the decentralized equilibrium allocations:

Proposition 5 *For any initial positions* $(k_0^j)_{j\in[0,1]}$, *an equilibrium exists. The allocation of production across firms is indeterminate, but the equilibrium is unique as regards aggregate* *and household allocations. The capital-labor ratio in the economy is given by* $\{k_t\}_{t=0}^{\infty}$ *such that*

$$k_{t+1} = G(k_t) \tag{2.29}$$

for all $t \geq 0$ *and* $k_0 = \int k_0^j dj$ *historically given, where* $G(k) \equiv sf(k) + (1 - \delta - n)k$*. Equilibrium growth is given by*

$$\gamma_t \equiv \frac{k_{t+1} - k_t}{k_t} = \gamma(k_t), \tag{2.30}$$

where $\gamma(k) \equiv s\phi(k) - (\delta + n)$, $\phi(k) \equiv f(k)/k$*. Finally, equilibrium prices are given by*

$$r_t = r(k_t) \equiv f'(k_t), \tag{2.31}$$

$$w_t = w(k_t) \equiv f(k_t) - f'(k_t)k_t, \tag{2.32}$$

where $r'(k) < 0 < w'(k)$*.*

Proof. We first characterize the equilibrium, assuming it exists.

Using $K_t^m = X_t L_t^m$ by (2.24), we can write the aggregate demand for capital as

$$\int_0^{M_t} K_t^m dm = X_t \int_0^{M_t} L_t^m dm$$

From the labor market clearing condition (2.28),

$$\int_0^{M_t} L_t^m dm = L_t.$$

Combining, we infer

$$\int_0^{M_t} K_t^m dm = X_t L_t,$$

and substituting in the capital market clearing condition (2.27), we conclude

$$X_t L_t = K_t,$$

where $K_t \equiv \int_0^{L_t} k_t^j dj$ denotes the aggregate capital stock. Equivalently, letting $k_t \equiv K_t/L_t$ denote the capital-labor ratio in the economy, we have

$$X_t = k_t. \tag{2.33}$$

That is, all firms use the same capital-labor ratio as the aggregate of the economy.

Substituting (2.33) into (2.22) and (2.23) we infer that equilibrium prices are given by

$$\begin{aligned} r_t &= r(k_t) \equiv f'(k_t) = F_K(k_t, 1) \\ w_t &= w(k_t) \equiv f(k_t) - f'(k_t)k_t = F_L(k_t, 1) \end{aligned}$$

Note that $r'(k) = f''(k) = F_{KK} < 0$ and $w'(k) = -f''(k)k = F_{LK} > 0$. That is, the interest rate is a decreasing function of the capital-labor ratio and the wage rate is an increasing function of the capital-labor ratio. The first properties reflects diminishing returns, the second reflects the complementarity of capital and labor.

Adding up the budget constraints of the households, we get

$$C_t + I_t = r_t K_t + w_t L_t + \int \pi_t^j dj,$$

where $C_t \equiv \int c_t^j dj$ and $I_t \equiv \int i_t^j dj$. Aggregate dividends must equal aggregate profits, $\int \pi_t^j dj = \int \Pi_t^m dj$. By (2.26), profits for each firm are zero. Therefore, $\int \pi_t^j dj = 0$, implying

$$C_t + I_t = Y_t = r_t K_t + w_t L_t$$

Equivalently, in per-capita terms,

$$c_t + i_t = y_t = r_t k_t + w_t.$$

From (2.25) and (2.33), or equivalently from (2.31) and (2.32),

$$r_t k_t + w_t = y_t = f(k_t)$$

We conclude that the household budgets imply

$$c_t + i_t = f(k_t),$$

which is simply the resource constraint of the economy.

Adding up the individual capital accumulation rules (2.16), we get the capital accumulation rule for the aggregate of the economy. In per-capita terms,

$$k_{t+1} = (1 - \delta - n)k_t + i_t$$

Adding up (2.19) across household, we similarly infer

$$i_t = sy_t = sf(k_t).$$

Combining, we conclude

$$k_{t+1} = sf(k_t) + (1 - \delta - n)k_t = G(k_t),$$

which is exactly the same as in the centralized allocation.

Finally, existence and uniqueness is now trivial. (2.29) maps any $k_t \in (0, \infty)$ to a unique $k_{t+1} \in (0, \infty)$. Similarly, (2.31) and (2.32) map any $k_t \in (0, \infty)$ to unique $r_t, w_t \in (0, \infty)$. Therefore, given any initial $k_0 = \int k_0^j dj$, there exist unique paths $\{k_t\}_{t=0}^{\infty}$ and $\{r_t, w_t\}_{t=0}^{\infty}$. Given $\{r_t, w_t\}_{t=0}^{\infty}$, the allocation $\{k_t^j, c_t^j, i_t^j\}$ for any household j is then uniquely determined by (2.16), (2.17), and (2.19). Finally, any allocation $(K_t^m, L_t^m)_{m \in [0, M_t]}$ of production across firms in period t is consistent with equilibrium as long as $K_t^m = k_t L_t^m$. **QED** ∎

- An immediate implication is that the decentralized market economy and the centralized dictatorial economy are isomorphic:

Proposition 6 *The aggregate and per-capita allocations in the competitive market economy coincide with those in the dictatorial economy.*

Proof. Follows directly from the fact that G is the same under both regimes, provided of course that (s, δ, n, f) are the same. **QED** ■

- Given this isomorphism, we can immediately translate the steady state and the transitional dynamics of the centralized plan to the steady state and the transitional dynamics of the decentralized market allocations:

Corollary 7 *Suppose $\delta + n \in (0,1)$ and $s \in (0,1)$. A steady state $(c^*, k^*) \in (0,\infty)^2$ for the competitive economy exists and is unique, and coincides with that of the social planner. k^* and y^* increase with s and decrease with δ and n, whereas c^* is non-monotonic with s and decreases with δ and n. Finally, $y^*/k^* = (\delta + n)/s$.*

Corollary 8 *Given any initial $k_0 \in (0,\infty)$, the competitive economy converges asymptotically to the steady state. The transition is monotonic. The equilibrium growth rate is positive and decreases over time towards zero if $k_0 < k^*$; it is negative and increases over time towards zero if $k_0 > k^*$.*

2.3 Shocks and Policies

- The Solow model can be interpreted also as a primitive RBC model. We can use the model to predict the response of the economy to productivity or taste shocks, or to shocks in government policies.

2.3.1 Productivity and Taste Shock

- *Productivity shocks:* Consider a positive (negative) shock in productivity, either temporary or permanent. The $\gamma(k)$ function shifts up (down), either termporarly or permanently. What are the effects on the steady state and the transitional dynamics, in either case?

- *Taste shocks:* Consider a temporary fall in the saving rate. The $\gamma(k)$ function shifts down for a while, and then return to its initial position. What the transitional dynamics?

2.3.2 Unproductive Government Spending

- Let us now introduce a *government* in the competitive market economy. The government spends resources without contributing to production or capital accumulation.

- The resource constraint of the economy now becomes

$$c_t + g_t + i_t = y_t = f(k_t),$$

where g_t denotes government consumption. It follows that the dynamics of capital are given by

$$k_{t+1} - k_t = f(k_t) - (\delta + n)k_t - c_t - g_t$$

- Government spending is financed with proportional income taxation, at rate $\tau \geq 0$. The government thus absorbs a fraction τ of aggregate output:

$$g_t = \tau y_t.$$

- Disposable income for the representative household is $(1-\tau)y_t$. We continue to assume that consumption and investment absorb fractions $1 - s$ and s of disposable income:

$$c_t = (1 - s)(1 - \tau)y_t.$$

- Combining the above, we conclude that the dynamics of capital are now given by

$$\gamma_t = \frac{k_{t+1} - k_t}{k_t} = s(1-\tau)\phi(k_t) - (\delta + n).$$

where $\phi(k) \equiv f(k)/k$. Given s and k_t, the growth rate γ_t decreases with τ.

- A steady state exists for any $\tau \in [0, 1)$ and is given by

$$k^* = \phi^{-1}\left(\frac{\delta + n}{s(1-\tau)}\right).$$

Given s, k^* decreases with τ.

- *Policy Shocks:* Consider a temporary shock in government consumption. What are the transitional dynamics?

2.3.3 Productive Government Spending

- Suppose now that production is given by

$$y_t = f(k_t, g_t) = k_t^\alpha g_t^\beta,$$

where $\alpha > 0$, $\beta > 0$, and $\alpha + \beta < 1$. Government spending can thus be interpreted as infrastructure or other productive services. The resource constraint is

$$c_t + g_t + i_t = y_t = f(k_t, g_t).$$

- We assume again that government spending is financed with proportional income taxation at rate τ, and that private consumption and investment are fractions $1 - s$ and s of disposable household income:

$$\begin{aligned} g_t &= \tau y_t. \\ c_t &= (1-s)(1-\tau)y_t \\ i_t &= s(1-\tau)y_t \end{aligned}$$

- Substituting $g_t = \tau y_t$ into $y_t = k_t^\alpha g_t^\beta$ and solving for y_t, we infer

$$y_t = k_t^{\frac{\alpha}{1-\beta}} \tau^{\frac{\beta}{1-\beta}} \equiv k_t^a \tau^b$$

  where $a \equiv \alpha/(1-\beta)$ and $b \equiv \beta/(1-\beta)$. Note that $a > \alpha$, reflecting the complementarity between government spending and capital.

- We conclude that the growth rate is given by

$$\gamma_t = \frac{k_{t+1} - k_t}{k_t} = s(1-\tau)\tau^b k_t^{a-1} - (\delta + n).$$

  The steady state is

$$k^* = \left(\frac{s(1-\tau)\tau^b}{\delta + n} \right)^{1/(1-a)}.$$

- Consider the rate τ that maximizes either k^*, or γ_t for any given k_t. This is given by

$$\frac{d}{d\tau}[(1-\tau)\tau^b] = 0 \Leftrightarrow$$
$$b\tau^{b-1} - (1+b)\tau^b = 0 \Leftrightarrow$$
$$\tau = b/(1+b) = \beta.$$

  That is, the "optimal" τ equals the elasticity of production with respect to government services. The more productive government services are, the higher their optimal provision.

2.4 Continuous Time and Conditional Convergence

2.4.1 The Solow Model in Continuous Time

- Recall that the basic growth equation in the discrete-time Solow model is

$$\frac{k_{t+1} - k_t}{k_t} = \gamma(k_t) \equiv s\phi(k_t) - (\delta + n).$$

We would expect a similar condition to hold under continuous time. We verify this below.

- The resource constraint of the economy is

$$C + I = Y = F(K, L).$$

In per-capita terms,

$$c + i = y = f(k).$$

- Population growth is now given by

$$\frac{\dot{L}}{L} = n$$

and the law of motion for aggregate capital is

$$\dot{K} = I - \delta K$$

- Let $k \equiv K/L$. Then,

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L}.$$

Substituting from the above, we infer

$$\dot{k} = i - (\delta + n)k.$$

Combining this with

$$i = sy = sf(k),$$

we conclude

$$\dot{k} = sf(k) - (\delta + n)k.$$

- Equivalently, the growth rate of the economy is given by

$$\frac{\dot{k}}{k} = \gamma(k) \equiv s\phi(k) - (\delta + n). \tag{2.34}$$

The function $\gamma(k)$ thus gives the growth rate of the economy in the Solow model, whether time is discrete or continuous.

2.4.2 Log-linearization and the Convergence Rate

- Define $z \equiv \ln k - \ln k^*$. We can rewrite the growth equation (2.34) as

$$\dot{z} = \Gamma(z),$$

where

$$\Gamma(z) \equiv \gamma(k^* e^z) \equiv s\phi(k^* e^z) - (\delta + n)$$

Note that $\Gamma(z)$ is defined for all $z \in \mathsf{R}$. By definition of k^*, $\Gamma(0) = s\phi(k^*) - (\delta + n) = 0$. Similarly, $\Gamma(z) > 0$ for all $z < 0$ and $\Gamma(z) < 0$ for all $z > 0$. Finally, $\Gamma'(z) = s\phi'(k^* e^z) k^* e^z < 0$ for all $z \in \mathsf{R}$.

- We next (log)linearize $\dot{z} = \Gamma(z)$ around $z = 0:$

$$\dot{z} = \Gamma(0) + \Gamma'(0) \cdot z$$

or equivalently

$$\dot{z} = \lambda z$$

where we substituted $\Gamma(0) = 0$ and let $\lambda \equiv \Gamma'(0)$.

- Straightforward algebra gives

$$\begin{aligned}\Gamma'(z) &= s\phi'(k^* e^z)k^* e^z < 0 \\ \phi'(k) &= \frac{f'(k)k - f(k)}{k^2} = -\left[1 - \frac{f'(k)k}{f(k)}\right]\frac{f(k)}{k^2} \\ sf(k^*) &= (\delta + n)k^*\end{aligned}$$

We infer

$$\Gamma'(0) = -(1 - \varepsilon_K)(\delta + n) < 0$$

where $\varepsilon_K \equiv F_K K/F = f'(k)k/f(k)$ is the elasticity of production with respect to capital, evaluated at the steady-state k.

- We conclude that

$$\frac{\dot{k}}{k} = \lambda \ln\left(\frac{k}{k^*}\right)$$

where

$$\lambda = -(1 - \varepsilon_K)(\delta + n) < 0$$

The quantity $-\lambda$ is called the *convergence rate.*

- Note that, around the steady state

$$\frac{\dot{y}}{y} = \varepsilon_K \cdot \frac{\dot{k}}{k}$$

and

$$\frac{y}{y^*} = \varepsilon_K \cdot \frac{k}{k^*}$$

It follows that

$$\frac{\dot{y}}{y} = \lambda \ln\left(\frac{y}{y^*}\right)$$

Thus, $-\lambda$ is the convergence rate for either capital or output.

- In the Cobb-Douglas case, $y = k^\alpha$, the convergence rate is simply

$$-\lambda = (1-\alpha)(\delta + n),$$

where α is the income share of capital. Note that as $\lambda \to 0$ as $\alpha \to 1$. That is, convergence becomes slower and slower as the income share of capital becomes closer and closer to 1. Indeed, if it were $\alpha = 1$, the economy would a balanced growth path.

- In the example with productive government spending, $y = k^\alpha g^\beta = k^{\alpha/(1-\beta)}\tau^{\beta/(1-\beta)}$, we get

$$-\lambda = \left(1 - \frac{\alpha}{1-\beta}\right)(\delta + n)$$

The convergence rate thus decreases with β, the productivity of government services. And $\lambda \to 0$ as $\beta \to 1-\alpha$.

- *Calibration:* If $\alpha = 35\%$, $n = 3\%$ ($= 1\%$ population growth+2% exogenous technological process), and $\delta = 5\%$, then $-\lambda = 6\%$. This contradicts the data. But if $\alpha = 70\%$, then $-\lambda = 2.4\%$, which matches the date.

2.5 Cross-Country Differences and Conditional Convergence.

2.5.1 Mankiw-Romer-Weil: Cross-Country Differences

- The Solow model implies that steady-state capital, productivity, and income are determined primarily by technology (f and δ), the national saving rate (s), and population growth (n).

- Suppose that countries share the same technology in the long run, but differ in terms of saving behavior and fertility rates. If the Solow model is correct, observed cross-country income and productivity differences should be explained by observed cross-country differences in s and n,

- Mankiw, Romer and Weil tests this hypothesis against the data. In it's simple form, the Solow model fails to explain the large cross-country dispersion of income and productivity levels.

- Mankiw, Romer and Weil then consider an extension of the Solow model, that includes two types of capital, physical capital (k) and human capital (h). Output is given by

$$y = k^{\alpha} h^{\beta},$$

  where $\alpha > 0, \beta > 0$, and $\alpha + \beta < 1$. The dynamics of capital accumulation are now given by

$$\begin{aligned} \dot{k} &= s_k y - (\delta + n)k \\ \dot{h} &= s_h y - (\delta + n)h \end{aligned}$$

  where s_k and s_h are the investment rates in physical capital and human capital, respectively. The steady-state levels of k, h, and y then depend on both s_k and s_h, as well as δ and n.

- Proxying s_h by education attainment levels in each country, Mankiw, Romer and Weil find that the Solow model extended for human capital does a pretty good job in explaining the cross-country dispersion of output and productivity levels.

2.5.2 Barro: Conditional Convergence

- Recall the log-linearization of the dynamics around the steady state:

$$\frac{\dot{y}}{y} = \lambda \ln \frac{y}{y^*}.$$

A similar relation will hold true in the neoclassical growth model a la Ramsey-Cass-Koopmans. $\lambda < 0$ reflects local diminishing returns. Such local diminishing returns occur even in endogenous-growth models. The above thus extends well beyond the simple Solow model.

- Rewrite the above as

$$\Delta \ln y = \lambda \ln y - \lambda \ln y^*$$

Next, let us proxy the steady state output by a set of country-specific controls X, which include s, δ, n, τ etc. That is, let

$$-\lambda \ln y^* \approx \beta' X.$$

We conclude

$$\Delta \ln y = \lambda \ln y + \beta' X + error$$

- The above represents a typical "Barro" conditional-convergence regression: We use cross-country data to estimate λ (the convergence rate), together with β (the effects of the saving rate, education, population growth, policies, etc.) The estimated convergence rate is about 2% per year.

- Discuss the effects of the other variables (X).

2.6 Miscellaneous

2.6.1 The Golden Rule and Dynamic Inefficiency

- *The Golden Rule:* Consumption at the steady state is given by

$$\begin{aligned} c^* &= (1-s)f(k^*) = \\ &= f(k^*) - (\delta+n)k^* \end{aligned}$$

Suppose the social planner chooses s so as to maximize c^*. Since k^* is a monotonic function of s, this is equivalent to choosing k^* so as to maximize c^*. Note that

$$c^* = f(k^*) - (\delta+n)k^*$$

is strictly concave in k^*. The FOC is thus both necessary and sufficient. c^* is thus maximized if and only if $k^* = k_{gold}$, where k_{gold} solve

$$f'(k_{gold}) - \delta = n.$$

Equivalently, $s = s_{gold}$, where s_{gold} solves

$$s_{gold} \cdot \phi\left(k_{gold}\right) = (\delta+n)$$

The above is called the "*golden rule*" for savings, after Phelps.

- *Dynamic Inefficiency:* If $s > s_{gold}$ (equivalently, $k^* > k_{gold}$), the economy is dynamically inefficient: If the saving raised is lowered to $s = s_{gold}$ for all t, then consumption in all periods will be higher!

- On the other hand, if $s < s_{gold}$ (equivalently, $k^* > k_{gold}$), then raising s towards s_{gold} will increase consumption in the long run, but at the cost of lower consumption in the short run. Whether such a trade-off between short-run and long-run consumption is desirable will depend on how the social planner weight the short run versus the long run.

