Problem Set #2 Solutions

Course 14.451 – Macro I

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Distributed: February 23, 2005

Due: Wednesday, March 2, 2005 [in class]

1. **Dynamic Programming – Analytic Solution**

Assume the following problem for the social planner:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} U_0 = \sum_{t=0}^{\infty} b^t U(c_t)$$
$$s.t. \quad c_t + k_{t+1} \leq f(k_t) \;\; \forall t \geq 0$$
$$c_t \geq 0, \; k_{t+1} \geq 0 \;\; \forall t \geq 0$$
$$k_0 > 0 \text{ given}$$

where U_0 is the lifetime utility of the representative agent, k_t is physical capital per unit of labor at time t, and c_t is consumption per unit of labor at time t. Assume that the labor supply of the agent is simply fixed at 1, and assume the following functional forms:

$$U(c) = \ln c$$
$$f(k) = A(1-t)k^a$$

where $a > 0$, and A is some constant greater than zero that captures technology in the economy. And finally, you should think of t as some government tax on output. And as most governments do in our world, this one throws the tax revenues in the ocean. ☺

(a) Re-express the above problem in the form of a dynamic programming problem. (i.e. Write out the Bellman Equation)

The above problem can be re-expressed as follows:

$$V(k_t) = \max_{c_t, k_{t+1}} \{\ln c_t + bV(k_{t+1})\}$$
$$s.t. \quad c_t + k_{t+1} = k_t^a$$

Notice that I've ignored the non-negativity constraints. By our Inada conditions, we know these will never bind. Dropping the time subscripts, and plugging in for the constraint, we have the following:

$$V(k) = \max_{0 \leq k' \leq (1-t)Ak^a} \left\{\ln\left(A(1-t)k^a - k'\right) + bV(k')\right\}$$

(b) Now, using a guess of $V(k) = E + F\ln k$ for your value function in part (a), solve for the optimal policy rules for consumption and capital.

The FOC is:

$$-\frac{1}{A(1-t)k^a - k'} + \frac{bF}{k'} = 0$$

Solving this, we have our policy function for capital:

$$k' = \frac{bF}{1+bF}(1-t)Ak^a$$

Using our budget constraint, we find the policy rule for consumption:

$$c = \frac{1}{1+bF}(1-t)Ak^a$$

(c) Plug your policy rules from part (b) into your original dynamic programming problem from part (a) to solve for the constants E and F. (This will take a bit of math on your part).

Our Bellman equation can be written as:

$$V(k) = \max\{\ln c + bV(k')\}$$

Plugging in with our guess $V(k) = E + F\ln k$ and our optimal policy rules $c(k)$ and $k'(k)$ from part (b), we have the following:

$$E + F\ln k = \ln(c(k)) + bE + bF\ln k'(k)$$
$$E + F\ln k = \ln\left(\frac{1}{1+bF}(1-t)Ak^a\right) + bE + bF\ln\left[\frac{bF}{1+bF}(1-t)Ak^a\right]$$

Okay, now here comes the fun part. With a little a little algebra, we can reduce this down to the following expression:

$$(1-b)E + F\ln k = -\ln(1+bF) + (1+bF)\ln((1-t)A) + bF\ln\left[\frac{bF}{1+bF}\right] + a(1+bF)\ln k$$

Notice that we have a $\ln k$ on each side of the expression. The only possible way for both these to drop out is if their coefficients are the same. i.e.

$$F = a(1 - bF)$$
$$F = \frac{a}{1-ab}$$

Now, plugging this into our expression, we have:

$$(1-b)E = -\ln\left(1 + b\left(\frac{a}{1-ab}\right)\right) + \left(1 + b\left(\frac{a}{1-ab}\right)\right)\ln\left((1-t)A\right) + b\left(\frac{a}{1-ab}\right)\ln\left[\frac{b\left(\frac{a}{1-ab}\right)}{1+b\left(\frac{a}{1-ab}\right)}\right]$$

Hopefully you can now see that we can solve for E. Doing this, and with a bit more manipulation, we have the following:

$$E = \left[\left(\frac{1}{1-ab}\right)\ln\left(A(1-t)\right) + \left(\frac{ab}{1-ab}\right)\ln ab + \ln(1-ab)\right](1-b)^{-1}$$

(d) What is the fraction of disposable income that the agent saves each period? How does it depend on b, and what is the intuition for this?

Plugging $F = \frac{a}{1-ab}$ into $k' = \frac{bF}{1+bF}(1-t)Ak^{a}$, we see that:

$$k' = ab(1-t)Ak^{a}$$

Thus, our savings rate is ab. When b is higher, the agents save more. The intuition is clear: Higher b implies the agent is more patient and values the future more. Thus, the agent saves more in order to consume more in the future.

(e) How do higher taxes affect the agent's happiness? What does better technology do for happiness? (1-2 sentences only… I just want to make sure you check that your answer makes intuitive sense before moving on).

Looking at our value function now that we've solved for E and F, it should be clear that higher taxes reduce the value of future utility for the agent for any given k, and more technology improves it.

2. **Dynamic Programming – Numerical Solution**

Write a program in MATLAB to solve the Dynamic Programming problem from part 1A using numerical iteration as I showed you in recitation last week. If you would like your solutions to match up closely to mine, feel free to use the following guidelines:

(i) Use a state vector of 50 possible states.
(ii) Center your state vector around the steady state of the economy using values in a range 10% above and below the steady state.
(iii) Stop the iteration when the absolute difference between all points of your old guess and new guess at the value function are less than .01

Finally, assume the following conditions:

$$A = 1$$
$$a = 0.35$$
$$b = 0.9$$
$$t = 0.3$$

(a) Using your numerical program, plot your value function $V(k)$ and policy functions $c(k)$ and $k'(k)$. Submit these graphs along with your MATLAB code.

(b) Now, again using MATLAB, plot your analytical solutions for the value function and policy functions from Question #1. Do your answers match up?

FOR SOLUTION: Download the MATLAB programs I've placed online. You need to put all the files in the same folder on your computer. Then, execute the "growth.m" file to see the solution.