Problem Set #3 Solutions

Course 14.454 – Macro IV

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This problem set does not need to be turned in

Question #1: Stock Prices, Dividends and Bubbles

Assume you are in an economy where the stock price, p_t, is given by the standard arbitrage equation (1) and the process for dividends at time t, d_t, is given by equation (2) below:

$$p_t = \frac{1}{1+r} E_t[p_{t+1}] + \frac{1}{1+r} E_t[d_{t+1}] \tag{1}$$

$$d_t - \bar{d} = \rho\left(d_{t-1} - \bar{d}\right) + \varepsilon_t, \quad \varepsilon_t \text{ i.i.d. and } E_{t-1}[\varepsilon_t] = 0 \tag{2}$$

(a) Use iterated expectations to solve for the price, p_t, as a function of ONLY future expected dividends. What assumption do you implicitly need to do this?

First, by plugging in for $E_t[p_{t+1}]$, we have:

$$p_t = \frac{1}{1+r} E_t\left[p_{t+1}\right] + \frac{1}{1+r} E_t\left[d_{t+1}\right]$$

$$p_t = \frac{1}{1+r} E_t\left[\frac{1}{1+r} E_{t+1}\left[p_{t+2}\right] + \frac{1}{1+r} E_{t+1}\left[d_{t+2}\right]\right] + \frac{1}{1+r} E_t\left[d_{t+1}\right]$$

Using the law of iterated expectations, we have:

$$p_t = \left(\frac{1}{1+r}\right)^2 E_t\left[p_{t+2}\right] + \frac{1}{1+r} E_t\left[d_{t+1}\right] + \left(\frac{1}{1+r}\right)^2 E_t\left[d_{t+2}\right]$$

Continuing this type of process of plugging in for future expected prices, we get:

$$p_t = \sum_{i=1}^{T} \left(\frac{1}{1+r}\right)^i E_t\left[d_{t+i}\right] + \left(\frac{1}{1+r}\right)^T E_t\left[p_{t+T}\right]$$

Taking T to ∞ and assuming that $\lim_{T\to\infty} \left(\frac{1}{1+r}\right)^T E_t\left[p_{t+T}\right] = 0$, we have our solution:

$$p_t = \sum_{i=1}^{\infty} \left(\frac{1}{1+r}\right)^i E_t\left[d_{t+i}\right]$$

(b) Assume that $\rho < 1/(1+r)$. Use iterated expectations to find an expression for the expectation (as of time t) for dividends at time $t+i$, $E_t[d_{t+i}]$, that is a function of only $\bar{d}$, ρ, and d_t.

From equation (2), we know,

$$d_{t+i} - \bar{d} = \rho\left(d_{t+i-1} - \bar{d}\right) + \varepsilon_{t+i}$$

Taking expectations and rearranging, we have,

$$E_t\left[d_{t+i}\right] = \rho\left(E_t\left[d_{t+i-1}\right] - \bar{d}\right) + \bar{d}$$

Plugging in for $E_t\left[d_{t+i-1}\right]$, we find:

$$\begin{aligned}
E_t\left[d_{t+i}\right] &= \rho\left(E_t\left[d_{t+i-1}\right] - \bar{d}\right) + \bar{d} \\
E_t\left[d_{t+i}\right] &= \rho\left(E_t\left[\rho\left(E_t\left[d_{t+i-1}\right] - \bar{d}\right) + \bar{d}\right] - \bar{d}\right) + \bar{d} \\
E_t\left[d_{t+i}\right] &= \rho^2\left(E_t\left[d_{t+i-2}\right] - \bar{d}\right) + \bar{d}
\end{aligned}$$

Continuing the iteration,

$$E_t\left[d_{t+i}\right] = \rho^i\left(d_t - \bar{d}\right) + \bar{d}$$

(c) Use your answers from (a) and (b) to find an expression for p_t, as a function of $\bar{d}$, ρ, and d_t. Call this solution to the arbitrage equation the fundamental price, p_t^*.