- *The Modified Golden Rule:* In the Ramsey model, this trade-off will be resolved when k^* satisfies the

$$f'(k^*) - \delta = n + \rho,$$

where $\rho > 0$ measures impatience (ρ will be called "the discount rate"). The above is called the "*modified golden rule.*" Naturally, the distance between the Ramsey-optimal k^* and the golden-rule k_{gold} increase with ρ.

- *Abel et. al.:* Note that the golden rule can be restated as

$$r - \delta = \frac{\dot{Y}}{Y}.$$

Dynamic inefficiency occurs when $r - \delta < \dot{Y}/Y$, dynamic efficiency is ensured if $r - \delta > \dot{Y}/Y$. Abel et al. use this relation to argue that, in reality, there is no evidence of dynamic inefficiency.

- *Bubbles:* If the economy is dynamically inefficient, there is room for bubbles.

2.6.2 Poverty Traps, Cycles, etc.

- Discuss the case of a general non-concave or non-monotonic G.
- Multiple steady states; unstable versus stable ones; poverty traps.
- Local versus global stability; local convergence rate.
- Oscillating dynamics; perpetual cycles.

2.6.3 Introducing Endogenous Growth

- What ensures that the growth rate asymptotes to zero in the Solow model (and the Ramsey model as well) is the vanishing marginal product of capital, that is, the Inada condition $\lim_{k\to\infty} f'(k) = 0$.

- Continue to assume that $f''(k) < 0$, so that $\gamma'(k) < 0$, but assume now that $\lim_{k\to\infty} f'(k) = A > 0$. This implies also $\lim_{k\to\infty} \phi(k) = A$. Then, as $k \to \infty$,

$$\gamma_t \equiv \frac{k_{t+1} - k_t}{k_t} \to sA - (n + \delta)$$

- If $sA < (n+\delta)$, then it is like before: The economy converges to k^* such that $\gamma(k^*) = 0$. But if $sA > (n + x + \delta)$, then the economy exhibits deminishing but not vanishing growth: γ_t falls with t, but $\gamma_t \to sA - (n + \delta) > 0$ as $t \to \infty$.

- Jones and Manuelli consider such a general convex technology: e.g., $f(k) = Bk^\alpha + Ak$. We then get both transitional dynamics in the short run and perpetual growth in the long run.

- In case that $f(k) = Ak$, the economy follows a balanced-growth path from the very beginning.

- We will later "endogenize" A in terms of policies, institutions, markets, etc.

- For example, Romer/Lucas: If we have human capital or spillover effects,

$$y = Ak^\alpha h^{1-\alpha}$$

  and $h = k$, then we get $y = Ak$.

- Reconcile conditional convergence with endogenous growth. Think of $\ln k - \ln k^*$ as a detrended measure of the steady-state allocation of resources (human versus physical capital, specialization pattern.); or as a measure of distance from technology frontier; etc.

Chapter 3

The Neoclassical Growth Model

- In the Solow model, agents in the economy (or the dictator) follow a simplistic linear rule for consumption and investment. In the Ramsey model, agents (or the dictator) choose consumption and investment optimally so as to maximize their individual utility (or social welfare).

3.1 The Social Planner

- In this section, we start the analysis of the neoclassical growth model by considering the optimal plan of a benevolent social planner, who chooses the static and intertemporal allocation of resources in the economy so as to maximize social welfare. We will later show that the allocations that prevail in a decentralized competitive market environment coincide with the allocations dictated by the social planner.

- Together with consumption and saving, we also endogenize labor supply.

3.1.1 Preferences

- Preferences are defined over streams of consumption and leisure $\{x_t\}_{t=0}^{\infty}$, where $x_t = (c_t, z_t)$, and are represented by a utility function $\mathcal{U} : \mathsf{X}^{\infty} \to \mathsf{R}$, where X is the domain of x_t, such that

$$\mathcal{U}\left(\{x_t\}_{t=0}^{\infty}\right) = \mathcal{U}\left(x_0, x_1, ...\right)$$

- We say that preferences are *recursive* if there is a function $W : \mathsf{X} \times \mathsf{R} \to \mathsf{R}$ such that, for all $\{x_t\}_{t=0}^{\infty}$,

$$\mathcal{U}\left(x_0, x_1, ...\right) = W\left[x_0, \mathcal{U}\left(x_1, x_2, ...\right)\right]$$

  We can then represent preferences as follows: A consumption-leisure stream $\{x_t\}_{t=0}^{\infty}$ induces a utility stream $\{\mathcal{U}_t\}_{t=0}^{\infty}$ according to the recursion

$$\mathcal{U}_t = W(x_t, \mathcal{U}_{t+1}).$$

  That is, utility in period t is given as a function of consumption in period t and utility in period $t+1$. W is called a *utility aggregator*. Finally, note that recursive preferences, as defined above, are both time-consistent and stationary.

- We say that preferences are *additively separable* if there are functions $\upsilon_t : \mathsf{X} \to \mathsf{R}$ such that

$$\mathcal{U}\left(\{x_t\}_{t=0}^{\infty}\right) = \sum_{t=0}^{\infty} \upsilon_t(x_t).$$

  We then interpret $\upsilon_t(x_t)$ as the utility enjoyed in period 0 from consumption in period $t+1$.

- Throughout our analysis, we will assume that preferences are both recursive and additively separable. In other words, we impose that the utility aggregator W is linear in

u_{t+1} : There is a function $U : \mathsf{R} \to \mathsf{R}$ and a scalar $\beta \in \mathsf{R}$ such that $W(x,u) = U(x) + \beta u$. We can thus represent preferences in recursive form as

$$\mathcal{U}_t = U(x_t) + \beta \mathcal{U}_{t+1}.$$

Alternatively,

$$\mathcal{U}_t = \sum_{\tau=0}^{\infty} \beta^{\tau} U(x_{t+\tau})$$

- β is called the *discount factor*. For preferences to be well defined (that is, for the infinite sum to converge) we need $\beta \in (-1,+1)$. Monotonicity of preferences imposes $\beta > 0$. Therefore, we restrict $\beta \in (0,1)$. The discount rate is given by ρ such that $\beta = 1/(1+\rho)$.

- U is sometimes called the per-period felicity or utility function. We let $\overline{z} > 0$ denote the maximal amount of time per period. We accordingly let $\mathsf{X} = \mathsf{R}_+ \times [0, \overline{z}]$. We finally impose that U is *neoclassical*, in that it satisfies the following properties:

  1. U is continuous and, although not always necessary, twice differentiable.

  2. U is strictly increasing and strictly concave:

$$\begin{aligned} U_c(c,z) &> 0 > U_{cc}(c,z) \\ U_z(c,z) &> 0 > U_{zz}(c,z) \\ U_{cz}^2 &< U_{cc} U_{zz} \end{aligned}$$

  3. U satisfies the Inada conditions

$$\begin{aligned} \lim_{c\to 0} U_c &= \infty \quad \text{and} \quad \lim_{c\to\infty} U_c = 0. \\ \lim_{z\to 0} U_z &= \infty \quad \text{and} \quad \lim_{z\to\overline{z}} U_z = 0. \end{aligned}$$

3.1.2 Technology and the Resource Constraint

- We abstract from population growth and exogenous technological change.
- The time constraint is given by

$$z_t + l_t \leq \overline{z}.$$

We usually normalize $\overline{z} = 1$ and thus interpret z_t and l_t as the fraction of time that is devoted to leisure and production, respectively.

- The resource constraint is given by

$$c_t + i_t \leq y_t$$

- Let $F(K, L)$ be a neoclassical technology and let $f(\kappa) = F(\kappa, 1)$ be the intensive form of F. Output in the economy is given by

$$y_t = F(k_t, l_t) = l_t f(\kappa_t),$$

where

$$\kappa_t = \frac{k_t}{l_t}$$

is the capital-labor ratio.

- Capital accumulates according to

$$k_{t+1} = (1 - \delta)k_t + i_t.$$

(Alternatively, interpret l as effective labor and δ as the effective depreciation rate.)

- Finally, we impose the following natural non-negativitly constraints:

$$c_t \geq 0, \quad z_t \geq 0, \quad l_t \geq 0, \quad k_t \geq 0.$$

- Combining the above, we can rewrite the *resource constraint* as

$$c_t + k_{t+1} \leq F(k_t, l_t) + (1-\delta)k_t,$$

and the time constraint as

$$z_t = 1 - l_t,$$

with

$$c_t \geq 0, \quad l_t \in [0,1], \quad k_t \geq 0.$$

3.1.3 The Ramsey Problem

- The social planner chooses a plan $\{c_t, l_t, k_{t+1}\}_{t=0}^{\infty}$ so as to maximize utility subject to the resource constraint of the economy, taking initial k_0 as given:

$$\max \mathcal{U}_0 = \sum_{t=0}^{\infty} \beta^t U(c_t, 1 - l_t)$$

$$c_t + k_{t+1} \leq (1-\delta)k_t + F(k_t, l_t), \quad \forall t \geq 0,$$

$$c_t \geq 0, \quad l_t \in [0,1], \quad k_{t+1} \geq 0., \quad \forall t \geq 0,$$

$$k_0 > 0 \ \ given.$$

3.1.4 Optimal Control

- Let μ_t denote the Lagrange multiplier for the resource constraint. The Lagrangian of the social planner's problem is

$$\mathcal{L}_0 = \sum_{t=0}^{\infty} \beta^t U(c_t, 1 - l_t) + \sum_{t=0}^{\infty} \mu_t \left[(1-\delta)k_t + F(k_t, l_t) - k_{t+1} - c_t\right]$$

- Define $\lambda_t \equiv \beta^t \mu_t$ and

$$\begin{aligned} H_t &\equiv H(k_t, k_{t+1}, c_t, l_t, \lambda_t) \equiv \\ &\equiv U(c_t, 1-l_t) + \lambda_t \left[(1-\delta)k_t + F(k_t, l_t) - k_{t+1} - c_t\right] \end{aligned}$$

  H is called the *Hamiltonian* of the problem.

- We can rewrite the Lagrangian as

$$\begin{aligned} \mathcal{L}_0 &= \sum_{t=0}^{\infty} \beta^t \left\{U(c_t, 1-l_t) + \lambda_t \left[(1-\delta)k_t + F(k_t, l_t) - k_{t+1} - c_t\right]\right\} = \\ &= \sum_{t=0}^{\infty} \beta^t H_t \end{aligned}$$

  or, in recursive form

$$\mathcal{L}_t = H_t + \beta \mathcal{L}_{t+1}.$$

- Given k_t, c_t and l_t enter only the period t utility and resource constraint; (c_t, l_t) thus appears only in H_t. Similarly, k_t,enter only the period t and $t+1$ utility and resource constraints; they thus appear only in H_t and H_{t+1}. Therefore,

Lemma 9 *If* $\{c_t, l_t, k_{t+1}\}_{t=0}^{\infty}$ *is the optimum and* $\{\lambda_t\}_{t=0}^{\infty}$ *the associated multipliers, then*

$$(c_t, l_t) = \arg\max_{c,l} \overbrace{H(k_t, k_{t+1}, c, l, \lambda_t)}^{H_t}$$

taking (k_t, k_{t+1}) *as given, and*

$$k_{t+1} = \arg\max_{k'} \overbrace{H(k_t, k', c_t, l_t, \lambda_t) + \beta H(k', k_{t+2}, c_{t+1}, l_{t+1}, \lambda_{t+1})}^{H_t + \beta\ H_{t+1}}$$

taking (k_t, k_{t+2}) *as given.*

Equivalently,

$$\begin{aligned}(c_t, l_t, k_{t+1}, c_{t+1}, l_{t+1},) &= \arg\max_{c,l,k',c',l'} [U(c,l) + \beta U(c', l')] \\ s.t. \quad c + k' &\leq (1-\delta)k_t + F(k_t, l) \\ c' + k_{t+2} &\leq (1-\delta)k' + F(k', l')\end{aligned}$$

taking (k_t, k_{t+2}) *as given.*

- We henceforth assume an interior solution. As long as $k_t > 0$, interior solution is indeed ensured by the Inada conditions on F and U.

- The FOC with respect to c_t gives

$$\frac{\partial \mathcal{L}_0}{\partial c_t} = \beta^t \frac{\partial H_t}{\partial c_t} = 0 \Leftrightarrow$$

$$\frac{\partial H_t}{\partial c_t} = 0 \Leftrightarrow$$

$$U_c(c_t, z_t) = \lambda_t$$

The FOC with respect to l_t gives

$$\frac{\partial \mathcal{L}_0}{\partial l_t} = \beta^t \frac{\partial H_t}{\partial l_t} = 0 \Leftrightarrow$$

$$\frac{\partial H_t}{\partial l_t} = 0 \Leftrightarrow$$

$$U_z(c_t, z_t) = \lambda_t F_L(k_t, l_t)$$

Finally, the FOC with respect to k_{t+1} gives

$$\frac{\partial \mathcal{L}_0}{\partial k_{t+1}} = \beta^t \left[\frac{\partial H_t}{\partial k_{t+1}} + \beta \frac{\partial H_{t+1}}{\partial k_{t+1}} \right] = 0 \Leftrightarrow$$

$$-\lambda_t + \beta \frac{\partial H_{t+1}}{\partial k_{t+1}} = 0 \Leftrightarrow$$

$$\lambda_t = \beta \left[1 - \delta + F_K(k_{t+1}, l_{t+1})\right] \lambda_{t+1}$$

- Combining the above, we get

$$\frac{U_z(c_t, z_t)}{U_c(c_t, z_t)} = F_L(k_t, l_t)$$

and

$$\frac{U_c(c_t, z_t)}{\beta U_c(c_{t+1}, z_{t+1})} = 1 - \delta + F_K(k_{t+1}, l_{t+1}).$$

- Both conditions impose equality between marginal rates of substitution and marginal rate of transformation. The first condition means that the marginal rate of substitution between consumption and leisure equals the marginal product of labor. The second condition means that the marginal rate of intertemporal substitution in consumption equals the marginal capital of capital net of depreciation (plus one). This last condition is called the *Euler condition*.

- The *envelope condition* for the Pareto problem is

$$\frac{\partial(\max \mathcal{U}_0)}{\partial k_0} = \frac{\partial \mathcal{L}_0}{\partial k_0} = \lambda_0 = U_c(c_0, z_0).$$

More generally,

$$\lambda_t = U_c(c_t, l_t)$$

represents the marginal utility of capital in period t and will equal the slope of the value function at $k = k_t$ in the dynamic-programming representation of the problem.

- Suppose for a moment that the horizon was finite, $T < \infty$. Then, the Lagrangian would be

$$\mathcal{L}_0 = \sum_{t=0}^{T} \beta^t H_t$$

and the Kuhn-Tucker condition with respect to k_{T+1} would give

$$\frac{\partial \mathcal{L}}{\partial k_{T+1}} = \beta^T \frac{\partial H_T}{\partial k_{T+1}} \geq 0 \quad \text{and} \quad k_{T+1} \geq 0, \text{ with complementary slackness;}$$

equivalently

$$\mu_T = \beta^T \lambda_T \geq 0 \quad \text{and} \quad k_{T+1} \geq 0, \quad \text{with} \quad \beta^T \lambda_T k_{T+1} = 0.$$

The latter means that either $k_{T+1} = 0$, or otherwise it better be that the shadow value of k_{T+1} is zero. When the horizon is infinite, the terminal condition $\beta^T \lambda_T k_{T+1} = 0$ is replaced by the transversality condition

$$\lim_{t\to\infty} \beta^t \lambda_t k_{t+1} = 0.$$

Equivalently, using $\lambda_t = U_c(c_t, z_t)$, we write the transversality condition as

$$\lim_{t\to\infty} \beta^t U_c(c_t, z_t) k_{t+1} = 0.$$

The above means that, as time passes, the (discounted) shadow value of capital converges to zero.

- We conclude:

Proposition 10 *The plan $\{c_t, l_t, k_t\}_{t=0}^{\infty}$ is a solution to the social planner's problem if and only if*

$$\frac{U_z(c_t, z_t)}{U_c(c_t, z_t)} = F_L(k_t, l_t), \tag{3.1}$$

$$\frac{U_c(c_t, z_t)}{\beta U_c(c_{t+1}, z_{t+1})} = 1 - \delta + F_K(k_{t+1}, l_{t+1}), \tag{3.2}$$

$$k_{t+1} = F(k_t, l_t) + (1 - \delta)k_t - c_t, \tag{3.3}$$

for all $t \geq 0$, and

$$k_0 > 0 \quad \textit{given,} \quad \textit{and} \quad \lim_{t\to\infty} \beta^t U_c(c_t, z_t) k_{t+1} = 0. \tag{3.4}$$

- *Remark:* We proved necessity of (3.1) and (3.2) essentially by a perturbation argument, and (3.3) is trivial. We did not prove necessity of (3.4), neither sufficiency of this set of conditions. One can prove both necessity and sufficiency using optimal-control techniques. Alternatively, we can use dynamic programming; the proof of the necessity and sufficiency of the Euler and transversality conditions is provided in Stokey and Lucas.

- Note that the (3.1) can be solved for We will later prove that

3.1.5 Dynamic Programing

Consider again the social planner's problem.

For any $k > 0$, define

$$V(k) \equiv \max \sum_{t=0}^{\infty} \beta^t U(c_t, 1 - l_t)$$

subject to

$$c_t + k_{t+1} \leq (1 - \delta)k_t + F(k_t, l_t), \quad \forall t \geq 0,$$

$$c_t, l_t, (1 - l_t), k_{t+1} \geq 0, \quad \forall t \geq 0,$$

$$k_0 = k \ \ given.$$

V is called the Value Function.

- Define $\overline{k}$ by the unique solution to

$$\overline{k} = (1 - \delta)\overline{k} + F(\overline{k}, 1)$$

  and note that $\overline{k}$ represents an upper bound on the level of capital that can be sustained in any steady state. Without serious loss of generality, we will henceforth restrict $k_t \in [0, \overline{k}]$.

- Let B be the set of continuous and bounded functions $v : [0, \bar{k}] \to \mathsf{R}$ and consider the mapping $\mathcal{T} : B \to B$ defined as follows:

$$\mathcal{T}v(k) = \max U(c, 1-l) + \beta v(k')$$
$$s.t. \qquad c + k' \leq (1-\delta)k + F(k,l)$$
$$k' \in [0, \bar{k}], \quad c \in [0, F(k,1)], \quad l \in [0,1].$$

The conditions we have imposed on U and F imply that $\mathcal{T}$ is a contraction mapping. It follows that $\mathcal{T}$ has a unique fixed point $V = \mathcal{T}V$ and this fixed point gives the solution to the planner's problem:

Proposition 11 *There is a unique V that solves the Bellman equation*

$$V(k) = \max U(c, 1-l) + \beta V(k')$$
$$s.t. \qquad c + k' \leq (1-\delta)k + F(k,l)$$
$$k' \in [0, \bar{k}], \quad c \in [0, F(k,1)], \quad l \in [0,1].$$

V is continuous, differentiable, and strictly concave. $V(k_0)$ gives the solution for the social planner's problem.

