

Problem Set #3 Solutions

Course 14.451 – Macro I

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1. Fiscal Policy in the Ramsey Model

Consider the standard Ramsey model of infinite-horizon households with the following set of preferences.

$$U = \int_0^{\infty} u[c(t)]e^{-\rho t} dt$$
$$u(c) = \frac{c^{1-\frac{1}{\rho}}}{1-\frac{1}{\rho}}$$

The population growth rate is n , there is the typical neoclassical production function, and technology grows at rate $x = 0$. The government now purchases goods and services of quantity G , imposes lump-sum taxes in the amount T , and has an outstanding quantity B of government bonds. The quantities G, T , and B -- which can vary over time -- are all measured in units of goods, and B starts at a given value, $B(0)$. Bonds are of infinitesimal maturity, pay the interest rate r , and are viewed by individual households as perfect substitutes for claims on capital or internal loans. (Assume that the government never defaults on its debts.) The government may provide public services that relate to the path of G , but the path of G is held fixed in this problem. Finally, for this question, please make sure you derive and express all of your solutions in continuous time.

- (a) What is the government's budget constraint? [Hint: You should write this in the form of a law of motion for B , and then write out the expression for \dot{b} , where $b = B/L$].**

For each period of time, it must be the case that what the government spends on purchases, G_t , and paying off its existing debt, $(1+r)B_t$, must equal the revenue it takes in from taxes, T_t , and the amount of new bonds it issues, B_{t+1} . Thus, we have:

$$G_t + (1+r)B_t = T_t + B_{t+1}$$

This can be re-expressed as:

$$B_{t+1} - B_t = G_t - T_t + rB_t$$

And, in continuous time it would be:

$$\dot{B} = G - T + rB$$

Now letting $g = G/L$, $t = T/L$, and $b = B/L$, we have:

$$\dot{b} = g - t + (r - n)b$$

- (b) What is the representative household's budget constraint? [Hint: Remember that the households take wages, w , and the interest rate, r , as given, where r is net of depreciation, such that $r = f'(k) - d$. Also make sure to express your constraint in per capita terms.]**

Again, I find it helpful to first think about the problem in discrete time. So, the individual has income $wL_t + (1+r)K_t + (1+r)B_t - T_t$. From this he/she either consumes, buys more bonds, or purchases more capital: $C_t + B_{t+1} + K_{t+1}$.

So, we have:

$$C_t + B_{t+1} + K_{t+1} = wL_t + (1+r)K_t + (1+r)B_t - T_t$$

Re-expressing this, we have:

$$(B_{t+1} - B_t) + (K_{t+1} - K_t) = wL_t + rK_t + rB_t - T_t - C_t$$

And, in continuous time, this would be:

$$\dot{B} + \dot{K} = wL + rK + rB - T - C$$

Finally, in per-capita, it would be:

$$\dot{b} + \dot{k} = w + (r - n)(b + k) - t - c$$

- (c) Combine the government's per capita budget constraint from part (a) with the individual's budget constraint in part (b) to find the law of motion for k . [Hint: Use the fact that $w + (r + d)k = f(k)$ in the competitive economy to substitute out w, r].**

Plugging our equation for \dot{b} , given by the government's budget constraint in part (a), into our consumer's budget constraint gives us the following:

$$\dot{k} = w + (r - n)k - c - g$$

Then, using $w + rk = f(k)$, we have:

$$\dot{k} = f(k) - (n + d)k - c - g$$

This is the resource constraint of our economy.

(i) How does additional government spending change the growth of capital and its steady state value in the economy? Why?

Higher government spending clearly reduces the growth of capital and its eventual steady state level, *holding c constant*, by removing sources from the economy and crowding out private investment. However, the answer is a bit more complicated because in actuality consumption will drop because of the increase in g , and if the drop in consumption is one-for-one, as will be the case in steady state, then there is no effect.

Because the drop in consumption is one-for-one in the steady state, additional government spending does not affect the steady state value of capital. This is seen explicitly in part (d), by setting $\dot{c} = 0$, and seeing that k^* is given by $f'(k^*) = d + r$, which is what we usually find. We get this result because the government's spending does not distort the return to capital, and hence, we will have the same steady state level of capital.

(ii) Does it matter whether the government uses taxes or bonds to finance its expenditures?

No. This model exhibits Ricardian Equivalence where the government's choice of bonds or taxes has no effect on the economy's equilibrium. All that matters is its total expenditure.

(d) Re-express the household's budget constraint from part (b) in terms of total per capita assets, a , where $a = b + k$, and solve the household's optimization problem to derive the growth rate of consumption in this economy.