Plugging $E_t\left[d_{t+i}\right] = \rho^i\left(d_t - \bar{d}\right) + \bar{d}$ into our answer from part (a), we have:

$$\begin{aligned}
p_t &= \sum_{i=1}^{\infty}\left[\left(\frac{\rho}{1+r}\right)^i\left(d_t - \bar{d}\right) + \left(\frac{1}{1+r}\right)^i \bar{d}\right] \\
p_t &= \left(\frac{1}{1+r}\right)\sum_{i=0}^{\infty}\left[\left(\frac{1}{1+r}\right)^i \rho^{i+1}\left(d_t - \bar{d}\right) + \left(\frac{1}{1+r}\right)^i \bar{d}\right] \\
p_t &= \left(\frac{\rho}{1+r}\right)\sum_{i=0}^{\infty}\left[\left(\frac{\rho}{1+r}\right)^i\right]\left(d_t - \bar{d}\right) + \frac{\bar{d}}{r} \\
p_t &= \left(\frac{\rho}{1+r-\rho}\right)\left(d_t - \bar{d}\right) + \frac{\bar{d}}{r}
\end{aligned}$$

(d) Now assume the price of the stock has a bubble component, b_t, where $b_t = (1+r)^t b_0$ and $b_0 > 0$. Prove that the price $p_t = p_t^* + b_t$ is also a solution to the arbitrage condition (1) and that our assumption from part (a) is no longer necessary.

Taking our initial arbitrage equation (1) and plugging in for $p_t = p_t^* + b_t$, we have:

$$p_t^* + b_t = \frac{1}{1+r} E_t\left[p_{t+1}^* + b_{t+1} \right] + \frac{1}{1+r} E_t\left[d_{t+1} \right]$$
$$p_t^* + b_t = \frac{1}{1+r} E_t\left[p_{t+1}^* \right] + \frac{1}{1+r} E_t\left[b_{t+1} \right] + \frac{1}{1+r} E_t\left[d_{t+1} \right]$$

But, since $E_t\left[b_{t+1}\right] = (1+r)b_t$, this equation reduces to:

$$p_t^* = \frac{1}{1+r} E_t\left[p_{t+1}^* \right] + \frac{1}{1+r} E_t\left[d_{t+1} \right]$$

Plugging in for our p_t^* and p_{t+1}^* using our answer from part (c), we have:

$$\left(\frac{\rho}{1+r-\rho}\right)\left(d_t - \bar{d}\right) + \frac{\bar{d}}{r} = \frac{1}{1+r} E_t\left[\left(\frac{\rho}{1+r-\rho}\right)\left(d_{t+1} - \bar{d}\right) + \frac{\bar{d}}{r}\right] + \frac{1}{1+r} E_t\left[d_{t+1}\right]$$

With a little algebra, we can reduce this expression in the following way:

$$(1+r)\left(\frac{\rho}{1+r-\rho}\right)\left(d_t - \bar{d}\right) + \bar{d} = E_t\left[\left(\frac{\rho}{1+r-\rho}\right)\left(d_{t+1} - \bar{d}\right)\right] + E_t\left[d_{t+1}\right]$$
$$(1+r)\left(\frac{\rho}{1+r-\rho}\right)\left(d_t\right) - r\left(\frac{\rho}{1+r-\rho}\right)\bar{d} + \bar{d} = \left(\frac{\rho}{1+r-\rho}\right) E_t\left[d_{t+1}\right] + E_t\left[d_{t+1}\right]$$
$$(1+r)\left(\frac{\rho}{1+r-\rho}\right)\left(d_t\right) - r\left(\frac{\rho}{1+r-\rho}\right)\bar{d} + \bar{d} = \left(\frac{1+r}{1+r-\rho}\right) E_t\left[d_{t+1}\right]$$
$$E_t\left[d_{t+1}\right] = \rho d_t - r\left(\frac{\rho}{1+r}\right)\bar{d} + \left(\frac{1+r-\rho}{1+r}\right)\bar{d}$$
$$E_t\left[d_{t+1}\right] = \rho\left(d_t - \bar{d}\right) + \bar{d}$$

This last expression is clearly true based our on answer from part (b). Thus, we have successfully shown that the arbitrage condition holds even in the presence of the bubble! Moreover, we were able to get this solution without the need for our earlier assumption from part (a) that:

$$\lim_{T\to\infty} \left(\frac{1}{1+r}\right)^T E_t\left[p_{t+T}\right] = 0$$

(e) Why are individuals willing to pay a higher price, p_t, for the stock than the fundamental price corresponding to the present value of the dividends, p_t^*?