Proposition 12 *Let*

$$[c(k), l(k), G(k)] = \arg\max\{...\}.$$

$c(k), l(k), G(k)$ are continuous; $c(k)$ and $G(k)$ are increasing. The plan $\{c_t, l_t, k_t\}_{t=0}^{\infty}$ is optimal if and only if it satisfies

$$\begin{aligned} c_t &= c(k_t) \\ l_t &= l(k_t) \\ k_{t+1} &= G(k_t) \end{aligned}$$

with k_0 historically given.

- *Remark:* The proofs of the above propositions, as well as the proof of the necessity and sufficiency of the Euler and transversality conditions, are provided in Stokey and Lucas. Because of time constraints, I will skip these proofs and concentrate on the characterization of the optimal plan.

- The Lagrangian for the DP problem is

$$\mathcal{L} = U(c, 1-l) + \beta V(k') + \lambda[(1-\delta)k + F(k,l) - k' - c]$$

The FOCs with respect to c, l and k' give

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c} &= 0 \Leftrightarrow U_c(c,z) = \lambda \\
\frac{\partial \mathcal{L}}{\partial l} &= 0 \Leftrightarrow U_z(c,z) = \lambda F_L(k,l) \\
\frac{\partial \mathcal{L}}{\partial k'} &= 0 \Leftrightarrow \lambda = \beta V_k(k')
\end{aligned}$$

The Envelope condition is

$$V_k(k) = \frac{\partial \mathcal{L}}{\partial k} = \lambda[1 - \delta + F_K(k,l)]$$

- Combining, we conclude

$$\frac{U_z(c_t, l_t)}{U_c(c_t, l_t)} = F_l(k_t, k_t)$$

and

$$\frac{U_c(c_t, l_t)}{U_c(c_{t+1}, l_{t+1})} = \beta\left[1 - \delta + F_K(k_{t+1}, l_{t+1})\right],$$

which are the same conditions we had derived with optimal control. Finally, note that we can state the Euler condition alternatively as

$$\frac{V_k(k_t)}{V_k(k_{t+1})} = \beta[1 - \delta + F_K(k_{t+1}, l_{t+1})].$$

3.2 Decentralized Competitive Equilibrium

3.2.1 Households

- Households are indexed by $j \in [0,1]$. There is one person per household and no population growth.

- The preferences of household j are given by

$$\mathcal{U}_0^j = \sum_{t=0}^{\infty} \beta^t U(c_t^j, z_t^j)$$

  In recursive form,

$$\mathcal{U}_t^j = U(c_t^j, z_t^j) + \beta \mathcal{U}_{t+1}^j$$

- The time constraint for household j can be written as

$$z_t^j = 1 - l_t^j.$$

- The budget constraint of household j is given by

$$c_t^j + i_t^j + x_t^j \leq y_t^j = r_t k_t^j + R_t b_t^j + w_t l_t^j + \alpha^j \Pi_t,$$

  where r_t denotes the rental rate of capital, w_t denotes the wage rate, R_t denotes the interest rate on risk-free bonds. Household j accumulates capital according to

$$k_{t+1}^j = (1-\delta) k_t^j + i_t^j$$

  and bonds according to

$$b_{t+1}^j = b_t^j + x_t^j$$

  In equilibrium, firm profits are zero, because of CRS. It follows that $\Pi_t = 0$ and we can rewrite the household budget as

$$c_t^j + k_{t+1}^j + b_{t+1}^j \leq (1 - \delta + r_t) k_t^j + (1 + R_t) b_t^j + w_t l_t^j.$$

- The natural non-negativity constraint

$$k_{t+1}^{j} \geq 0$$

is imposed on capital holdings, but no short-sale constraint is imposed on bond holdings. That is, household can either lend or borrow in risk-free bonds. We only impose the following *natural borrowing limit*

$$-(1+R_{t+1})b_{t+1}^{j} \leq (1-\delta+r_{t+1})k_{t+1}^{j} + \sum_{\tau=t+1}^{\infty} \frac{q_\tau}{q_{t+1}} w_\tau.$$

where

$$q_t \equiv \frac{1}{(1+R_0)(1+R_1)...(1+R_t)} = (1+R_t)q_{t+1}.$$

This constraint simply requires that the net debt position of the household does not exceed the present value of the labor income he can attain by working all time.

- Note that simple arbitrage between bonds and capital implies that, in any equilibrium,

$$R_t = r_t - \delta.$$

That is, the interest rate on riskless bonds must equal the rental rate of capital net of depreciation. If $R_t < r_t - \delta$, all individuals would like to short-sell bonds (up to their borrowing constraint) and invest into capital. If $R_t > r_t - \delta$, capital would be dominated by bonds, and nobody in the economy would invest in capital. In the first case, there would be excess supply for bonds in the aggregate. In the second case, there would be excess demand for bonds and no investment in the aggregate. In equilibrium, R_t and r_t must adjust so that $R_t = r_t - \delta$.

- Provided that $R_t = r_t - \delta$, the household is indifferent between bonds and capital. The "portfolio" choice between k_t^j and b_t^j is thus indeterminate. What is pinned down is only the total asset position, $a_t^j = b_t^j + k_t^j$. The budget constraint then reduces to

$$c_t^j + a_{t+1}^j \leq (1 + R_t)a_t^j + w_t l_t^j,$$

and the natural borrowing constraint then becomes

$$a_{t+1}^j \geq \underline{a}_{t+1},$$

where

$$\underline{a}_{t+1} \equiv -\frac{1}{q_t} \sum_{\tau=t+1}^{\infty} q_\tau w_\tau$$

- Note that $\underline{a}_t$ is bounded away from $-\infty$ as long as q_t is bounded away from 0 and $\sum_{\tau=t}^{\infty} q_\tau w_\tau$ is bounded away from $+\infty$. If $\sum_{\tau=t+1}^{\infty} q_\tau w_\tau$ was infinite at any t, the agent could attain infinite consumption in every period $\tau \geq t+1$. That this is not the case is ensured by the restriction that

$$\sum_{t=0}^{\infty} q_t w_t < +\infty.$$

- Given a price sequence $\{R_t, w_t\}_{t=0}^{\infty}$, household j chooses a plan $\{c_t^j, l_t^j, k_{t+1}^j\}_{t=0}^{\infty}$ so as to maximize lifetime utility subject to its budget constraints

$$\max \quad \mathcal{U}_0^j = \sum_{t=0}^{\infty} \beta^t U(c_t^j, 1 - l_t^j)$$

$$s.t. \;\; c_t^j + a_{t+1}^j \leq (1 + R_t)a_t^j + w_t l_t^j$$

$$c_t^j \geq 0, \quad l_t^j \in [0,1], \quad a_{t+1}^j \geq \underline{a}_{t+1}$$

- Let $\mu_t^j = \beta^t \lambda_t^j$ be the Lagrange multiplier for the budget constraint, we can write the Lagrangian as

$$\begin{aligned} \mathcal{L}_0^j &= \sum_{t=0}^{\infty} \beta^t \left\{ U(c_t^j, 1 - l_t^j) + \lambda_t^j \left[(1 + R_t)a_t^j + w_t l_t^j - a_{t+1}^j - c_t^j \right] \right\} \\ &= \sum_{t=0}^{\infty} \beta^t H_t^j \end{aligned}$$

where

$$H_t^j = U(c_t^j, 1 - l_t^j) + \lambda_t^j \left[(1 + R_t)a_t^j + w_t l_t^j - a_{t+1}^j - c_t^j\right]$$

- The FOC with respect to c_t^j gives

$$\frac{\partial \mathcal{L}_0^j}{\partial c_t^j} = \beta^t \frac{\partial H_t^j}{\partial c_t^j} = 0 \Leftrightarrow$$
$$U_c(c_t^j, z_t^j) = \lambda_t^j$$

The FOC with respect to l_t^j gives

$$\frac{\partial \mathcal{L}_0^j}{\partial l_t^j} = \beta^t \frac{\partial H_t^j}{\partial l_t^j} = 0 \Leftrightarrow$$
$$U_z(c_t^j, z_t^j) = \lambda_t^j w_t$$

Combining, we get

$$\frac{U_z(c_t^j, z_t^j)}{U_c(c_t^j, z_t^j)} = w_t.$$

That is, households equate their marginal rate of substitution between consumption and leisure with the (common) wage rate.

- The Kuhn-Tucker condition with respect to a_{t+1}^j gives

$$\frac{\partial \mathcal{L}_0^j}{\partial a_{t+1}^j} = \beta^t \left[\frac{\partial H_t^j}{\partial a_{t+1}^j} + \beta \frac{\partial H_{t+1}^j}{\partial a_{t+1}^j}\right] \leq 0 \Leftrightarrow$$
$$\lambda_t^j \geq \beta \left[1 + R_t\right] \lambda_{t+1}^j,$$

with equality whenever $a_{t+1}^j > \underline{a}_{t+1}$. That is, the complementary slackness condition is

$$\left[\lambda_t^j - \beta \left[1 + R_t\right] \lambda_{t+1}^j\right] \left[a_{t+1}^j - \underline{a}_{t+1}\right] = 0.$$

- Finally, if time was finite, the terminal condition would be

$$\mu_T^j \geq 0, \quad a_{T+1}^j \geq \underline{a}_{T+1}, \quad \mu_T^j \left[a_{T+1}^j - \underline{a}_{T+1}\right] = 0,$$

where $\mu_t^j \equiv \beta^t \lambda_t^j$. Now that time is infinite, the transversality condition is

$$\lim_{t \to 0} \beta^t \lambda_t^j \left[a_{t+1}^j - \underline{a}_{t+1}\right] = 0.$$

- Using $\lambda_t^j = U_c(c_t^j, z_t^j)$, we can restate the Euler condition as

$$U_c(c_t^j, z_t^j) \geq \beta[1 + R_t]U_c(c_{t+1}^j, z_{t+1}^j),$$

with equality whenever $a_{t+1}^j > \underline{a}_{t+1}$. That is, as long as the borrowing constraint does not bind, households equate their marginal rate of intertemporal substitution with the (common) return on capital. On the other hand, if the borrowing constraint is binding, the marginal utility of consumption today may exceed the marginal benefit of savings: The household would like to borrow, but it's not capable of.

- For general borrowing limit $\underline{a}_t$, there is nothing to ensure that the Euler condition must be satisfied with equality. For example, if we had specified $\underline{a}_t = 0$, it likely the borrowing constraint will bind, especially if $\beta(1+R_t) < 1$ and w_t is low as compared to its long-run mean. But if $\underline{a}_t$ is the natural borrowing limit, and the utility satisfies the Inada condition $U_c \to \infty$ as $c \to 0$, then a simple argument ensures that the borrowing constraint can never bind: Suppose that $a_{t+1} = \underline{a}_{t+1}$. Then $c_\tau^j = z_\tau^j = 0$ for all $\tau \geq t$, implying $U_c(c_{t+1}^j, z_{t+1}^j) = \infty$ and therefore necessarily $U_c(c_t^j, z_t^j) < \beta[1 + R_t]U_c(c_t^j, z_t^j)$, unless $c_t^j = 0$ which would be optimal only if $a_t = \underline{a}_t$. Therefore, unless $a_0 = \underline{a}_0$ to start with, the borrowing which would contradict the Euler condition. Therefore, $a_t > \underline{a}_t$ at all dates, and the Euler condition is satisfied with equality:

- Moreover, if the borrowing constraint never binds, iterating $\lambda_t^j = \beta\left[1+R_t\right]\lambda_{t+1}^j$ implies

$$\beta^t \lambda_t^j = q_t \lambda_0^j.$$

We can therefore rewrite the transversality as

$$\lim_{t\to\infty} \beta^t \lambda_t^j a_{t+1}^j = \lim_{t\to\infty} \beta^t \lambda_t^j \underline{a}_{t+1} = \lambda_0^j \lim_{t\to\infty} q_t \underline{a}_{t+1}$$

But note that

$$q_t \underline{a}_{t+1} = \sum_{\tau=t}^{\infty} q_\tau w_\tau$$

and $\sum_{\tau=0}^{\infty} q_\tau w_\tau < \infty$ implies $\sum_{\tau=t}^{\infty} q_\tau w_\tau = 0$. Therefore, the transversality condition reduces to

$$\lim_{t\to\infty} \beta^t \lambda_t^j a_{t+1}^j = 0$$

Equivalently,

$$\lim_{t\to\infty} \beta^t U_c(c_t^j, z_t^j) a_{t+1}^j = 0.$$

- It is useful to restate the household problem in a "static" format (that's essentially assuming complete Arrow-Debreu markets). As long as the borrowing constraint does not bind and the Inada conditions hold, we can rewrite the household problem as

$$\max \sum_{t=0}^{\infty} \beta^t U(c_t^j, z_t^j)$$

$$s.t. \ \sum_{t=0}^{\infty} q_t \cdot c_t^j + \sum_{t=0}^{\infty} q_t w_t \cdot z_t^j \leq \overline{x}$$

where

$$\overline{x} \equiv q_0(1+R_0)a_0 + \sum_{t=0}^{\infty} q_t w_t < \infty.$$

The constraint follows by integrating the per-period budgets for all $t \geq 0$ and is called the *intertemporal budget constraint*. Note that, by assumption,

$$\sum_{t=0}^{\infty} q_t < \infty \quad \text{and} \quad \sum_{t=0}^{\infty} q_t w_t < \infty,$$

which ensures that the set of feasible $\{c_t^j, z_t^j\}_{t=0}^{\infty}$ is compact. The FOCs give

$$\begin{aligned} \beta^t U_c(c_t^j, z_t^j) &= \mu q_t, \\ \beta^t U_z(c_t^j, z_t^j) &= \mu q_t w_t, \end{aligned}$$

where $\mu > 0$ is Lagrange multiplier associated to the intertermporal budget. You can check that these conditions coincide with the one derived before.

- Finally, note that the objective is strictly concave and the constraint is linear. Therefore, the FOCs together with the transversality are both necessary and sufficient. We conclude:

Proposition 13 *Suppose the price sequence* $\{R_t, r_t, w_t\}_{t=0}^{\infty}$ *satisfies* $R_t = r_t - \delta$ *for all* t, $\sum_{t=0}^{\infty} q_t < \infty$, *and* $\sum_{t=0}^{\infty} q_t w_t < \infty$,. *The plan* $\{c_t^j, l_t^j, a_t^j\}_{t=0}^{\infty}$ *solves the individual household's problem if and only if*

$$\frac{U_z(c_t^j, z_t^j)}{U_c(c_t^j, z_t^j)} = w_t,$$
$$\frac{U_c(c_t^j, z_t^j)}{\beta U_c(c_{t+1}^j, z_{t+1}^j)} = 1 + R_t,$$
$$c_t^j + a_{t+1}^j = (1 + R_t) a_t^j + w_t l_t^j,$$

for all $t \geq 0$, *and*

$$a_0^j > 0 \quad \textit{given,} \quad \textit{and} \quad \lim_{t \to \infty} \beta^t U_c(c_t^j, z_t^j) a_{t+1}^j = 0.$$

Given $\{a_t^j\}_{t=1}^{\infty}$, *an optimal portfolio is any* $\{k_t^j, b_t^j\}_{t=1}^{\infty}$ *such that* $k_t^j \geq 0$ *and* $b_t^j = a_t^j - k_t^j$.

- *Remark:* For a more careful discussion on the necessity and sufficiency of the FOCs and the transversality condition, check Stokey and Lucas.

3.2.2 Firms

- There is an arbitrary number M_t of firms in period t, indexed by $m \in [0, M_t]$. Firms employ labor and rent capital in competitive labor and capital markets, have access to the same neoclassical technology, and produce a homogeneous good that they sell competitively to the households in the economy.

- Let K_t^m and L_t^m denote the amount of capital and labor that firm m employs in period t. Then, the profits of that firm in period t are given by

$$\Pi_t^m = F(K_t^m, L_t^m) - r_t K_t^m - w_t L_t^m.$$

- The firms seeks to maximize profits. The FOCs for an interior solution require

$$\begin{aligned} F_K(K_t^m, L_t^m) &= r_t. \\ F_L(K_t^m, L_t^m) &= w_t. \end{aligned}$$

  You can think of the first condition as the firm's demand for labor and the second condition as the firm's demand for capital.

- As we showed before in the Solow model, under CRS, an interior solution to the firms' problem to exist if and only if r_t and w_t imply the same K_t^m / L_t^m. This is the case if and only if there is some $X_t \in (0, \infty)$ such that

$$\begin{aligned} r_t &= f'(X_t) \\ w_t &= f(X_t) - f'(X_t) X_t \end{aligned}$$

where $f(k) \equiv F(k, 1)$. Provided so, firm profits are zero

$$\Pi_t^m = 0$$

and the FOCs reduce to

$$K_t^m = X_t L_t^m.$$

That is, the FOCs pin down the capital labor ratio for each firm (K_t^m / L_t^m), but not the size of the firm (L_t^m). Moreover, because all firms have access to the same technology, they use exactly the same capital-labor ratio. (See our earlier analysis in the Solow model for more details.)

3.2.3 Market Clearing

- There is no exogenous aggregate supply of riskless bonds. Therefore, the *bond market* clears if and only if

$$0 = \int_0^{L_t} b_t^j dj.$$

- The *capital market* clears if and only if

$$\int_0^{M_t} K_t^m dm = \int_0^1 k_t^j dj$$

Equivalently,

$$\int_0^{M_t} K_t^m dm = k_t$$

where $k_t = K_t \equiv \int_0^1 k_t^j dj$ is the aggregate and per-head supply of capital in the economy.

- The *labor market*, on the other hand, clears if and only if

$$\int_0^{M_t} L_t^m dm = \int_0^{L_t} l_t^j dj$$

Equivalently,

$$\int_0^{M_t} L_t^m dm = l_t$$

where $l_t = L_t \equiv \int_0^{L_t} l_t^j dj$ is the aggregate and per-head supply of labor force in the economy.

3.2.4 General Equilibrium: Definition

- The definition of a *general equilibrium* is quite natural:

Definition 14 *An equilibrium of the economy is an allocation* $\{(c_t^j, l_t^j, k_{t+1}^j, b_{t+1}^j)_{j\in[0,L_t]}, (K_t^m, L_t^m)_{m\in[0,M_t]}\}_{t=}^{\infty}$ *and a price path* $\{R_t, r_t, w_t\}_{t=0}^{\infty}$ *such that*

(i) Given $\{R_t, r_t, w_t\}_{t=0}^{\infty}$*, the path* $\{c_t^j, l_t^j, k_{t+1}^j, b_{t+1}^j\}$ *maximizes the utility of of household* j*, for every* j*.*

(ii) Given (r_t, w_t)*, the pair* (K_t^m, L_t^m) *maximizes firm profits, for every* m *and* t*.*

(iii) The bond, capital and labor markets clear in every period

- *Remark:* In the above definition we surpassed the distribution of firm profits (or the stock market). As we explained before in the Solow model, this is without any serious loss of generality because firm profits (and thus firm value) is zero.