Re-expressing the household's budget constraint, we have:

$$\dot{a} = w + (r - n)a - t - c$$

The Hamiltonian can thus be expressed as:

$$H(a, c, \mathbf{m}, t) = u(c)e^{(n-r)t} + \mathbf{m}(t)[w + (r - n)a - c - t]$$

The FOCs are simply:

$$\frac{\partial H}{\partial c} = u'(c)e^{(n-r)t} - \mathbf{m}(t) = 0$$

$$\frac{\partial H}{\partial a} = (r - n)\mathbf{m}(t) = -\dot{\mathbf{m}}$$

Manipulating these two FOCs and plugging in for our utility function, we find the usual Ramsey equation for consumption growth.

$$\frac{\dot{c}}{c} = q(r - r)$$

Or, equivalently:

$$\frac{\dot{c}}{c} = q(f'(k) - d - r)$$

Moreover, if we wanted to be more explicit, we could have written out our non-Ponzi condition as:

$$\lim_{t \rightarrow \infty} a(t)e^{-\tau(t)t} \geq 0$$

Where,

$$\bar{r}(t) = \int_0^t [r(s) - n] ds$$

And, this would give us a transversality condition of:

$$\lim_{t \rightarrow \infty} m(t)a(t) = 0$$

(i) Why can we re-write and solve the problem using just total assets?

We can rewrite the problem in terms of total assets because individuals are indifferent between holding bonds and capital since they both provide exactly the same return.

(ii) How does the government affect the growth rate of consumption?

At face value, the government has no effect on the growth rate of consumption in this economy. However, if the government's expenditures aren't perfectly offset by a reduction in consumption outside of the steady state, then the interest rate of the economy will be higher along the transition, and this would increase the growth rate of consumption.

(e) Using your answers above, briefly describe (in words) how differences in $B(0)$ or in the path of B and T affect the transitional dynamics and steady-state values of the variables c , k , y , and r ? What about the effect of G ?

This model exhibits Ricardian Equivalence where the government's choice of bonds or taxes has no effect on the economy's equilibrium. Neither does the initial level of bonds, $B(0)$. All that matters is its total expenditure, G , where more government expenditures lowers the steady state level of consumption.

2. Government and Growth in the Ramsey Model

Consider the household-production version of the Ramsey model. The government taxes output at the rate τ_Y , taxes labor at the rate τ_L (a lump-sum tax), provides per capita lump-sum transfers in the amount v , and purchases goods and services in the per capita amount g . The production function is Cobb-Douglas, (i.e. $Y = AK^a L^{1-a}$) and there is no technological progress. Thus, households maximize utility given by:

$$U = \int_0^{\infty} u(c(t))e^{-rt} dt$$

$$u(c) = \frac{c^{1-\frac{1}{q}}}{1-\frac{1}{q}}$$

subject to the budget constraint,

$$\dot{k} = (1-\tau_Y)AK^a - \tau_L - c - (n+d)k + v$$

where $k(0)$ is given. Suppose that the government uses its goods and services to blow up Pacific islands, actions that neither provide utility nor enhance productivity. All of the government's remaining tax revenues are remitted to households as lump-sum transfers.

(a) What is the government's budget constraint (in per capita terms)?

The government's budget constraint is:

$$G = \tau_Y AK^a L^{1-a} + \tau_L L - vL$$

In per capita terms, we have:

$$g = \tau_Y AK^a + \tau_L - v$$

(b) What are the household's first-order optimization conditions, assuming that the representative household takes τ_Y, τ_L, v and g as given?

The constrained optimization will be,

$$L = \int_0^{\infty} u(c(t))e^{(n-r)t} dt + \int_0^{\infty} \mathbf{m}(t) \left((1-\tau_Y)AK^a - \tau_L - c - (n+d)k + v - \dot{k} \right) dt + \mathbf{l}(t) e^{-r(t)}$$

The Hamiltonian can thus be written as:

$$u(c(t))e^{(n-r)t} + \mathbf{m}(t) \left[(1-\tau_Y)AK^a - \tau_L - c - (n+d)k + v \right]$$

The FOCs are:

$$\begin{aligned}\frac{\partial H}{\partial c} &= u'(c)e^{(n-r)t} - \mathbf{m}(t) = 0 \\ \frac{\partial H}{\partial a} &= (\mathbf{a}(1-t_Y)Ak^{a-1} - (n+d))\mathbf{m}(t) = -\dot{\mathbf{m}}(t) \\ \mathbf{I}e^{-r(t)} &= \mathbf{m}(t)\end{aligned}$$

- (c) Use the household's FOCs to solve for the growth rate of consumption. What is the steady state level of k in this economy?**

Taking the time derivative of our first FOC, we have:

$$\dot{c}u''(c)e^{(n-r)t} + (n-r)u'(c)e^{(n-r)t} = \dot{\mathbf{m}}(t)$$

Plugging this into our second FOC, we have:

$$\begin{aligned}(\mathbf{a}(1-t_Y)Ak^{a-1} - (n+d))u'(c)e^{(n-r)t} &= -\dot{c}u''(c)e^{(n-r)t} - (n-r)u'(c)e^{(n-r)t} \\ (\mathbf{a}(1-t_Y)Ak^{a-1} - (n+d))u'(c) &= -\dot{c}u''(c) - (n-r)u'(c) \\ (\mathbf{a}(1-t_Y)Ak^{a-1} - (r+d)) &= -\frac{cu''(c)\dot{c}}{u'(c)c}\end{aligned}$$