The arbitrage condition can be satisfied despite the bubble, because individuals are willing to pay a higher price for the stock than the fundamental price (which represents the discounted future dividends) because they correctly anticipate that the price will continue to rise because of the bubble component of the price. The rising price yields capital gains that exactly offset the lower dividend to price ratio.

Question #2: Markups via Sticky Prices in the Goods Market

This problem is based heavily on sections 1, 4.1, and 4.3 of Rotemberg and Woodford's Handbook chapter. You may find it very helpful to read these sections of the chapter before proceeding with this problem set.

PART 1: -- Markups: What are they and why do they matter?

Consider a continuum 1 of imperfectly competitive firms. Let the marginal cost of each firm i be given by $Pc(y_i)$, where y_i is the quantity supplied, P is the general price level, and $c'(y) > 0$. Since $c'(y) > 0$, an increase in the quantity supplied by industry i, will be associated with an increase in marginal cost. Because of imperfect competition, each firm faces a downward sloping demand for their good, and can charge a price greater than marginal cost. The markup, μ, of a firm is simply given as its price over marginal cost. In this example, the markup by firm i is given by $\mu_i = P_i/[Pc(y_i)]$. So, if a firm wishes to increase its output and maintain a constant markup, it will need to raise its relative price (P_i/P). However, in a symmetric equilibrium, we know that it must be that:

$$\frac{1}{\mu} = c(Y) \qquad (3)$$

where the common level of output (and hence aggregate) will be given by Y, and μ is the common (and hence average) markup.

(a) Analyze equation (3). What is the intuition for why average markups and aggregate output move in opposite directions in this equation? If we want to propagate/amplify the business cycle, what type of movement in markups must our models generate?

To see where (3) comes from, simply note that in a symmetric equilibrium, it must be that $y_i = Y, P_i = P$ for all i. Thus,

$$\mu_i = \frac{P_i}{Pc(y_i)} \quad \text{implies} \quad \mu = \frac{P}{Pc(Y)} = \frac{1}{c(Y)}$$

The average output and aggregate output move in the opposite directions for the following reason: If all firms try to increase their output, it will not be possible for them all to increase their relative price as a means of maintaining their markup. Why? Because if all firms increased their relative price, then the general price of all firms would also rise, thus increasing marginal costs and bringing the markups back down. So, in general equilibrium, an increase in output by each firm can only be possible if the firms allow their markup to fall.

To propagate or amplify the business cycle, we will want to generate countercyclical movements in the markup. For instance, if markups fall during economic booms, then there will be an even greater increase in the aggregate level of output as seen in equation (3).

Again, assume monopolistic competition among a large number of suppliers of differentiated goods. Each firm i faces a downward-sloping demand curve for its product of the form:

$$Y_t^i = D\left(\frac{P_t^i}{P_t}\right)Y_t \qquad \textbf{(4)}$$

where P_t^i is the price of firm i at time t, P_t is an aggregate price index, Y_t is an index of aggregate sales at time t, and D is a decreasing function. Assume a constant elasticity of demand, $\varepsilon_D = -xD'(x)/D(x) > 1$, and assume each firm faces the same level of (nominal) marginal costs C_t in a given period. Neglecting fixed costs, profits of firm i at time t are given by:

$$\Pi_t^i = \left(P_t^i - C_t\right)D\left(\frac{P_t^i}{P_t}\right)Y_t \qquad \textbf{(5)}$$

(b) Assume completely flexible prices: Maximize the firm's profits to find its optimal markup, μ^*, as a function of the elasticity of demand. Is the markup an increasing or decreasing function of the elasticity? Explain the intuition of this result.