3.2.5 General Equilibrium: Existence, Uniqueness, and Characterization

- In the Solow model, we had showed that the decentralized market economy and the centralized dictatorial economy were isomorphic. A similar result applies in the Ramsey model. The following proposition combines the first and second fundamental welfare theorems, as applied in the Ramsey model:

Proposition 15 *The set of competitive equilibrium allocations for the market economy coincide with the set of Pareto allocations for the social planner.*

Proof. I will sketch the proof assuming that **(a)** in the market economy, $k_0^j + b_0^j$ is equal across all j; and **(b)** the social planner equates utility across agents. For the more general case, we need to extend the social planner's problem to allow for an unequal distribution of consumption and wealth across agents. The set of competitive equilibrium allocations coincides with the set of Pareto optimal allocations, each different competitive equilibrium allocation corresponding to a different system of Pareto weights in the utility of the social planner. I also surpass the details about the boundedness of prices. For a more careful analysis, see Stokey and Lucas.

a. We first consider how the solution to the social planner's problem can be implemented as a competitive equilibrium. The social planner's optimal plan is given by $\{c_t, l_t, k_t\}_{t=0}^{\infty}$ such that

$$\frac{U_z(c_t, 1-l_t)}{U_c(c_t, 1-l_t)} = F_L(k_t, l_t), \quad \forall t \geq 0,$$

$$\frac{U_c(c_t, 1-l_t)}{U_c(c_{t+1}, 1-l_{t+1})} = \beta[1-\delta+F_K(k_{t+1}, l_{t+1})], \quad \forall t \geq 0,$$

$$c_t + k_{t+1} = (1-\delta)k_t + F(k_t, l_t), \quad \forall t \geq 0,$$

$$k_0 > 0 \text{ given, and } \lim_{t\to\infty} \beta^t U_c(c_t, 1-l_t)k_{t+1} = 0.$$

Choose a price path $\{R_t, r_t, w_t\}_{t=0}^{\infty}$ such that

$$\begin{aligned} R_t &= r_t - \delta, \\ r_t &= F_K(k_t, l_t) = f'(\kappa_t), \\ w_t &= F_L(k_t, l_t) = f(\kappa_t) - f'(\kappa_t)\kappa_t, \end{aligned}$$

where $\kappa_t \equiv k_t/l_t$. Trivially, these prices ensure that the FOCs are satisfied for every household and every firm if we set $c_t^j = c_t$, $l_t^j = l_t$ and $K_t^m/L_t^m = k_t$ for all j and m. Next, we need to verify that the proposed allocation satisfies the budget constraints of each household. From the resource constraint of the economy,

$$c_t + k_{t+1} = F(k_t, l_t) + (1-\delta)k_t.$$

From CRS and the FOCs for the firms,

$$F(k_t, l_t) = r_t k_t + w_t l_t.$$

Combining, we get

$$c_t + k_{t+1} = (1-\delta + r_t)k_t + w_t l_t.$$

As long as $c_t^j = c_t$, $l_t^j = l_t$, and $a_t^j = k_t^j + b_t^j = k_t$ for all j, t, and $R_t = r_t - \delta$ for all t, it follows that

$$c_t^j + k_{t+1}^j + b_{t+1}^j = (1-\delta + r_t)k_t^j + (1+R_t)b_t^j + w_t l_t^j,$$

which proves that the budget constraint is satisfied for every j, t. Finally, it is trivial to that the proposed allocations clear the bond, capital, and labor markets.

b. We next consider the converse, how a competitive equilibrium coincides with the Pareto solution. Because agents have the same preferences, face the same prices, and are endowed with identical level of initial wealth, and because the solution to the individual's problem is essentially unique (where essentially means unique with respect to c_t^j, l_t^j, and $a_t^j = k_t^j + b_t^j$ but indeterminate with respect to the portfolio choice between k_t^j and b_t^j), every agent picks the same allocations: $c_t^j = c_t$, $l_t^j = l_t$ and $a_t^j = a_t$ for all j, t. By the FOCs to the individual's problem, it follows that $\{c_t, l_t, a_t\}_{t=0}^{\infty}$ satisfies

$$\frac{U_z(c_t, 1-l_t)}{U_c(c_t, 1-l_t)} = w_t, \quad \forall t \geq 0,$$

$$\frac{U_c(c_t, 1-l_t)}{U_c(c_{t+1}, 1-l_{t+1})} = \beta[1-\delta+r_t], \quad \forall t \geq 0,$$

$$c_t + a_{t+1} = (1-\delta+r_t)a_t + w_t l_t, \quad \forall t \geq 0,$$

$$a_0 > 0 \text{ given, and } \lim_{t\to\infty} \beta^t U_c(c_t, 1-l_t)a_{t+1} = 0.$$

From the market clearing conditions for the capital and bond markets, the aggregate supply of bonds is zero and thus

$$a_t = k_t.$$

Next, by the FOCs for the firms,

$$\begin{aligned} r_t &= F_K(k_t, l_t) \\ w_t &= F_L(k_t, l_t) \end{aligned}$$

and by CRS

$$r_t k_t + w_t l_t = F(k_t, l_t)$$

Combining the above with the "representative" budget constraints gives

$$c_t + k_{t+1} = F(k_t, l_t) + (1-\delta)k_t, \quad \forall t \geq 0,$$

which is simply the resource constraint of the economy. Finally, $a_0 = k_0$, and $\lim_{t\to\infty} \beta^t U_c(c_t, 1-l_t)a_{t+1} = 0$ with $a_{t+1} = k_{t+1}$ reduces the social planner's transversality condition. This concludes the proof that the competitive equilibrium coincides with the social planner's optimal plan. **QED** ■

- Following the above, we have:

Proposition 16 *An equilibrium always exists. The allocation of production across firms is indeterminate, and the portfolio choice of each household is also indeterminate, but the equilibrium is unique as regards prices, aggregate allocations, and the distribution of consumption, labor and wealth across households. If initial wealth $k_0^j + b_0^j$ is equal across all agent j, then $c_t^j = c_t$, $l_t^j = l_t$ and $k_t^j + b_t^j = k_t$ for all j. The equilibrium is then given by an allocation $\{c_t, l_t, k_t\}_{t=0}^{\infty}$ such that, for all $t \geq 0$,*

$$\frac{U_z(c_t, 1 - l_t)}{U_c(c_t, 1 - l_t)} = F_L(k_t, l_t),$$
$$\frac{U_c(c_t, 1 - l_t)}{U_c(c_{t+1}, 1 - l_{t+1})} = \beta[1 - \delta + F_K(k_{t+1}, l_{t+1})],$$
$$k_{t+1} = F(k_t, l_t) + (1 - \delta)k_t - c_t,$$

and such that

$$k_0 > 0 \quad \textit{given}, \quad \textit{and} \quad \lim_{t\to\infty} \beta^t U_c(c_t, 1 - l_t)k_{t+1} = 0.$$

Finally, equilibrium prices are given by

$$\begin{aligned} R_t &= R(k_t) \equiv f'(k_t) - \delta, \\ r_t &= r(k_t) \equiv f'(k_t), \\ w_t &= w(k_t) \equiv f(k_t) - f'(k_t)k_t, \end{aligned}$$

where $R'(k) = r'(k) < 0 < w'(k)$.

Proof. The characterization of the equilibrium follows from our previous analysis. Existence and uniqueness of the equilibrium follow directly from existence and uniqueness of the social planner's optimum, given the coincidence of competitive and Pareto allocations. See Stokey and Lucas for more details. **QED** ■

3.3 Steady State and Transitional Dynamics

3.3.1 Steady State

- A steady state is a fixed point (c, l, k) of the dynamic system. A trivial steady state is at $c = l = k = 0$. We now consider interior steady states.

Proposition 17 *There exists a unique steady state $(c^*, l^*, k^*) > 0$. The steady-state values of the capital-labor ratio, the productivity of labor, the output-capital ratio, the consumption-capital ratio, the wage rate, the rental rate of capital, and the interest rate are all independent of the utility function U and are pinned down uniquely by the technology F, the depreciation rate δ, and the discount rate ρ. In particular, the capital-labor ratio $\kappa^* \equiv k^*/l^*$ equates the net-of-depreciation MPK with the discount rate,*

$$f'(\kappa^*) - \delta = \rho,$$

and is a decreasing function of $\rho + \delta$, where $\rho \equiv 1/\beta - 1$. Similarly,

$$R^* = \rho, \quad r^* = \rho + \delta,$$

$$w^* = F_L(\kappa^*, 1) = \frac{U_z(c^*, 1 - l^*)}{U_c(c^*, 1 - l^*)},$$

$$\frac{y^*}{l^*} = f(\kappa^*), \quad \frac{y^*}{k^*} = \phi(\kappa^*), \quad \frac{c^*}{k^*} = \frac{y^*}{k^*} - \delta,$$

where $f(\kappa) \equiv F(\kappa, 1)$ and $\phi(\kappa) \equiv f(\kappa)/\kappa$.

Proof. (c^*, l^*, k^*) must solve

$$\frac{U_z(c^*, 1 - l^*)}{U_c(c^*, 1 - l^*)} = F_L(k^*, l^*),$$

$$1 = \beta[1 - \delta + F_K(k^*, l^*)],$$

$$c^* = F(k^*, l^*) - \delta k^*,$$

Let $\kappa \equiv k/l$ denote the capital-labor ratio at the stead state. By CRS,

$$\begin{aligned} F(k,l) &= lf(\kappa) \\ F_K(k,l) &= f'(\kappa) \\ F_L(k,l) &= f(\kappa) - f'(\kappa)\kappa \\ \frac{F(k,l)}{k} &= \phi(\kappa) \end{aligned}$$

where $f(\kappa) \equiv F(\kappa, 1)$ and $\phi(\kappa) \equiv f(\kappa)/\kappa$. The Euler condition then reduces to

$$1 = \beta[1 - \delta + f'(\kappa^*)]$$

That is, the capital-labor ratio is pinned down uniquely by the equation of the MPK, net of depreciation, with the discount rate

$$f'(\kappa^*) - \delta = \rho$$

where $\rho \equiv 1/\beta - 1$ or, equivalently, $\beta \equiv 1/(1+\rho)$. The gross rental rate of capital and the net interest rate are thus

$$r^* = \rho + \delta \quad \text{and} \quad R^* = \rho,$$

while the wage rate is

$$w^* = F_{L:}(\kappa^*, 1)$$

The average product of labor and the average product of capital are given by

$$\frac{y^*}{l^*} = f(\kappa^*) \quad \text{and} \quad \frac{y^*}{k^*} = \phi(\kappa^*),$$

while, by the resource constraint, the consumption-capital ratio is given by

$$\frac{c^*}{k^*} = \phi(\kappa^*) - \delta = \frac{y^*}{k^*} - \delta.$$

The comparative statics are then trivial. **QED** ∎

3.3.2 Transitional Dynamics

- Consider the condition that determined labor supply:

$$\frac{U_z(c_t, 1-l_t)}{U_c(c_t, 1-l_t)} = F_L(k_t, l_t).$$

We can solve this for l_t as a function of contemporaneous consumption and capital:

$$l_t = l(c_t, k_t).$$

Substituting then into the Euler condition and the resource constraint, we conclude:

$$\begin{aligned} \frac{U_c(c_t, 1-l(c_t, k_t))}{U_c(c_t, 1-l(c_t, k_t))} &= \beta[1-\delta+F_K(k_{t+1}, l(c_{t+1}, k_{t+1}))] \\ k_{t+1} &= F(k_t, l(c_t, k_t)) + (1-\delta)k_t - c_t \end{aligned}$$

This is a system of two first-order difference equation in c_t and k_t. Together with the initial condition (k_0 given) and the transversality condition, this system pins down the path of $\{c_t, k_t\}_{t=0}^{\infty}$.

3.4 The Neoclassical Growth Model with Exogenous Labor

3.4.1 Steady State and Transitional Dynamics

- Suppose that leisure is not valued, or that the labor supply is exogously fixed. Either way, let $l_t = 1$ for all t. Suppose further that preferences exhibit constant elasticity of intertemporal substitution:

$$U(c) = \frac{c^{1-1/\theta} - 1}{1 - 1/\theta},$$

$\theta > 0$ is reciprocal of the elasticity of the marginal utility of consumption and is called the elasticity of intertemporal substitution. Under these restrictions, the dynamics reduce to

$$\left(\frac{c_{t+1}}{c_t}\right)^{\theta} = \beta[1 + f'(k_{t+1}) - \delta] = \beta[1 + R_t],$$

$$k_{t+1} = f(k_t) + (1 - \delta)k_t - c_t.$$

- Finally, we know that the transversality condition is satisfied if and only if the path converges to the steady state, and we can also so that the capital stock converges monotonically to its steady state value. We conclude:

Proposition 18 *Suppose that labor is exogenously fixed and preferences exhibit CEIS. The path* $\{c_t, k_t\}_{t=0}^{\infty}$ *is the equilibrium path of the economy (and the solution to the social planner's problem) if and only if*

$$\frac{c_{t+1}}{c_t} = \{\beta[1 + f'(k_{t+}) - \delta]\}^{\theta},$$

$$k_{t+1} = f(k_t) + (1 - \delta)k_t - c_t,$$

for all t*, with*

$$k_0 \text{ given and } \lim_{t\to\infty} k_t = k^*,$$

where k^* *is the steady state value of capital:*

$$f'(k^*) = \rho + \delta$$

For any initial $k_0 < k^*$ $(k_0 > k^*)$*, the capital stock* k_t *is increasing (respectively, decreasing) over time and converges to asymptotically to* k^**. Similarly, the rate of per-capita consumption growth* c_{t+1}/c_t *is positive and decreasing (respectively, negative and increasing) over time and converges monotonically to* 0.

Proof. The policy rule $k_{t+1} = G(k_t)$ is increasing, continuous, and intersects with the 45^o only at $k = 0$ and $k = k^*$. See Lucas and Stokey for the complete proof. The same argument as in the Solow model then implies that $\{k_t\}_{t=0}^{\infty}$ is monotonic and converges to k^*. The monotonicity and convergence of $\{c_{t+1}/c_t\}_{t=0}^{\infty}$ then follows immediately from the monotonicity and convergence of $\{k_t\}_{t=0}^{\infty}$ together with the fact that $f'(k)$ is decreasing. ■

- We will see these results also graphically in the phase diagram, below.

3.4.2 Continuous Time

- Taking logs of the Euler condition and approximating $\ln\beta = -\ln(1+\rho) \approx -\rho$ and $\ln[1-\delta+f'(k_t)] \approx f'(k_t)-\delta$, we can write the Euler condition as

$$\ln c_{t+1} - \ln c_t \approx \theta[f'(k_{t+1}) - \delta - \rho].$$

  We can also rewrite the resource constraint as

$$k_{t+1} - k_t = f(k_t) - \delta k_t - c_t.$$

- The approximation turns out to be exact when time is continuous:

Proposition 19 *Suppose that time is continuous. Like before, assume that labor is exogenously fixed and preferences exhibit CEIS. The path $\{c_t, k_t\}_{t\in\mathbb{R}_+}$ is the equilibrium path of the economy (and the solution to the social planner's problem) if and only if*

$$\frac{\dot{c}_t}{c_t} = \theta[f'(k_t) - \delta - \rho] = \theta[R_t - \rho],$$

$$\dot{k}_t = f(k_t) - \delta k_t - c_t,$$

for all t, with

$$k_0 \text{ given and } \lim_{t\to\infty} k_t = k^*,$$

where k^* is the steady state value of capital:

$$f'(k^*) = \rho + \delta$$

Proof. See Barro and Sala-i-Martin for details. ■

3.4.3 Phase Diagram (Figure 1)

- We can now use the phase diagram to describe the transitional dynamics of the economy. See **Figure 1.**

- The $\dot{k} = 0$ locus is given by (c, k) such that

$$\dot{k} = f(k) - \delta k - c = 0 \Leftrightarrow$$
$$c = f(k) - \delta k$$

  On the other hand, the $\dot{c} = 0$ locus is given by (c, k) such that

$$\dot{c} = c\theta[f'(k) - \delta - \rho] = 0 \Leftrightarrow$$
$$k = k^*$$

  Remark: Obviously, $c = 0$ also ensures $\dot{c} = 0$, but this corresponds to the trivial and unstable steady state $c = 0 = k$, so I will ignore it for the rest of the discussion.

- The steady state is simply the intersection of the two loci:

$$\dot{c} = \dot{k} = 0 \quad \Leftrightarrow$$
$$\left\{ \begin{array}{c} k = k^* \equiv (f')^{-1}(\rho + \delta) \\ c = c^* \equiv f(k^*) - \delta k^* \end{array} \right\} \text{ or } \{c = k = 0\}$$

- The $\dot{c} = 0$ and $\dot{k} = 0$ loci are depicted in Figure 1. Note that the two loci partition the (c, k) space in four regions. We now examine what is the direction of change in c and k in each of these four regions.

- Consider first the direction of $\dot{c}$. If $0 < k < k^*$ $[k > k^*]$, then and only then $\dot{c} > 0$ $[\dot{c} < 0]$. That is, c increases [decreases] with time whenever (c, k) lies the left [right] of the $\dot{c} = 0$ locus. The direction of $\dot{c}$ is represented by the vertical arrows in Figure 1.

- Consider next the direction of $\dot{k}$. If $c < f(k) - \delta k$ $[c > f(k) - \delta k]$, then and only then $\dot{k} > 0$ $[\dot{k} < 0]$. That is, k increases [decreases] with time whenever (c, k) lies below [above] the $\dot{k} = 0$ locus. The direction of $\dot{k}$ is represented by the horizontal arrows in Figure 1.

- We can now draw the time path of $\{k_t, c_t\}$ starting from any arbitrary (k_0, c_0), as in Figure 1. Note that there are only two such paths that go through the steady state. The one with positive slope represents the stable manifold or saddle path. The other corresponds to the unstable manifold.

- The equilibrium path of the economy for any initial k_0 is given by the stable manifold. That is, for any given k_0, the equilibrium c_0 is the one that puts the economy on the saddle path.

- To understand why the saddle path is the optimal path when the horizon is infinite, note the following:

  - Any c_0 that puts the economy *above* the saddle path leads to zero capital and zero consumption in finite time, thus violating the Euler condition at that time. Of course, if the horizon was finite, such a path would have been the equilibrium path. But with infinite horizon it is better to consume less and invest more in period 0, so as to never be forced to consume zero at finite time.