Plugging in for the derivatives of our utility function, we arrive at:

$$\frac{\dot{c}}{c} = \mathbf{q} (\mathbf{a}(1-t_Y)Ak^{a-1} - (r+d))$$

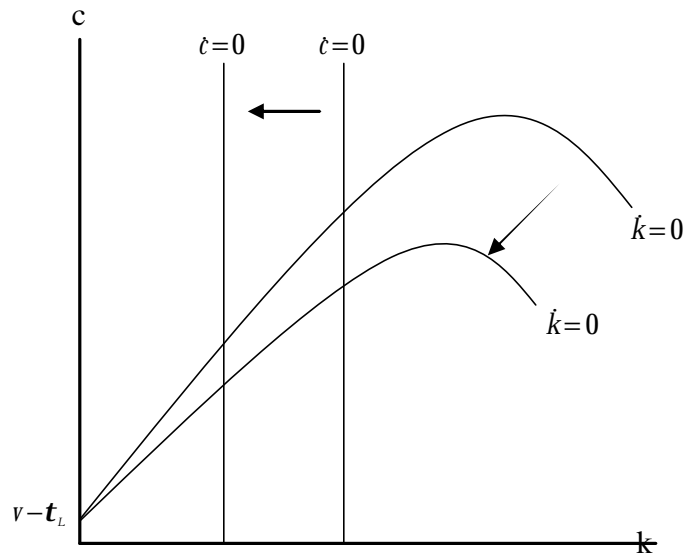
The steady state of the economy occurs when \dot{c} and \dot{k} both equal zero.

Setting $\dot{c} = 0$, we find that:

$$k^* = \left(\frac{\mathbf{a}(1-t_Y)A}{d+r} \right)^{\frac{1}{1-a}}$$

- (d) Use a phase-diagram in (k, c) space to show how the paths of k and c change when the government surprises people by permanently raising the values of t_Y and g (without changing t_L and v). What will be the immediate impact on consumption? What happens to the steady state values of k and c ?**

The phase diagram is the following:

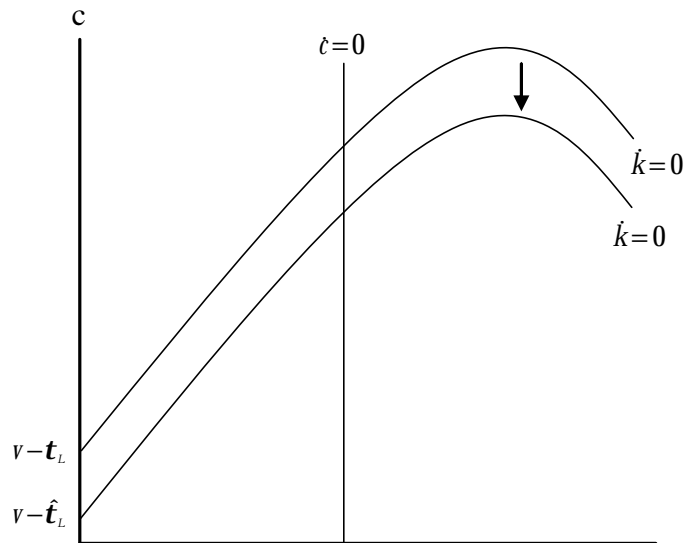


When t_y increases, the private marginal return to capital decreases so that the $\dot{c} = 0$ shifts inward. At the same time, a higher t_y also decreases capital accumulation, and therefore, the $\dot{k} = 0$ curve shifts downward. It is then clear that the steady state level of capital will be lower in the new equilibrium, as will consumption.

The immediate effect on consumption, however, is ambiguous. It could either jump up or down depending on whether the substitution or income effect dominates. It will all depend on whether the saddle path of the new equilibrium lies above or below the old steady state (not drawn here).

- (e) Redo part (d) for the case in which the government raises t_L and g (without changing t_y or v) What happens to the steady state value of k ? Explain in 1-2 why the effect of this policy change is different from what we saw in part (d).**

The phase diagram is the following:



In this case, the steady state value of k does not change. This is because the increase in the lump sum tax has no distortion on the marginal product of capital. Therefore, individuals will still choose to invest in the same amount of capital as before, and the only effect of the tax is to decrease consumption.

- (f) Redo part (d) for the case in which the government raises t_L and v (without changing t_Y and g). What happens to the steady-state value of k ? Explain why this policy change is different from those in parts (d) and (e).**

Using our government budget constraint from part (a), we clearly see that the changes in t_L and v must be equal but of different signs. Therefore, our phase diagram will not change. Basically, the government takes away with the one hand, but returns it with the other. The steady state of the economy does not change.

This differs from part (d) in that the tax and subsidy changes do not distort the marginal return to capital. This policy change is also different from part (e) in that there are no changes in the household's after tax income.