As usual, monopolistic firms maximize their profits by choose their price:

$$\max_{P_t^i}\left(P_t^i - C_t\right)D\left(\frac{P_t^i}{P_t}\right)Y_t$$

The FOC will be:

$$D\left(\frac{P_t^i}{P_t}\right)Y_t + \frac{\left(P_t^i - C_t\right)}{P_t}D'\left(\frac{P_t^i}{P_t}\right)Y_t = 0$$

$$D\left(\frac{P_t^i}{P_t}\right) = -\frac{\left(P_t^i - C_t\right)}{P_t^i}\frac{P_t^i}{P_t}D'\left(\frac{P_t^i}{P_t}\right)$$

$$\frac{P_t^i}{P_t^i - C_t} = -\frac{\frac{P_t^i}{P_t}D'\left(\frac{P_t^i}{P_t}\right)}{D\left(\frac{P_t^i}{P_t}\right)} = \varepsilon_D$$

$$\mu^* \equiv \frac{P_t^i}{C_t} = \frac{\varepsilon_D}{\varepsilon_D - 1}$$

Thus, we can clearly see that the optimal markup is a function only of the elasticity of the substitution, and it is a decreasing function of the elasticity. This makes sense: As customers become more elastic and more responsive to prices changes, the monopolistic firm will not be able to charge as high of a markup. If customers are infinitely elastic, a monopolistic firm will have no ability to charge a markup on its product.

PART 2 – Generating movement in Markups via Sticky Prices, Deriving the Math

Now, we are going to look at sticky price model that will generate movements in the markups charged by firms over the business cycle. Now assume that in each period t, a fraction $(1-\alpha)$ of firms are able to change their prices while the rest must keep their prices constant. A firm that changes its price, chooses it in order to maximize:

$$E_t \sum_{j=0}^{\infty} \alpha^j \beta^j \frac{\Pi_{t+j}}{P_{t+j}} \tag{6}$$

where β is the discount factor per period of time and α^j represents the probability that this prices will still apply *j* periods later. The profit function is the same as in equation (5).

(c) Denote X_t as the new price chosen at date t by any firms that choose then. Prove that the first order condition for their optimization problem is:

$$E_t \sum (\alpha\beta)^j \frac{Y_{t+j}}{P_{t+j}} D'\left(\frac{X_t}{P_{t+j}}\right) \frac{X_t}{P_{t+j}} \left[1 - \frac{1}{\varepsilon_D} - \frac{C_{t+j}}{X_t}\right] = 0 \tag{7}$$

The maximization problem of these firms can be written as:

$$\max_{X_t^i} E_t \sum_{j=0}^{\infty} \alpha^j \beta^j \frac{\left(X_t^i - C_{t+j}\right) D\left(\frac{X_t^i}{P_{t+j}}\right) Y_{t+j}}{P_{t+j}}$$

Taking the derivative with respect to X_t^i, we have:

$$E_t \sum_{j=0}^{\infty} (\alpha\beta)^j \left[\frac{D\left(\frac{X_t^i}{P_{t+j}}\right) Y_{t+j}}{P_{t+j}} + \frac{\left(X_t^i - C_{t+j}\right) D'\left(\frac{X_t^i}{P_{t+j}}\right) Y_{t+j}}{\left(P_{t+j}\right)^2} \right] = 0$$

$$E_t \sum_{j=0}^{\infty} (\alpha\beta)^j \left(\frac{Y_{t+j}}{P_{t+j}}\right) D'\left(\frac{X_t^i}{P_{t+j}}\right) \left[\frac{D\left(\frac{X_t^i}{P_{t+j}}\right)}{D'\left(\frac{X_t^i}{P_{t+j}}\right)} + \frac{\left(X_t^i - C_{t+j}\right)}{P_{t+j}} \right] = 0$$