- On the other hand, any c_0 that puts the economy *below* the saddle path leads to so much capital accumulation in the limit that the transversality condition is violated. Actually, in finite time the economy has cross the golden-rule and will henceforth become dynamically inefficient. One the economy reaches k_{gold}, where $f'(k_{gold}) - \delta = 0$, continuing on the path is dominated by an alternative feasible path, namely that of investing nothing in new capital and consuming $c = f(k_{gold}) - \delta k_{gold}$ thereafter. In other words, the economy is wasting too much resources in investment and it would better increase consumption.

- Let the function $c(k)$ represent the saddle path. In terms of dynamic programming, $c(k)$ is simply the optimal policy rule for consumption given capital k. Equivalently, the optimal policy rule for capital accumulation is given by

$$\dot{k} = f(k) - \delta k - c(k),$$

or in discrete time

$$k_{t+1} \approx G(k_t) \equiv f(k_t) + (1-\delta)k_t - c(k_t).$$

- Finally, note that, no matter what is the form of $U(c)$, you could also write the dynamics in terms of k and λ:

$$\begin{aligned} \frac{\dot{\lambda}_t}{\lambda_t} &= f'(k_t) - \delta - \rho \\ \dot{k}_t &= f(k_t) - \delta k_t - c(\lambda_t), \end{aligned}$$

where $c(\lambda)$ solves $U_c(c) = \lambda$, that is, $c(\lambda) \equiv U_c^{-1}(\lambda)$. Note that $U_{cc} < 0$ implies $c'(\lambda) < 0$. As an exercise, you can draw the phase diagram and analyze the dynamics in terms of k and λ.

3.5 Comparative Statics and Impulse Responses

3.5.1 Additive Endowment (Figure 2)

- Suppose that each household receives an endowment $e > 0$ from God. Then, the household budget is

$$c_t^j + k_{t+1}^j = w_t + r_t k_t^j + (1-\delta)k_t^j + e$$

Optimal consumption growth is thus given again by

$$\frac{U_c(c_t^j)}{U_c(c_{t+1}^j)} = \beta[1 + r_{t+1} - \delta]$$

which together with $r_t = f'(k_t)$ implies

$$\frac{c_{t+1}}{c_t} = \{\beta[1 + f'(k_{t+1}) - \delta]\}^\theta$$

On the other hand, adding up the budget across households gives the resource constraint of the economy

$$k_{t+1} - k_t = f(k_t) - \delta k_t - c_t + e$$

- We conclude that the phase diagram becomes

$$\frac{\dot{c}_t}{c_t} = \theta[f'(k_t) - \delta - \rho],$$

$$\dot{k}_t = f(k_t) - \delta k_t - c_t + e.$$

- In the steady state, k^* is independent of e and c^* moves one to one with e.

- Consider a permanent increase in e by Δe. This leads to a parallel shift in the $\dot{k} = 0$ locus, but no change in the $\dot{c} = 0$ locus. If the economy was initially at the steady state, then k stays constant and c simply jumps by exactly e. On the other hand, if the economy was below the steady state, c will initially increase but by less that e, so that both the level and the rate of consumption growth will increase along the transition. See **Figure 2.**

3.5.2 Taxation and Redistribution (Figures 3 and 4)

- Suppose that labor and capital income are taxed at a flat tax rate $\tau \in (0,1)$. The government redistributes the proceeds from this tax uniformly across households. Let T_t be the transfer in period t. Then, the household budget is

$$c_t^j + k_{t+1}^j = (1-\tau)(w_t + r_t k_t^j) + (1-\delta)k_t^j + T_t,$$

implying

$$\frac{U_c(c_t^j)}{U_c(c_{t+1}^j)} = \beta[1 + (1-\tau)r_{t+1} - \delta].$$

That is, the tax rate decreases the private return to investment. Combining with $r_t = f'(k_t)$ we infer

$$\frac{c_{t+1}}{c_t} = \{\beta[1 + (1-\tau)f'(k_{t+1}) - \delta]\}^\theta .$$

Adding up the budgets across household gives

$$c_t + k_{t+1} = (1-\tau)f(k_{t+1}) + (1-\delta)k_t + T_t$$

The government budget on the other hand is

$$T_t = \tau \int_j (w_t + r_t k_t^j) = \tau f(k_t)$$

Combining we get the resource constraint of the economy:

$$k_{t+1} - k_t = f(k_t) - \delta k_t - c_t$$

Observe that, of course, the tax scheme does not appear in the resource constraint of the economy, for it is only redistributive and does not absorb resources.

- We conclude that the phase diagram becomes

$$\frac{\dot{c}_t}{c_t} = \theta[(1-\tau)f'(k_t) - \delta - \rho],$$

$$\dot{k}_t = f(k_t) - \delta k_t - c_t.$$

- In the steady state, k^* and therefore c^* are decreasing functions of τ.

A. Unanticipated Permanent Tax Cut

- Consider an unanticipated permanent tax cut that is enacted immediately. The $\dot{k} = 0$ locus does not change, but the $\dot{c} = 0$ locus shifts right. The saddle path thus shifts right. See **Figure 3.**

- A permanent tax cut leads to an immediate negative jump in consumption and an immediate positive jump in investment. Capital slowly increases and converges to a higher k^*. Consumption initially is lower, but increases over time, so soon it recovers and eventually converges to a higher c^*.

B. Anticipated Permanent Tax Cut

- Consider an permanent tax cut that is (credibly) announced at date 0 to be enacted at some date $\widehat{t} > 0$. The difference from the previous exercise is that $\dot{c} = 0$ locus now does not change immediately. It remains the same for $t < \widehat{t}$ and shifts right only for $t > \widehat{t}$. Therefore, the dynamics of c and k will be dictated by the "old" phase diagram (the one corresponding to high τ) for $t < \widehat{t}$ and by the "new" phase diagram (the one corresponding to low τ) for $t > \widehat{t}$,

- At $t = \widehat{t}$ and on, the economy must follow the saddle path corresponding to the new low τ, which will eventually take the economy to the new steady state. For $t < \widehat{t}$, the economy must follow a path dictated by the old dynamics, but at $t = \widehat{t}$ the economy must exactly reach the new saddle path. If that were not the case, the consumption path would have to jump at date $\widehat{t}$, which would violate the Euler condition (and thus be suboptimal). Therefore, the equilibrium c_0 is such that, if the economy follows a path dictated by the old dynamics, it will reach the new saddle path exactly at $t = \widehat{t}$. See **Figure 4.**

- Following the announcement, consumption jumps down and continues to fall as long as the tax cut is not initiated. The economy is building up capital in anticipation of the tax cut. As soon as the tax cut is enacted, capital continues to increase, but consumption also starts to increase. The economy then slowly converges to the new higher steady state.

3.5.3 Productivity Shocks: A prelude to RBC (Figures 5 and 6)

- We now consider the effect of a shock in total factor productivity (TFP). The reaction of the economy in our deterministic framework is similar to the impulse responses we get in a stochastic Real Business Cycle (RBC) model. Note, however, that here we consider the case that labor supply is exogenously fixed. The reaction of the economy will be somewhat different with endogenous labor supply, whether we are in the deterministic or the stochastic case.

- Let output be given by

$$y_t = A_t f(k_t)$$

where A_t denotes TFP. Note that

$$r_t = A_t f'(k_t)$$
$$w_t = A_t[f(k_t) - f'(k_t)k_t]$$

so that both the return to capital and the wage rate are proportional to TFP.

- We can then write the dynamics as

$$\frac{\dot{c}_t}{c_t} = \theta[A_t f'(k_t) - \delta - \rho],$$
$$\dot{k}_t = A_t f(k_t) - \delta k_t - c_t.$$

Note that TFP A_t affects both the production possibilities frontier of the economy (the resource constrain) and the incentives to accumulate capital (the Euler condition).

- In the steady state, both k^* and c^* are increasing in A.

A. Unanticipated Permanent Productivity Shock

- The $\dot{k} = 0$ locus shifts up and the $\dot{c} = 0$ locus shifts right, permanently.
- c_0 may either increase or fall, depending on whether wealth or substitution effect dominates. Along the transition, both c and k are increasing towards the new higher steady state. See **Figure 5** for the dynamics.

B. Unanticipated Transitory Productivity Shock

- The $\dot{k} = 0$ locus shifts up and the $\dot{c} = 0$ locus shifts right, but only for $t \in [0, \widehat{t}]$ for some finite $\widehat{t}$.
- Again, c_0 may either increase or fall, depending on whether wealth or substitution effects dominates. I consider the case that c_0 increases. A typical transition is depicted in **Figure 6**.

3.5.4 Government Spending (Figure 7 and 8)

- We now introduce a government that collects taxes in order to finance some exogenous level of government spending.

A. Lump Sum Taxation

- Suppose the government finances its expenditure with lump-sum taxes. The household budget is

$$c_t^j + k_{t+1}^j = w_t + r_t k_t^j + (1-\delta)k_t^j - T_t,$$

implying

$$\frac{U_c(c_t^j)}{U_c(c_{t+1}^j)} = \beta[1 + r_{t+1} - \delta] = \beta[1 + f'(k_{t+1}) - \delta]$$

That is, taxes do not affect the savings choice. On the other hand, the government budget is

$$T_t = g_t,$$

where g_t denotes government spending. The resource constraint of the economy becomes

$$c_t + g_t + k_{t+1} = f(k_t) + (1-\delta)k_t$$

- We conclude

$$\frac{\dot{c}_t}{c_t} = \theta[f'(k_t) - \delta - \rho],$$

$$\dot{k}_t = f(k_t) - \delta k_t - c_t - g_t$$

- In the steady state, k^* is independent of g and c^* moves one-to-one with $-g$. Along the transition, a permanent increase in g both decreases c and slows down capital accumulation. See **Figure 7.**

- Note that the effect of government spending financed with lump-sum taxes is isomorphic to a negative endowment shock.

B. Distortionary Taxation

- Suppose the government finances its expenditure with distortionary income taxation. The household budget is

$$c_t^j + k_{t+1}^j = (1-\tau)(w_t + r_t k_t^j) + (1-\delta)k_t^j,$$

implying

$$\frac{U_c(c_t^j)}{U_c(c_{t+1}^j)} = \beta[1+(1-\tau)r_{t+1} - \delta] = \beta[1+(1-\tau)f'(k_{t+1}) - \delta].$$

That is, taxes now distort the savings choice. On the other hand, the government budget is

$$g_t = \tau f(k_t)$$

and the resource constraint of the economy is again

$$c_t + g_t + k_{t+1} = f(k_t) + (1-\delta)k_t.$$

- We conclude

$$\frac{\dot{c}_t}{c_t} = \theta[(1-\tau)f'(k_t) - \delta - \rho],$$
$$\dot{k}_t = (1-\tau)f(k_t) - \delta k_t - c_t.$$

Government spending is now isomorphic to a negative TFP change.

- In the steady state, k^* is a decreasing function of g (equivalently, τ) and c^* decreases more than one-to-one with g. Along the transition, a permanent increase in g (and τ) drastically slows down capital accumulation. The immediate See **Figure 7.**

- Note that the effect of government spending financed with distortionary taxes is isomorphic to a negative TFP shock.

3.6 Endogenous Labor Supply, the RBC Propagation Mechanism, and Beyond

3.6.1 The Phase Diagram with Endogenous Labor Supply

- Solve for labor supply as a function of k and c :

$$\frac{U_x(c_t, 1-l_t)}{U_c(c_t, 1-l_t)} = F_L(k_t, l_t) \Rightarrow l_t = l(k_t, c_t)$$

  Note that l increases with k, but less than one-to-one (or otherwise F_L would fall). This reflects the substitution effect. On the other hand, l falls with c, reflecting the wealth effect.

- Substitute back into the dynamic system for k and c, assuming CEIS preferences:

$$\frac{\dot{c}_t}{c_t} = \theta[f'(k_t/l(k_t, c_t)) - \delta - \rho],$$
$$\dot{k}_t = f(k_t, l(k_t, c_t)) - \delta k_t - c_t,$$

  which gives a system in k_t and c_t alone.

- Draw suggestive phase diagram. See Figure **??**.

- Note that the $\dot{c}$ is now negatively sloped, not vertical as in the model with exogenously fixed labor. This reflects the wealth effect on labor supply. Lower c corresponds to lower effective wealth, which results to higher labor supply for any given k (that is, for any given wage).

3.6.2 Impulse Responces Revisited

- Note that the endogeneity of labor supply makes the Euler condition (the $\dot{c}$ locus) sensitive to wealth effects, but also mitigates the impact of wealth effects on the resource constraint (the $\dot{k}$ locus).

- Reconsider the impulse responces of the economy to shocks in productivity or governement spending.

- Government spending.... If financed with lump sum taxes, an increase in g has a negative wealth effect, which increases labor supply. This in turn leads an increase in the MPK and stimulates more investment. At the new steady state the capital-labor ratio remains the same, as it is simply the one that equates the MPK with the discount rate, but both employment and the stock of capital go up...

- Note that the above is the supply-side effect of government spending. Contrast this with the demand-side effect in Keynesian models (e.g., IS-LM).

- Productivity shocks....

3.6.3 The RBC Propagation Mechanism, and Beyond

- Just as we can use the model to "explain" the variation of income and productivity levels in the cross-section of countries (i.e., do the Mankiw-Romer-Weil exercise), we can also use the model to "explain" the variation of income, productivity, investment and employment in the time-series of any given country. Hence, the RBC paradigm.

- The heart of the RBC propagation mechanism is the interaction of consumption smoothing and deminishing returns to capital accumulation. Explain....

- This mechanism generates endogenous *persistence* and *amplification*. Explain...

- Enogenous persistence is indeed the other face of conditional convergence. But just as the model fails to generate a substantially low rate of conditional convergence, it also fails to generate either substantial persistence or substantial amplification. For the model to match the data, we then need to assume that exogenous productivity (the Solow residual) is itself very volatile and persistent. But then we partly answer and partly peg the question.

- Hence the search for other endogenous propagation mechanisms.

- Discuss Keynesian models and monopolistic competition... Discuss the potential role financial markets...

to be completed

Chapter 4

Applications

4.1 Arrow-Debreu Markets and Consumption Smoothing

4.1.1 The Intertemporal Budget

- For any given sequence $\{R_t\}_{t=0}^{\infty}$, pick an arbitrary $q_0 > 0$ and define q_t recursively by

$$q_t = \frac{q_0}{(1+R_0)(1+R_1)...(1+R_t)}.$$

  q_t represents the price of period$-t$ consumption relative to period-0 consumption.

- Multiplying the period-t budget by q_t and adding up over all t, we get

$$\sum_{t=0}^{\infty} q_t \cdot c_t^j \leq q_0 \cdot x_0^j$$

where

$$x_0^j \equiv (1+R_0)a_0 + h_0^j,$$
$$h_0^j \equiv \sum_{t=0}^{\infty} \frac{q_t}{q_0}[w_t l_t^j - T_t^j].$$

The above represents the intertemporal budget constraint. $(1+R_0)a_0^j$ is the household's *financial wealth* as of period 0. T_t^j is a lump-sum tax obligation, which may depend on the identity of household but not on its choices. h_0^j is the present value of labor income as of period 0 net of taxes; we often call h_0^j the household's *human wealth* as of period 0. The sum $x_0^j \equiv (1+R_0)a_0^j + h_0^j$ represents the household's *effective wealth*.

- Note that the sequence of per-period budgets and the intertemporal budget constraint are equivalent.

We can then write household's consumption problem as follows

$$\max \sum_{t=0}^{\infty} \beta^t U(c_t^j, z_t^j)$$
$$s.t. \ \sum_{t=0}^{\infty} q_t \cdot c_t^j \leq q_0 \cdot x_0^j$$

4.1.2 Arrow-Debreu versus Radner

- We now introduce uncertainty...

- Let $q(s^t)$ be the period-0 price of a unite of the consumable in period t and event s^t and $w(s^t)$ the period-t wage rate in terms of period-t consumables for a given event s^t. $q(s^t)w(s^t)$ is then the period-t and event-s^t wage rate in terms of period-0 consunambles. We can then write household's consumption problem as follows

$$\max \sum_t \sum_{s^t} \beta^t \pi(s^t) U\left(c^j(s^t), z^j(s^t)\right)$$
$$s.t. \sum_t \sum_{s^t} q(s^t) \cdot c^j(s^t) \leq q_0 \cdot x_0^j$$

where

$$x_0^j \equiv (1+R_0)a_0 + h_0^j,$$
$$h_0^j \equiv \sum_{t=0}^{\infty} \frac{q(s^t)w(s^t)}{q_0}[l^j(s^t) - T^j(s^t)].$$

$(1+R_0)a_0^j$ is the household's *financial wealth* as of period 0. $T^j(s^t)$ is a lump-sum tax obligation, which may depend on the identity of household but not on its choices. h_0^j is the present value of labor income as of period 0 net of taxes; we often call h_0^j the household's *human wealth* as of period 0. The sum $x_0^j \equiv (1+R_0)a_0^j + h_0^j$ represents the household's *effective wealth*.

4.1.3 The Consumption Problem with CEIS

- Suppose for a moment that preferences are separable between consumption and leisure and are homothetic with respect to consumption:

$$U(c,z) = u(c) + v(z).$$
$$u(c) = \frac{c^{1-1/\theta}}{1-1/\theta}$$

- Letting μ be the Lagrange multiplier for the intertemporal budget constraint, the FOCs imply

$$\beta^t \pi(s^t) u'\left(c^j(s^t)\right) = \mu q(s^t)$$

for all $t \geq 0$. Evaluating this at $t = 0$, we infer $\mu = u'(c_0^j)$. It follows that

$$\frac{q(s^t)}{q_0} = \frac{\beta^t \pi(s^t) u'\left(c^j(s^t)\right)}{u'(c_0^j)} = \beta^t \pi(s^t) \left(\frac{c^j(s^t)}{c_0^j}\right)^{-1/\theta}.$$

That is, the price of a consumable in period t relative to period 0 equals the marginal rate of intertemporal substitution between 0 and t.

- Solving $q_t/q_0 = \beta^t \pi(s^t) \left[c^j(s^t)/c_0^j\right]^{-1/\theta}$ for $c^j(s^t)$ gives

$$c^j(s^t) = c_0^j \left[\beta^t \pi(s^t)\right]^{\theta} \left[\frac{q(s^t)}{q_0}\right]^{-\theta}.$$

It follows that the present value of consumption is given by

$$\sum_t \sum_{s^t} q(s^t) c^j(s^t) = q_0^{-\theta} c_0^j \sum_{t=0}^{\infty} \left[\beta^t \pi(s^t)\right]^{\theta} q(s^t)^{1-\theta}$$

Substituting into the resource constraint, and solving for c_0, we conclude

$$c_0^j = m_0 \cdot x_0^j$$

where

$$m_0 \equiv \frac{1}{\sum_{t=0}^{\infty} \left[\beta^t \pi(s^t)\right]^{\theta} \left[q(s^t)/q_0\right]^{1-\theta}}.$$

Consumption is thus linear in effective wealth. m_0 represent the MPC out of effective wealth as of period 0.

4.1.4 Intertemporal Consumption Smoothing, with No Uncertainty

- Consider for a moment the case that there is no uncertainty, so that $c^j(s^t) = c_t^j$ and $q(s^t) = q_t$ for all s^t.

- Then, the riskless bond and the Arrow securities satisfy the following arbitrage condition
$$q_t = \frac{q_0}{(1+R_0)(1+R_1)...(1+R_t)}.$$
Alternatively,
$$q_t = q_0 \left[1+\widetilde{R}_{0,t}\right]^{-t}$$
where $\widetilde{R}_{0,t}$ represents the "average" interest rate between 0 and t. Next, note that m_0 is decreasing (increasing) in q_t if and only if $\theta > 1$ ($\theta < 1$). It follows that the marginal propensity to save in period 0, which is simply $1 - m_0$, is decreasing (increasing) in $\widetilde{R}_{0,t}$, for any $t \geq 0$, if and only if $\theta > 1$ ($\theta < 1$).

- A similar result applies for all $t \geq 0$. We conclude

Proposition 20 *Suppose preferences are seperable between consumption and leisure and homothetic in consumption (CEIS). Then, the optimal consumption is linear in contemporaneous effective wealth:*
$$c_t^j = m_t \cdot x_t^j$$
where
$$x_t^j \equiv (1+R_t)a_t^j + h_t^j,$$
$$h_t^j \equiv \sum_{\tau=t}^{\infty} \frac{q_t}{q_t}[w_\tau l_\tau^j - T_\tau^j],$$
$$m_t \equiv \frac{1}{\sum_{\tau=t}^{\infty} \beta^{\theta(\tau-t)}(q_\tau/q_t)^{1-\theta}}.$$
m_t is a decreasing (increasing) function of q_τ for any $\tau \geq t$ if and only $\theta > 1$ ($\theta < 1$). That is, the marginal propensity to save out of effective wealth is increasing (decreasing) in future interest rates if and only if the elasticity of intertemporal substitution is higher (lower)

than unit. Moreover, for given prices, the optimal consumption path is independent of the timining of either labor income or taxes.

- Obviously, a similar result holds with uncertainty, as long as there are complete Arrow-Debreu markets.

- Note that any expected change in income has no effect on consumption as long as it does not affect the present value of labor income. Also, if there is an innovation (unexpected change) in income, consumption will increase today and for ever by an amount proportional to the innovation in the annuity value of labor income.

- To see this more clearly, suppose that the interest rate is constant and equal to the discount rate: $R_t = R = 1/\beta - 1$ for all t. Then, the marginal propensity to consume is

  $$m = 1 - \beta^{\theta}(1+R)^{1-\theta} = 1 - \beta,$$

  the consumption rule in period 0 becomes

  $$c_0^j = m \cdot \left[(1+R)a_0 + h_0^j\right]$$

  and the Euler condition reduces to

  $$c_t^j = c_0^j$$

  Therefore, the consumer choose a totally flat consumption path, no matter what is the time variation in labor income. And any unexpected change in consumption leads to a parallel shift in the path of consumption by an amount equal to the annuity value of the change in labor income. This is the manifestation of *intertemporal consumption smoothing.*

- More generally, if the interest rate is higher (lower) than the discount rate, the path of consumption is smooth but has a positive (negative) trend. To see this, note that the Euler condition is

$$\log c_{t+1} \approx \theta[\beta(1+R)]^{\theta} + \log c_t.$$

4.1.5 Incomplete Markets and Self-Insurance

- The above analysis has assumed no uncertainty, or that markets are complete. Extending the model to introduce idiosyncratic uncertainty in labor income would imply an Euler condition of the form

$$u'(c_t^j) = \beta(1+R)\mathsf{E}_t u'(c_{t+1}^j)$$

Note that, because of the convexity of u', as long as $Var_t[c_{t+1}^j] > 0$, we have $\mathsf{E}_t u'(c_{t+1}^j) > u'(\mathsf{E}_t c_{t+1}^j)$ and therefore

$$\frac{\mathsf{E}_t c_{t+1}^j}{c_t^j} > [\beta(1+R)]^{\theta}$$

This extra kick in consumption growth reflects the *precautionary motive for savings.* It remains true that transitory innovations in income result to persistent changes in consumption (because of consumption smoothing). At the same time, consumers find it optimal to accumulate a *buffer stock,* as a vehicle for self-insurance.

4.2 Aggregation and the Representative Consumer

- Consider a deterministic economy populated by many heterogeneous households. Households differ in their initial asset positions and (perhaps) their streams of labor income, but not in their preferences. They all have CEIS preferences, with identical θ.

- Following the analysis of the previous section, consumption for individual j is given by

$$c_t^j = m_t \cdot x_t^j.$$

Note that individuals share the same MPC out of effective wealth because they have identical θ.

- Adding up across households, we infer that aggregate consumption is given by

$$c_t = m_t \cdot x_t$$

where

$$x_t \equiv (1+R_t)a_t + h_t,$$
$$h_t \equiv \sum_{\tau=t}^{\infty} \frac{q_\tau}{q_t}[w_\tau l_\tau - T_\tau],$$
$$m_t \equiv \frac{1}{\sum_{\tau=t}^{\infty} \beta^{\theta(\tau-t)}(q_\tau/q_t)^{1-\theta}}.$$

- Next, recall that individual consumption growth satisfies

$$\frac{q_t}{q_0} = \frac{\beta^t u'(c_t^j)}{u'(c_0^j)} = \beta^t \left(\frac{c_t^j}{c_0^j}\right)^{-1/\theta},$$

for every j. But if all agents share the same consumption growth rate, this should be the aggregate one. Therefore, equilibrium prices and aggregate consumption growth satisfy

$$\frac{q_t}{q_0} = \beta^t \left(\frac{c_t}{c_0}\right)^{-1/\theta}$$

Equivalently,

$$\frac{q_t}{q_0} = \frac{\beta^t u'(c_t)}{u'(c_0)}.$$

- Consider now an economy that has a single consumer, who is endowed with wealth x_t and has preferences

$$U(c) = \frac{c^{1-1/\theta}}{1-1/\theta}$$

The Euler condition for this consumer will be

$$\frac{q_t}{q_0} = \frac{\beta^t u'(c_t)}{u'(c_0)}.$$

Moreover, this consumer will find it optimal to choose consumption

$$c_t = m_t \cdot x_t.$$

But these are exactly the aggregative conditions we found in the economy with many agents.

- That is, the two economies share exactly the same equilibrium prices and allocations. It is in this sense that we can think of the single agent of the second economy as the "representative" agent of the first multi-agent economy.

- Note that here we got a stronger result than just the existence of a representative agent. Not only a representative agent existed, but he also had exactly the same preferences as each of the agents of the economy. This was true only because agents had identical preference to start with and their preferences were homothetic. If either condition fails, the preferences of the representative agent will be "weighted average" of the population preferences, with the weights depending on the wealth distribution.

- Finally, note that these aggregation results extend easily to the case of uncertainty as long as markets are complete.

4.3 Fiscal Policy

4.3.1 Ricardian Equilivalence

- The intertemporal budget for the representative household is given by

$$\sum_{t=0}^{\infty} q_t c_t \leq q_0 x_0$$

where

$$x_0 = (1+R_0)a_0 + \sum_{t=0}^{\infty} \frac{q_t}{q_0}[w_t l_t - T_t]$$

and $a_0 = k_0 + b_0$.

- On the other hand, the intertemporal budget constraint for the government is

$$\sum_{t=0}^{\infty} q_t g_t + q_0(1+R_0)b_0 = \sum_{t=0}^{\infty} q_t T_t$$

- Substituting the above into the formula for x_0, we infer

$$x_0 = (1+R_0)k_0 + \sum_{t=0}^{\infty} \frac{q_t}{q_0} w_t l_t - \sum_{t=0}^{\infty} \frac{q_t}{q_0} g_t$$

That is, aggregate household wealth is independent of either the outstanding level of public debt or the timing of taxes.

- We can thus rewrite the representative household's intertemporal budget as

$$\sum_{t=0}^{\infty} q_t[c_t + g_t] \leq q_0(1+R_0)k_0 + \sum_{t=0}^{\infty} q_t w_t l_t$$

Since the representative agent's budget constraint is independent of either b_0 or $\{T_t\}_{t=0}^{\infty}$, his consumption and labor supply will also be independent. But then the resource constraint implies that aggregate investment will be unaffected as well. Therefore, the aggregate path $\{c_t, k_t\}_{t=0}^{\infty}$ is independent of either b_0 or $\{T_t\}_{t=0}^{\infty}$. All that matter is the stream of government spending, not the way this is financed.

- More generally, consider now arbitrary preferences and endogenous labor supply, but suppose that the tax burden and public debt is uniformly distributed across households. Then, for *every* individual j, effective wealth is independent of either the level of public debt or the timing of taxes:

$$x_0^j = (1 + R_0)k_0^j + \sum_{t=0}^{\infty} \frac{q_t}{q_0} w_t l_t^j - \sum_{t=0}^{\infty} \frac{q_t}{q_0} g_t,$$

Since the individual's intertemporal budget is independent of either b_0 or $\{T_t\}_{t=0}^{\infty}$, her optimal plan $\{c_t^j, l_t^j, a_t^j\}_{t=0}^{\infty}$ will also be independent of either b_0 or $\{T_t\}_{t=0}^{\infty}$ for any given price path. But if individual behavior does not change for given prices, markets will continue to clear for the same prices. That is, equilibrium prices are indeed also independent of either b_0 or $\{T_t\}_{t=0}^{\infty}$. We conclude

Proposition 21 *Equilibrium prices and allocations are independent of either the intial level of public debt, or the mixture of deficits and (lump-sum) taxes that the government uses to finance governement spending.*

- *Remark:* For Ricardian equivalence to hold, it is critical both that markets are complete (so that agents can freely trade the riskless bond) and that horizons are infinite (so that the present value of taxes the household expects to pay just equals the amount of public debt it holds). If either condition fails, such as in OLG economies or economies with borrowing constraints, Ricardian equivalence will also fail. Ricardian equivalence may also fail if there are

4.3.2 Tax Smoothing and Debt Management

topic covered in class

notes to be completed

4.4 Risk Sharing and CCAPM

4.4.1 Risk Sharing

topic covered in class

notes to be completed

4.4.2 Asset Pricing and CCAPM

topic covered in class

notes to be completed

4.5 Ramsey Meets Tobin: Adjustment Costs and q

topic covered in recitation

notes to be completed

4.6 Ramsey Meets Laibson: Hyperbolic Discounting

4.6.1 Implications for Long-Run Savings

topic covered in class

notes to be completed

4.6.2 Implications for Self-Insurance

topic covered in class

notes to be completed

Chapter 5

Overlapping Generations Models

5.1 OLG and Life-Cycle Savings

5.1.1 Households

- Consider a household born in period t, living in periods t and $t+1$. We denote by c_t^y his consumption when young and c_{t+1}^o his consumption when old.

- Preferences are given by

$$u(c_t^y) + \beta u(c_{t+1}^o)$$

  where β denotes a discount factor and u is a neoclassical utility function.

- The household is born with zero initial wealth, saves only for life-cycle consumption smoothing, and dies leaving no bequests to future generations. The household receives labor income possibly in both periods of life. We denote by l^y and l^o the endowments of effective labor when young and when old, respectively. The budget constraint during the first period of life is thus

$$c_t^y + a_t \leq w_t l^y,$$

whereas the budget constraint during the second period of life is

$$c_{t+1}^o \leq w_{t+1} l^o + (1 + R_{t+1}) a_t.$$

Adding up the two constraints (and assuming that the household can freely borrow and lend when young, so that a_t can be either negative or positive), we derive the intertemporal budget constraint of the household:

$$c_t^y + \frac{c_{t+1}^o}{1 + R_{t+1}} \leq h_t \equiv w_t l^y + \frac{w_{t+1} l^o}{1 + R_{t+1}}$$

- The household choose consumption and savings so as to maximize life utility subject to his intertemporal budget:

$$\max \left[u(c_t^y) + \beta u(c_{t+1}^o) \right]$$

$$s.t. \quad c_t^y + \frac{c_{t+1}^o}{1 + R_{t+1}} \leq h_t.$$

The Euler condition gives:

$$u'(c_t^y) = \beta(1 + r_{t+1}) u'(c_{t+1}^o).$$

In words, the household chooses savings so as to smooth (the marginal utility of) consumption over his life-cycle.

- With CEIS preferences, $u(c) = c^{1-1/\theta}/(1 - 1/\theta)$, the Euler condition reduces to

$$\frac{c_{t+1}^o}{c_t^y} = [\beta(1 + R_{t+1})]^{\theta}.$$

Life-cycle consumption growth is thus an increasing function of the return on savings and the discount factor. Combining with the intertemporal budget, we infer

$$h_t = c_t^y + \frac{c_{t+1}^o}{1 + R_{t+1}} = c_t^y + \beta^\theta (1 + R_{t+1})^{\theta - 1} c_t^y$$

and therefore optimal consumption during youth is given by

$$c_t^y = m(r_{t+1}) \cdot h_t$$

where

$$m(R) \equiv \frac{1}{1 + \beta^\theta (1 + R)^{\theta - 1}}.$$

Finally, using the period-1 budget, we infer that optimal life-cycle saving are given by

$$a_t = w_t l^y - m(R_{t+1}) h_t = [1 - m(R_{t+1})] w_t l^y - m(R_{t+1}) \frac{w_{t+1} l^o}{1 + R_{t+1}}$$

5.1.2 Population Growth

- We denote by N_t the size of generation t and assume that population grows at constant rate n :

$$N_{t+1} = (1 + n) N_t$$

- It follows that the size of the labor force in period t is

$$L_t = N_t l^y + N_{t-1} l^o = N_t \left[l^y + \frac{l^o}{1 + n} \right]$$

We henceforth normalize $l^y + l^o/(1 + n) = 1$, so that $L_t = N_t$.

- *Remark:* As always, we can reinterpret N_t as effective labor and n as the growth rate of exogenous technological change.

5.1.3 Firms and Market Clearing

- Let $k_t = K_t/L_t = K_t/N_t$. The FOCs for competitive firms imply:

$$\begin{aligned} r_t &= f'(k_t) \equiv r(k_t) \\ w_t &= f(k_t) - f'(k_t)k_t \equiv w(k_t) \end{aligned}$$

On the other hand, the arbitrage condition between capital and bonds implies $1+R_t = 1+r_t-\delta$, and therefore

$$R_t = f'(k_t) - \delta \equiv r(k_t) - \delta$$

- Total capital is given by the total supply of savings:

$$K_{t+1} = a_t N_t$$

Equivalently,

$$(1+n)k_{t+1} = a_t.$$

5.1.4 General Equilibrium

- Combining $(1+n)k_{t+1} = a_t$ with the optimal rule for savings, and substituting $r_t = r(k_t)$ and $w_t = w(k_t)$, we infer the following general-equilibrium relation between savings and capital in the economy:

$$(1+n)k_{t+1} = [1 - m(r(k_{t+1}) - \delta)]w(k_t)l^y - m(r(k_{t+1}) - \delta)\frac{w(k_{t+1})l^o}{1 + r(k_{t+1}) - \delta}.$$

- We rewrite this as an implicit relation between k_{t+1} and k_t :

$$\Phi(k_{t+1}, k_t) = 0.$$

Note that

$$\begin{aligned}\Phi_1 &= (1+n) + h\frac{\partial m}{\partial R}\frac{\partial r}{\partial k} + ml^o\frac{\partial}{\partial k}\left(\frac{w}{1+r}\right), \\ \Phi_2 &= -(1-m)\frac{\partial w}{\partial k}l^y.\end{aligned}$$

Recall that $\frac{\partial m}{\partial R} \lesseqgtr 0 \Leftrightarrow \theta \gtreqless 1$, whereas $\frac{\partial r}{\partial k} = F_{KK} < 0$, $\frac{\partial w}{\partial k} = F_{LK} > 0$, and $\frac{\partial}{\partial k}\left(\frac{w}{1+r}\right) > 0$. It follows that Φ_2 is necessarily negative, but Φ_1 may be of either sign:

$$\Phi_2 < 0 \quad \text{but} \quad \Phi_1 \lesseqgtr 0.$$

We can thus always write k_t as a function of k_{t+1}, but to write k_{t+1} as a function of k_t,we need Φ to be monotonic in k_{t+1}.

- A sufficient condition for the latter to be the case is that savings are non-decreasing in real returns:

  $$\theta \geq 1 \Rightarrow \frac{\partial m}{\partial r} \geq 0 \Rightarrow \Phi_1 > 0$$

  In that case, we can indeed express k_{t+1} as a function of k_t :

  $$k_{t+1} = G(k_t).$$

  Moreover, $G' = -\frac{\Phi_2}{\Phi_1} > 0$, and therefore k_{t+1} increases monotonically with k_t. However, there is no guarantee that $G' < 1$. Therefore, in general there can be multiple steady states (and poverty traps). See **Figure 1**.

- On the other hand, if θ is sufficiently lower than 1, the equation $\Phi(k_{t+1}, k_t) = 0$ may have multiple solutions in k_{t+1} for given k_t. That is, it is possible to get *equilibrium indeterminacy*. Multiple equilibria indeed take the form of *self-fulfilling prophesies*. The anticipation of a high capital stock in the future leads agents to expect a low return on savings, which in turn motivates high savings (since $\theta < 1$) and results to a high capital stock in the future. Similarly, the expectation of low k in period $t+1$ leads to high returns and low savings in the period t, which again vindicates initial expectations. See **Figure 2**.

5.2 Some Examples

5.2.1 Log Utility and Cobb-Douglas Technology

- Assume that the elasticity of intertemporal substitution is unit, that the production technology is Cobb-Douglas, and that capital fully depreciates over the length of a generation:

$$u(c) = \ln c, \quad f(k) = k^\alpha, \quad \text{and} \quad \delta = 1.$$

- It follows that the MPC is constant,

$$m = \frac{1}{1+\beta}$$

and one plus the interest rate equals the marginal product of capital,

$$1 + R = 1 + r(k) - \delta = r(k)$$

where

$$\begin{aligned} r(k) &= f'(k) = \alpha k^{\alpha-1} \\ w(k) &= f(k) - f'(k)k = (1-\alpha)k^\alpha. \end{aligned}$$

- Substituting into the formula for G, we conclude that the law of motion for capital reduces to

$$k_{t+1} = G(k_t) = \frac{f'(k_t)k_t}{\zeta(1+n)} = \frac{\alpha k_t^\alpha}{\zeta(1+n)}$$

where the scalar $\zeta > 0$ is given by

$$\zeta \equiv \frac{(1+\beta)\alpha + (1-\alpha)l^o/(1+n)}{\beta(1-\alpha)l^y}$$

Note that ζ is increasing in l^o, decreasing in l^y, decreasing in β, and increasing in α (decreasing in $1-\alpha$). Therefore, G (savings) decreases with an increase in l^o and a decrease in l^y, with an decrease in β, or with an increase in α.

5.2.2 Steady State

- The steady state is any fixed point of the G mapping:

$$k_{olg} = G(k_{olg})$$

Using the formula for G, we infer

$$f'(k_{olg}) = \zeta(1+n)$$

and thus $k_{olg} = (f')^{-1}(\zeta(1+n))$.

- Recall that the golden rule is given by

$$f'(k_{gold}) = \delta + n,$$

and here $\delta = 1$. That is, $k_{gold} = (f')^{-1}(1+n)$.

- Pareto optimality requires

$$k_{olg} < k_{gold} \Leftrightarrow r > \delta + n \Leftrightarrow \zeta > 1,$$

while Dynamic Inefficiency occurs when

$$k_{olg} > k_{gold} \Leftrightarrow r < \delta + n \Leftrightarrow \zeta < 1.$$

Note that

$$\zeta = \frac{(1+\beta)\alpha + (1-\alpha)l^o/(1+n)}{\beta(1-\alpha)l^y}$$

is increasing in l^o, decreasing in l^y, decreasing in β, and increasing in α (decreasing in $1-\alpha$). Therefore, inefficiency is less likely the higher l^o, the lower l^y, the lower is β, and the higher α.

- Provide intuition...

- In general, ζ can be either higher or lower than 1. There is thus no guarantee that there will be no dynamic inefficiency. But, Abel et al argue that the empirical evidence suggests $r > \delta + n$, and therefore no evidence of dynamic inefficiency.

5.2.3 No Labor Income When Old: The Diamond Model

- Assume $l^o = 0$ and therefore $l^y = 1$. That is, household work only when young. This case corresponds to Diamond's OLG model.

- In this case, ζ reduces to

$$\zeta = \frac{(1+\beta)\alpha}{\beta(1-\alpha)}.$$

ζ is increasing in α and $\zeta = 1 \Leftrightarrow \alpha = \frac{1}{2+1/\beta}$. Therefore,

$$r \gtreqless n + \delta \Leftrightarrow \zeta \gtreqless 1 \Leftrightarrow \alpha \gtreqless (2+1/\beta)^{-1}$$

Note that, if $\beta \in (0,1)$, then $(2+1/\beta)^{-1} \in (0,1/3)$ and therefore dynamic inefficiency is possible only if α is sufficiently lower than 1/3. This suggests that dynamic inefficiency is rather unlikely. However, in an OLG model β can be higher than 1, and the higher β the more likely to get dynamic inefficiency in the Diamond model.

- Finally, note that dynamic inefficiency becomes *less* likely as we increase l^o, that is, as we increase income when old (hint: retirement benefits).

5.2.4 Perpetual Youth: The Blanchard Model

- We now reinterpret n as the rate of exogenous technological growth. We assume that household work the same amount of time in every period, meaning that in effective terms $l^o = (1+n)l^y$. Under the normalization $l^y + l^o/(1+n) = 1$, we infer $l^y = l^o/(1+n) = 1/2$.

- The scalar ζ reduces to

$$\zeta = \frac{2(1+\beta)\alpha + (1-\alpha)}{\beta(1-\alpha)}$$

Note that ζ is increasing in α, and since $\alpha > 0$, we have

$$\zeta > \frac{2(1+\beta)0 + (1-0)}{\beta(1-0)} = \frac{1}{\beta}.$$

- If $\beta \in (0,1)$, it is necessarily the case that $\zeta > 1$. It follows that necessarily $r > n + \delta$ and thus

$$k_{blanchard} < k_{gold},$$

meaning that it is impossible to get dynamic inefficiency.

- Moreover, recall that the steady state in the Ramsey model is given by

$$\beta[1 + f'(k_{ramsey}) - \delta] = 1 + n \Leftrightarrow$$

$$f'(k_{ramsey}) = (1+n)/\beta \Leftrightarrow$$

$$k_{ramsey} = (f')^{-1}((1+n)/\beta)$$

while the OLG model has

$$f'(k_{blanchard}) = \zeta(1+n) \Leftrightarrow$$
$$k_{blanchard} = (f')^{-1}(\zeta(1+n))$$

Since $\zeta > 1/\beta$, we conclude that the steady state in Blanchard's model is necessarily lower than in the Ramsey model. We conclude

$$k_{blanchard} < k_{ramsey} < k_{gold}.$$

- Discuss the role of "perpetual youth" and "new-comers".

5.3 Ramsey Meets Diamond: The Blanchard Model

topic covered in recitation

notes to be completed

5.4 Various Implications

- Dynamic inefficiency and the role of government
- Ricardian equivalence breaks, public debt crowds out investment.
- Fully-funded social security versus pay-as-you-go.
- Bubbles

notes to be completed

Chapter 6

Endogenous Growth I: AK, H, and G

6.1 The Simple AK Model

6.1.1 Pareto Allocations

- Total output in the economy is given by

$$Y_t = F(K_t, L_t) = AK_t,$$

where $A > 0$ is an exogenous parameter. In intensive form,

$$y_t = f(k_t) = Ak_t.$$

- The social planner's problem is the same as in the Ramsey model, except for the fact that output is linear in capital:

$$\max \sum_{t=0}^{\infty} u(c_t)$$

$$s.t. \quad c_t + k_{t+1} \leq f(k_t) + (1-\delta)k_t$$

- The Euler condition gives
$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta\,(1 + A - \delta)$$
Assuming CEIS, this reduces to
$$\frac{c_{t+1}}{c_t} = [\beta\,(1 + A - \delta)]^{\theta}$$
or
$$\ln c_{t+1} - \ln c_t = \theta(R - \rho)$$
where $R = A - \delta$ is the net social return on capital. That is, consumption growth is proportional to the difference between the real return on capital and the discount rate. Note that now the real return is a constant, rather than diminishing with capital accumulation.

- Note that the resource constraint can be rewritten as
$$c_t + k_{t+1} = (1 + A - \delta)k_t.$$
Since total resources (the RHS) are linear in k, an educated guess is that optimal consumption and investment are also linear in k. We thus propose
$$c_t = (1 - s)(1 + A - \delta)k_t$$
$$k_{t+1} = s(1 + A - \delta)k_t$$
where the coefficient s is to be determined and must satisfy $s \in (0, 1)$ for the solution to exist.

- It follows that
$$\frac{c_{t+1}}{c_t} = \frac{k_{t+1}}{k_t} = \frac{y_{t+1}}{y_t}$$

so that consumption, capital and income all grow at the same rate. To ensure perpetual growth, we thus need to impose

$$\beta\,(1 + A - \delta) > 1,$$

or equivalently $A - \delta > \rho$. If that condition were not satisfied, and instead $A - \delta < \rho$, then the economy would shrink at a constant rate towards zero.

- From the resource constraint we then have

$$\frac{c_t}{k_t} + \frac{k_{t+1}}{k_t} = (1 + A - \delta),$$

implying that the consumption-capital ratio is given by

$$\frac{c_t}{k_t} = (1 + A - \delta) - [\beta\,(1 + A - \delta)]^{\theta}$$

Using $c_t = (1 - s)(1 + A - \delta)k_t$ and solving for s we conclude that the optimal saving rate is

$$s = \beta^{\theta}\,(1 + A - \delta)^{\theta-1}\,.$$

Equivalently, $s = \beta^{\theta}(1 + R)^{\theta-1}$, where $R = A - \delta$ is the net social return on capital. Note that the saving rate is increasing (decreasing) in the real return if and only if the EIS is higher (lower) than unit, and $s = \beta$ for $\theta = 1$. Finally, to ensure $s \in (0, 1)$, we impose

$$\beta^{\theta}\,(1 + A - \delta)^{\theta-1} < 1.$$

This is automatically ensured when $\theta \leq 1$ and $\beta\,(1 + A - \delta) > 1$, as then $s = \beta^{\theta}\,(1 + A - \delta)^{\theta-1} \leq \beta < 1$. But when $\theta > 1$, this puts an upper bound on A. If A exceeded that upper bound, then the social planner could attain infinite utility, and the problem is not well-defined.

- We conclude that

Proposition 22 *Consider the social planner's problem with linear technology $f(k) = Ak$ and CEIS preferences. Suppose $(\beta, \theta, A, \delta)$ satisfy $\beta(1 + A - \delta) > 1 > \beta^{\theta}(1 + A - \delta)^{\theta - 1}$. Then, the economy exhibits a balanced growth path. Capital, output, and consumption all grow at a constant rate given by*

$$\frac{k_{t+1}}{k_t} = \frac{y_{t+1}}{y_t} = \frac{c_{t+1}}{c_t} = [\beta(1 + A - \delta)]^{\theta} > 1.$$

while the investment rate out of total resources is given by

$$s = \beta^{\theta}(1 + A - \delta)^{\theta - 1}.$$

The growth rate is increasing in the net return to capital, increasing in the elasticity of intertemporal substitution, and decreasing in the discount rate.

6.1.2 The Frictionless Competitive Economy

- Consider now how the social planner's allocation is decentralized in a competitive market economy.

- Suppose that the same technology that is available to the social planner is available to each single firm in the economy. Then, the equilibrium rental rate of capital and the equilibrium wage rate will be given simply

$$r = A \quad \text{and} \quad w = 0.$$

- The arbitrage condition between bonds and capital will imply that the interest rate is

$$R = r - \delta = A - \delta.$$

- Finally, the Euler condition for the household will give

$$\frac{c_{t+1}}{c_t} = \left[\beta\left(1+R\right)\right]^{\theta}.$$

- We conclude that the competitive market allocations coincide with the Pareto optimal plan. Note that this is true only because the private and the social return to capital coincide.

6.2 A Simple Model of Human Capital

6.2.1 Pareto Allocations

- Total output in the economy is given by

$$Y_t = F(K_t, H_t) = F(K_t, h_t L_t),$$

  where F is a neoclassical production function, K_t is aggregate capital in period t, h_t is human capital per worker, and $H_t = h_t L_t$ is effective labor.

- Note that, due to CRS, we can rewrite output per capita as

$$y_t = F(k_t, h_t) = F\left(\frac{k_t}{h_t}, 1\right)\frac{h_t}{k_t + h_t}[k_t + h_t] =$$

  or equivalently

$$y_t = F(k_t, h_t) = A\left(\kappa_t\right)[k_t + h_t],$$

  where $\kappa_t = k_t/h_t = K_t/H_t$ is the ratio of physical to human capital, $k_t + h_t$ measures total capital, and

$$A\left(\kappa\right) \equiv \frac{F\left(\kappa, 1\right)}{1+\kappa} \equiv \frac{f(\kappa)}{1+\kappa}$$

  represents the return to total capital.

- Total output can be used for consumption or investment in either type of capital, so that the resource constraint of the economy is given by

$$c_t + i_t^k + i_t^h \leq y_t.$$

The laws of motion for two types of capital are

$$\begin{aligned} k_{t+1} &= (1-\delta_k)k_t + i_t^k \\ h_{t+1} &= (1-\delta_h)h_t + i_t^h \end{aligned}$$

As long as neither i_t^k nor i_t^h are constrained to be positive, the resource constraint and the two laws of motion are equivalent to a single constraint, namely

$$c_t + k_{t+1} + h_{t+1} \leq F(k_t, h_t) + (1-\delta_k)k_t + (1-\delta_h)h_t$$

- The social planner's problem thus becomes

$$\max \sum_{t=0}^{\infty} u(c_t)$$

$$s.t. \quad c_t + k_{t+1} + h_{t+1} \leq F(k_t, h_t) + (1-\delta_k)k_t + (1-\delta_h)h_t$$

- Since there are two types of capital, we have two Euler conditions, one for each type of capital. The one for physical capital is

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta \left[1 + F_k(k_{t+1}, h_{t+1}) - \delta_k\right],$$

while the one for human capital is

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta \left[1 + F_h(k_{t+1}, h_{t+1}) - \delta_h\right].$$

- Combining the two Euler condition, we infer

$$F_k(k_{t+1}, h_{t+1}) - \delta_k = F_h(k_{t+1}, h_{t+1}) - \delta_h.$$

Remember that F is CRS, implying that both F_k and F_h are functions of the ratio $\kappa_{t+1} = k_{t+1}/h_{t+1}$. In particular, F_k is decreasing in κ and F_h is increasing in κ. The above condition therefore determines a unique optimal ratio κ^* such that

$$\frac{k_{t+1}}{h_{t+1}} = \kappa_{t+1} = \kappa^*$$

for all $t \geq 0$. For example, if $F(k,h) = k^{\alpha}h^{1-\alpha}$ and $\delta_k = \delta_h$, then $\frac{F_k}{F_h} = \frac{\alpha}{1-\alpha}\frac{h}{k}$ and therefore $\kappa^* = \frac{\alpha}{1-\alpha}$. More generally, the optimal physical-to-human capital ratio is increasing in the relative productivity of physical capital and decreasing in the relative depreciation rate of physical capital.

- Multiplying the Euler condition for k with $k_{t+1}/(k_{t+1} + h_{t+1})$ and the one for h with $h_{t+1}/(k_{t+1} + h_{t+1})$, and summing the two together, we infer the following "weighted" Euler condition:

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta \left\{ 1 + \frac{k_{t+1}[F_k(k_{t+1}, h_{t+1}) - \delta_k] + h_{t+1}[F_h(k_{t+1}, h_{t+1}) - \delta_h]}{k_{t+1} + h_{t+1}} \right\}$$

By CRS, we have

$$F_k(k_{t+1}, h_{t+1})k_{t+1} + F_h(k_{t+1}, h_{t+1})h_{t+1} = F(k_{t+1}, h_{t+1}) = A(\kappa_{t+1})[k_{t+1} + h_{t+1}]$$

It follows that

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta \left\{ 1 + A(\kappa_{t+1}) - \frac{\delta_k k_{t+1} + \delta_h h_{t+1}}{k_{t+1} + h_{t+1}} \right\}$$

Using the fact that $\kappa_{t+1} = \kappa^*$, and letting

$$A^* \equiv A(\kappa^*) \equiv \frac{F(\kappa^*, 1)}{1 + \kappa^*}$$

represent the "effective" return to total capital and

$$\delta^* \equiv \frac{\kappa^*}{1+\kappa^*}\delta_k + \frac{1}{1+\kappa^*}\delta_h$$

the "effective" depreciation rate of total capital, we conclude that the "weighted" Euler condition evaluated at the optimal physical-to-human capital ratio is

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta\left[1 + A^* - \delta^*\right].$$

- Assuming CEIS, this reduces to

$$\frac{c_{t+1}}{c_t} = \left[\beta\left(1 + A^* - \delta^*\right)\right]^\theta$$

or

$$\ln c_{t+1} - \ln c_t = \theta(A^* - \delta^* - \rho)$$

where$A^* - \delta^*$ is the net social return to total savings. Note that the return is constant along the balanced growth path, but it is not exogenous. It instead depends on the ratio of physical to human capital. The latter is determined optimally so as to maximize the net return on total savings. To see this, note that $k_{t+1}/h_{t+1} = \kappa^*$ indeed solves the following problem

$$\max \;\; F(k_{t+1}, h_{t+1}) - \delta_k k_{t+1} - \delta_h h_{t+1}$$

$$s.t. \quad k_{t+1} + h_{t+1} = constant$$

- Given the optimal ratio κ^*, the resource constraint can be rewritten as

$$c_t + [k_{t+1} + h_{t+1}] = (1 + A^* - \delta^*)[k_t + h_t].$$

Like in the simple Ak model, an educated guess is then that optimal consumption and total investment are also linear in total capital:

$$c_t = (1-s)(1+A^* - \delta^*)[k_t + h_t],$$
$$k_{t+1} + h_{t+1} = s(1+A^* - \delta^*)[k_t + h_t].$$

The optimal saving rate s is given by

$$s = \beta^\theta \left(1 + A^* - \delta^*\right)^{\theta - 1}.$$

- We conclude that

Proposition 23 *Consider the social planner's problem with CRS technology $F(k,h)$ over physical and human capital and CEIS preferences. Let κ^* be the ratio k/h that maximizes $F(k,h) - \delta_k k - \delta_h h$ for any given $k+h$, and let*

$$A^* \equiv \frac{F(\kappa^*, 1)}{1+\kappa^*} \quad \text{and} \quad \delta^* \equiv \frac{\kappa^*}{1+\kappa^*}\delta_k + \frac{1}{1+\kappa^*}\delta_h$$

Suppose $(\beta, \theta, F, \delta_k, \delta_h)$ satisfy $\beta\left(1 + A^* - \delta^*\right) > 1 > \beta^\theta \left(1 + A^* - \delta^*\right)^{\theta-1}$. Then, the economy exhibits a balanced growth path. Physical capital, human capital, output, and consumption all grow at a constant rate given by

$$\frac{y_{t+1}}{y_t} = \frac{c_{t+1}}{c_t} = \left[\beta\left(1 + A^* - \delta^*\right)\right]^\theta > 1.$$

while the investment rate out of total resources is given by $s = \beta^\theta \left(1 + A^* - \delta^*\right)^{\theta-1}$ and the optimal ratio of physical to human capital is $k_{t+1}/h_{t+1} = \kappa^*$. The growth rate is increasing in the productivity of either type of capital, increasing in the elasticity of intertemporal substitution, and decreasing in the discount rate.

6.2.2 Market Allocations

- Consider now how the social planner's allocation is decentralized in a competitive market economy.

- The household budget is given by

$$c_t + i_t^k + i_t^h + b_{t+1} \leq y_t + (1 + R_t) b_t.$$

  and the laws of motion for the two types of capital are

$$\begin{aligned} k_{t+1} &= (1 - \delta_k) k_t + i_t^k \\ h_{t+1} &= (1 - \delta_h) h_t + i_t^h \end{aligned}$$

  We can thus write the household budget as

$$c_t + k_{t+1} + h_{t+1} + b_{t+1} \leq (1 + r_t - \delta_k) k_t + (1 + w_t - \delta_h) h_t + (1 + R_t) b_t.$$

  Note that $r_t - \delta_k$ and $w_t - \delta_h$ represent the market returns to physical and human capital, respectively.

- Suppose that the same technology that is available to the social planner is available to each single firm in the economy. Then, the equilibrium rental rate of capital and the equilibrium wage rate will be given simply

$$r_t = F_k(\kappa_t, 1) \quad \text{and} \quad w_t = F_h(\kappa_t, 1),$$

  where $\kappa_t = k_t / h_t$.

- The arbitrage condition between bonds and the two types of capital imply that

$$R_t = r_t - \delta_k = w_t - \delta_h.$$

Combining the above with the firms' FOC, we infer

$$\frac{F_k(\kappa_t, 1)}{F_h(\kappa_t, 1)} = \frac{r_t}{w_t} = \frac{\delta_h}{\delta_k}$$

and therefore $\kappa_t = \kappa^*$, like in the Pareto optimum. It follows then that

$$R_t = A^* - \delta^*,$$

where A^* and δ^* are defined as above.

- Finally, the Euler condition for the household is

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta \left(1 + R_t\right).$$

Using $R_t = A^* - \delta^*$, we conclude

$$\frac{y_{t+1}}{y_t} = \frac{c_{t+1}}{c_t} = \left[\beta \left(1 + A^* - \delta^*\right)\right]^{\theta}$$

- Hence, the competitive market allocations once again coincide with the Pareto optimal plan. Note that again this is true only because the private and the social return to *each* type of capital coincide.

6.3 Learning by Education (Ozawa and Lucas)

see problem set

notes to be completed

6.4 Learning by Doing and Knowledge Spillovers (Arrow and Romer)

6.4.1 Market Allocations

- Output for firm m is given by

$$Y_t^m = F\left(K_t^m, h_t L_t^m\right)$$

where h_t represents the aggregate level of human capital or knowledge. h_t is endogenously determined in the economy (we will specify in a moment how), but it is taken as exogenous from either firms or households.

- Firm profits are given by

$$\Pi_t^m = F\left(K_t^m, h_t L_t^m\right) - r_t K_t^m - w_t L_t^m$$

The FOCs give

$$\begin{aligned} r_t &= F_K\left(K_t^m, h_t L_t^m\right) \\ w_t &= F_L\left(K_t^m, h_t L_t^m\right) h_t \end{aligned}$$

Using the market clearing conditions for physical capital and labor, we infer $K_t^m/L_t^m = k_t$, where k_t is the aggregate capital labor ratio in the economy. We conclude that, given k_t and h_t, market prices are given by

$$\begin{aligned} r_t &= F_K(k_t, h_t) = f'(\kappa_t) \\ w_t &= F_L(k_t, h_t) h_t = [f(\kappa_t) - f'(\kappa_t)\kappa_t] h_t \end{aligned}$$

where $f(\kappa) \equiv F(\kappa, 1)$ is the production function in intensive form and $\kappa_t = k_t/h_t$.

- Households, like firms, take w_t, r_t and h_t as exogenously given. The representative household maximizes utility subject to the budget constraint

$$c_t + k_{t+1} + b_{t+1} \leq w_t + (1 + r_t - \delta)k_t + (1 + R_t)b_t.$$

Arbitrage between bonds and capital imply $R_t = r_t - \delta$ and the Euler condition gives

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta\,(1 + R_t) = \beta(1 + r_t - \delta).$$

- To close the model, we need to specify how h_t is determined. Following Arrow and Romer, we assume that knowledge accumulation is the unintentional by-product of learning-by-doing in production. We thus let the level of knowledge to be proportional to either the level of output, or the level of capital:

$$h_t = \eta k_t,$$

for some constant $\eta > 0$.

- It follows that the ratio $k_t/h_t = \kappa_t$ is pinned down by $\kappa_t = 1/\eta$. Letting the constants A and ω be defined

$$A \equiv f'(1/\eta) \quad \text{and} \quad \omega \equiv f(1/\eta)\eta - f'(1/\eta),$$

we infer that equilibrium prices are given by

$$r_t = A \quad \text{and} \quad w_t = \omega k_t.$$

Substituting into the Euler condition gives

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta\,(1 + A - \delta)\,.$$

Finally, it is immediate that capital and output grow at the same rate as consumption. We conclude

Proposition 24 *Let* $A \equiv f'(1/\eta)$ *and* $\omega \equiv f(1/\eta)\eta - f'(1/\eta)$, *and suppose* $\beta(1 + A - \delta) > 1 > \beta^{\theta}(1 + A - \delta)^{\theta - 1}$. *Then, the market economy exhibits a balanced growth path. Physical capital, knowledge, output, wages, and consumption all grow at a constant rate given by*

$$\frac{y_{t+1}}{y_t} = \frac{c_{t+1}}{c_t} = [\beta(1 + A - \delta)]^{\theta} > 1.$$

The wage rate is given by $w_t = \omega k_t$, *while the investment rate out of total resources is given by* $s = \beta^{\theta}(1 + A - \delta)^{\theta - 1}$.

6.4.2 Pareto Allocations and Policy Implications

- Consider now the Pareto optimal allocations. The social planner recognizes that knowledge in the economy is proportional to physical capital and internalizes the effect of learning-by-doing. He thus understands that output is given by

$$y_t = F(k_t, h_t) = A^* k_t$$

  where $A^* \equiv f(1/\eta)\eta = A + \omega$ represents the *social* return on capital. It is therefore as if the social planner had access to a linear technology like in the simple Ak model, and therefore the Euler condition for the social planner is given by

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta(1 + A^* - \delta).$$

- Note that the social return to capital is higher than the private (market) return to capital:

$$A^* > A = r_t$$

  The difference is actually ω, the fraction of the social return on savings that is "wasted" as labor income.

Proposition 25 *Let* $A^* \equiv A+\omega \equiv f(1/\eta)\eta$ *and suppose* $\beta\left(1+A^*-\delta\right)>1>\beta^\theta\left(1+A^*-\delta\right)^{\theta-1}$. *Then, the Pareto optimal plan exhibits a balanced growth path. Physical capital, knowledge, output, wages, and consumption all grow at a constant rate given by*

$$\frac{y_{t+1}}{y_t}=\frac{c_{t+1}}{c_t}=\left[\beta\left(1+A^*-\delta\right)\right]^\theta>1.$$

Note that $A<A^*$, *and therefore the market growth rate is lower than the Pareto optimal one.*

- *Exercise:* Reconsider the market allocation and suppose the government intervenes by subsidizing either private savings or firm investment. Find, in each case, what is the subsidy that implements the optimal growth rate. Is this subsidy the optimal one, in the sense that it maximizes social welfare?

6.5 Government Services (Barro)

notes to be completed

Chapter 7

Endogenous Growth II: R&D and Technological Change

7.1 Expanding Product Variety (Romer)

- There are three production sectors in the economy: A final-good sector, an intermediate good sector, and an R&D sector.

- The final good sector is perfectly competitive and thus makes zero profits. Its output is used either for consumption or as input in each of the other two sector.

- The intermediate good sector is monopolistic. There is product differentiation. Each intermediate producer is a quasi-monopolist with respect to his own product and thus enjoys positive profits. To become an intermediate producer, however, you must first acquire a "blueprint" from the R&D sector. A "blueprint" is simply the technology or know-how for transforming final goods to differentiated intermediate inputs.

- The R&D sector is competitive. Researchers produce "blueprints", that is, the technology for producing an new variety of differentiated intermediate goods. Blueprints are protected by perpetual patents. Innovators auction their blueprints to a large number of potential buyers, thus absorbing all the profits of the intermediate good sector. But there is free entry in the R&D sector, which drive net profits in that sector to zero as well.

7.1.1 Technology

- The technology for final goods is given by a neoclassical production function of labor L and a composite factor Z:

$$Y_t = F(\mathcal{X}_t, L_t) = A(L_t)^{1-\alpha}(\mathcal{X}_t)^{\alpha}.$$

The composite factor is given by a CES aggregator of intermediate inputs:

$$\mathcal{X}_t = \left[\int_0^{N_t} (X_{t,j})^{\varepsilon} dj\right]^{1/\varepsilon},$$

where N_t denotes the number of different intermediate goods available in period t and $X_{t,j}$ denotes the quantity of intermediate input j employed in period t.

- In what follows, we will assume $\varepsilon = \alpha$, which implies

$$Y_t = A(L_t)^{1-\alpha} \int_0^{N_t} (X_{t,j})^{\alpha} dj$$

Note that $\varepsilon = \alpha$ means that the elasticity of substitution between intermediate inputs is 1 and therefore the marginal product of each intermediate input is independent of the quantity of other intermediate inputs:

$$\frac{\partial Y_t}{\partial X_{t,j}} = \alpha A \left(\frac{L_t}{X_{t,j}}\right)^{1-\alpha}.$$

More generally, intermediate inputs could be either complements or substitutes, in the sense that the marginal product of input j could depend either positively or negatively on X_t.

- We will interpret intermediate inputs as capital goods and therefore let aggregate "capital" be given by the aggregate quantity of intermediate inputs:

$$K_t = \int_0^{N_t} X_{t,j} dj.$$

- Finally, note that if $X_{t,j} = X$ for all j and t, then

$$Y_t = AL_t^{1-\alpha} N_t X^\alpha$$

and

$$K_t = N_t X,$$

implying

$$Y_t = A(N_t L_t)^{1-\alpha}(K_t)^\alpha$$

or, in intensive form, $y_t = AN_t^{1-\alpha} k_t^\alpha$. Therefore, to the extent that all intermediate inputs are used in the same quantity, the technology is linear in knowledge N and capital K. Therefore, if both N and K grow at a constant rate, as we will show to be the case in equilibrium, the economy will exhibit long run growth like in an Ak model.

7.1.2 Final Good Sector

- The final good sector is perfectly competitive. Firms are price takers.

- Final good firms solve

$$\max \; Y_t - w_t L_t - \int_0^{N_t} (p_{t,j} X_{t,j}) dj$$

where w_t is the wage rate and $p_{t,j}$ is the price of intermediate good j.

- Profits in the final good sector are zero, due to CRS, and the demands for each input are given by the FOCs

$$w_t = \frac{\partial Y_t}{\partial L_t} = (1-\alpha)\frac{Y_t}{L_t}$$

and

$$p_{t,j} = \frac{\partial Y_t}{\partial X_{t,j}} = \alpha A \left(\frac{L_t}{X_{t,j}}\right)^{1-\alpha}$$

for all $j \in [0, N_t]$.

7.1.3 Intermediate Good Sector

- The intermediate good sector is monopolistic. Firms understand that they face a downward sloping demand for their output.

- The producer of intermediate good j solves

$$\max\ \Pi_{t,j} = p_{t,j}X_{t,j} - \kappa(X_{t,j})$$

subject to the demand curve

$$X_{t,j} = L_t \left(\frac{\alpha A}{p_{t,j}}\right)^{\frac{1}{1-\alpha}},$$

where $\kappa(X)$ represents the cost of producing X in terms of final-good units.

- We will let the cost function be linear:

$$\kappa(X) = X.$$

The implicit assumption behind this linear specification is that technology of producing intermediate goods is identical to the technology of producing final goods. Equivalently, you can think of intermediate good producers buying final goods and transforming them to intermediate inputs. What gives them the know-how for this transformation is precisely the blueprint they hold.

- The FOCs give

$$p_{t,j} = p \equiv \frac{1}{\alpha} > 1$$

for the optimal price, and

$$X_{t,j} = xL$$

for the optimal supply, where

$$x \equiv A^{\frac{1}{1-\alpha}} \alpha^{\frac{2}{1-\alpha}}.$$

- Note that the price is higher than the marginal cost ($p = 1/\alpha > \kappa'(X) = 1$), the gap representing the mark-up that intermediate-good firms charge to their customers (the final good firms). Because there are no distortions in the economy other than monopolistic competition in the intermediate-good sector, the price that final-good firms are willing to pay represents the social product of that intermediate input and the cost that intermediate-good firms face represents the social cost of that intermediate input. Therefore, the mark-up $1/\alpha$ gives the gap between the social product and the social cost of intermediate inputs. (*Hint:* The social planner would like to correct for this distortion. How?)

- The resulting maximal profits are

$$\Pi_{t,j} = \pi L$$

where

$$\pi \equiv (p-1)x = \tfrac{1-\alpha}{\alpha} x = \tfrac{1-\alpha}{\alpha} A^{\frac{1}{1-\alpha}} \alpha^{\frac{2}{1-\alpha}}.$$

7.1.4 The Innovation Sector

- The present value of profits of intermediate good j from period t and on is given by

$$V_{t,j} = \sum_{\tau=t} \frac{q_\tau}{q_t} \Pi_{\tau,j}$$

or recursively

$$V_{t,j} = \Pi_{t,j} + \frac{V_{t+1,j}}{1 + R_{t+1}}$$

- We know that profits are stationary and identical across all intermediate goods: $\Pi_{t,j} = \pi L$ for all t, j. As long as the economy follows a balanced growth path, we expect the interest rate to be stationary as well: $R_t = R$ for all t. It follows that the present value of profits is stationary and identical across all intermediate goods:

$$V_{t,j} = V = \frac{\pi L}{R}.$$

Equivalently, $RV = \pi L$, which has a simple interpretation: The opportunity cost of holding an asset which has value V and happens to be a "blueprint", instead of investing in bonds, is RV; the dividend that this asset pays in each period is πL; arbitrage then requires the dividend to equal the opportunity cost of the asset, namely $RV = \pi L$.

- New blueprints are also produced using the same technology as final goods. In effect, innovators buy final goods and transform them to blueprints at a rate $1/\eta$.

- Producing an amount ΔN of new blueprints costs $\eta \cdot \Delta N$, where $\eta > 0$ measures the cost of R&D in units of output. On the other hand, the value of these new blueprints is $V \cdot \Delta N$, where $V = \pi L/R$. Net profits for a research firm are thus given by

$$(V - \eta) \cdot \Delta N$$

Free entry in the sector of producing blueprints then implies

$$V = \eta.$$

7.1.5 Households

- Households solve

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$s.t. \quad c_t + a_{t+1} \leq w_t + (1 + R_t)a_t$$

- As usual, the Euler condition gives

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta(1 + R_{t+1}).$$

  And assuming CEIS, this reduces to

$$\frac{c_{t+1}}{c_t} = \left[\beta(1 + R_{t+1})\right]^{\theta}.$$

7.1.6 Resource Constraint

- Final goods are used either for consumption by households, or for production of intermediate goods in the intermediate sector, or for production of new blueprints in the innovation sector. The resource constraint of the economy is given by

$$C_t + K_t + \eta \cdot \Delta N_t = Y_t,$$

  where $C_t = c_t L$, $\Delta N_t = N_{t+1} - N_t$, and $K_t = \int_0^{N_t} X_{t,j} dj$.

7.1.7 General Equilibrium

- Combining the formula for the value of innovation with the free-entry condition, we infer $\pi L/R = V = \eta$. It follows that the equilibrium interest rate is

$$R = \frac{\pi L}{\eta} = \tfrac{1-\alpha}{\alpha} A^{\frac{1}{1-\alpha}} \alpha^{\frac{2}{1-\alpha}} L/\eta,$$

which verifies our earlier claim that the interest rate is stationary.

- The resource constraint reduces to

$$\frac{C_t}{N_t} + \eta \cdot \left[\frac{N_{t+1}}{N_t} - 1\right] + X = \frac{Y_t}{N_t} = AL^{1-\alpha} X^{\alpha},$$

where $X = xL = K_t/N_t$. It follows that C_t/N_t is constant along the balanced growth path, and therefore $C_t, N_t, K_t,$ and Y_t all grow at the same rate, γ.

- Combining the Euler condition with the equilibrium condition for the real interest rate, we conclude that the equilibrium growth rate is given by

$$1 + \gamma = \beta^{\theta} \left[1 + R\right]^{\theta} = \beta^{\theta} \left[1 + \tfrac{1-\alpha}{\alpha} A^{\frac{1}{1-\alpha}} \alpha^{\frac{2}{1-\alpha}} L/\eta\right]^{\theta}$$

- Note that the equilibrium growth rate of the economy decreases with η, the cost of expanding product variety or producing new "knowledge".

- The growth rate is also increasing in L or any other factor that increases the "scale" of the economy and thereby raises the profits of intermediate inputs and the demand for innovation. This is the (in)famous "scale effect" that is present in many models of endogenous technological change. Discuss....

7.1.8 Pareto Allocations and Policy Implications

- Consider now the problem of the social planner. Obviously, due to symmetry in production, the social planner will choose the same quantity of intermediate goods for all varieties: $X_{t,j} = X_t = x_t L$ for all j. Using this, we can write the problem of the social planner simply as maximizing utility,

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t),$$

subject to the resource constraint

$$C_t + N_t \cdot X_t + \eta \cdot (N_{t+1} - N_t) = Y_t = AL^{1-\alpha} N_t X_t^{\alpha},$$

where $C_t = c_t L$.

- The FOC with respect to X_t gives

$$X_t = x^* L,$$

where

$$x^* = A^{\frac{1}{1-\alpha}} \alpha^{\frac{1}{1-\alpha}}$$

represents the optimal level of production of intermediate inputs.

- The Euler condition, on the other hand, gives the optimal growth rate as

$$1 + \gamma^* = \beta^\theta \left[1 + R^*\right]^\theta = \beta^\theta \left[1 + \tfrac{1-\alpha}{\alpha} A^{\frac{1}{1-\alpha}} \alpha^{\frac{1}{1-\alpha}} L/\eta\right]^\theta,$$

where

$$R^* = \tfrac{1-\alpha}{\alpha} A^{\frac{1}{1-\alpha}} \alpha^{\frac{1}{1-\alpha}} L/\eta$$

represents that social return on savings.

- Note that

$$x^* = x \cdot \alpha^{-\frac{1}{1-\alpha}} > x$$

That is, the optimal level of production of intermediate goods is higher in the Pareto optimum than in the market equilibrium. This reflects simply the fact that, due to the monopolistic distortion, production of intermediate goods is inefficiently low in the market equilibrium. Naturally, the gap x^*/x is an increasing function of the mark-up $1/\alpha$.

- Similarly,

$$R^* = R \cdot \alpha^{-\frac{1}{1-\alpha}} > R.$$

That is, the market return on savings (R) falls short of the social return on savings (R^*), the gap again arising because of the monopolistic distortion in the intermediate good sector. It follows that

$$1 + \gamma^* > 1 + \gamma,$$

so that the equilibrium growth rate is too low as compared to the Pareto optimal growth rate.

- *Policy exercise:* Consider three different types of government intervention: A subsidy on the production of intermediate inputs; an subsidy on the production of final goods (or the demand for intermediate inputs); and a subsidy on R&D. Which of these policies could achieve an increase in the market return and the equilibrium growth rate? Which of these policies could achieve an increases in the output of the intermediate good sector? Which one, or which combination of these policies can implement the first best allocations as a market equilibrium?

7.1.9 Introducing Skilled Labor and Human Capital

notes to be completed

7.1.10 International Trade, Technology Diffusion, and other implications

notes to be completed

7.2 Increasing Product Quality (Aghion-Howitt)

topic covered in recitation

notes to be completed