$$E_t \sum_{j=0}^{\infty} (\alpha\beta)^j \left(\frac{Y_{t+j}}{P_{t+j}}\right) D'\left(\frac{X_t^i}{P_{t+j}}\right) \left(\frac{X_t^i}{P_{t+j}}\right) \left[\frac{D\left(\frac{X_t^i}{P_{t+j}}\right)}{\left(\frac{X_t^i}{P_{t+j}}\right) D'\left(\frac{X_t^i}{P_{t+j}}\right)} + \frac{\left(X_t^i - C_{t+j}\right)}{X_t^i} \right] = 0$$

$$E_t \sum_{j=0}^{\infty} (\alpha\beta)^j \left(\frac{Y_{t+j}}{P_{t+j}}\right) D'\left(\frac{X_t^i}{P_{t+j}}\right) \left(\frac{X_t^i}{P_{t+j}}\right) \left[1 - \frac{1}{\varepsilon_D} - \frac{C_{t+j}}{X_t^i}\right] = 0$$

NOTE: Part (d) asks you to do a log-linearization. If you unfamiliar with this type of calculation, please go to the course web-site and download the file "Log-Linearization Handout" found on the webpage where you can download the problem sets. This handout should help familiarize you with log-linearization.

(d) We now want to take a log-linear approximation of the first-order condition (7) around a steady state in which all prices are constant over time and equal to one another, marginal cost is similarly constant, and the constant ratio of price to marginal cost equals μ^*. Let $\hat{x}_t$, $\hat{\pi}_t$, and $\hat{c}_t$ denote the percentage deviations of the variables X_t / P_t, P_t / P_{t-1} and C_t / P_t, respectively, from their steady state values. Do the log linearization of equation (7) to get equation (8) below, and interpret this equation.

$$E_t\sum_{j=0}^{\infty}(\alpha\beta)^j\left\{\left[\hat{x}_t-\sum_{k=1}^{j}\hat{\pi}_{t+k}\right]-\hat{c}_{t+j}\right\}=0 \tag{8}$$

The proof of this equation is rather difficult. So, let me first give the intuition of the result. Note that the term

$$\left[\hat{x}_t-\sum_{k=1}^{j}\hat{\pi}_{t+k}\right]$$

represents the firms relative price at time $t+j$. (You will see this in the proof below). And, $\hat{c}_{t+j}$ is the relative marginal cost at time $t+j$. Given these interpretations, we see that the equation is simply telling us that that deviations in the firm's optimal steady state relative price are expected to be proportional to the deviations in marginal cost of production *on average* over the time that the price chosen at date t is *expected* to apply. Thus, if the firm anticipates higher average marginal costs or inflation over the period it expects its prices to be fixed, it will choose a higher price.

Now, let's do the log-linearization to show this result:

The first thing I will do is rewrite equation (7) in terms of the variables we are going to log-linearize around:

$$E_t\sum_{j=0}^{\infty}(\alpha\beta)^j\frac{Y_{t+j}}{P_{t+j}}D'\left(\frac{X_t}{P_{t+j}}\right)\frac{X_t}{P_{t+j}}\left[1-\frac{1}{\varepsilon_D}-\frac{C_{t+j}}{X_t}\right]=0$$

$$E_t\sum_{j=0}^{\infty}(\alpha\beta)^j\left(\frac{Y_{t+j}}{P_{t+j}}\right)D\left(\frac{X_t}{P_{t+j}}\right)\left(\frac{D'\left(\frac{X_t}{P_{t+j}}\right)\frac{X_t}{P_{t+j}}}{D\left(\frac{X_t}{P_{t+j}}\right)}\right)\left[1-\frac{1}{\varepsilon_D}-\frac{C_{t+j}}{X_t}\right]=0$$

$$E_t\sum_{j=0}^{\infty}(\alpha\beta)^j\left(\frac{Y_{t+j}}{P_{t+j}}\right)D\left(\frac{X_t}{P_{t+j}}\right)(-\varepsilon_D)\left[\frac{\varepsilon_D-1}{\varepsilon_D}-\frac{C_{t+j}}{X_t}\right]=0$$

Multiplying through by X_t, and rearranging, we get:

$$\begin{aligned}&E_t\sum_{j=0}^{\infty}(\alpha\beta)^j D\left(\frac{X_t}{P_{t+j}}\right)Y_{t+j}\left[\frac{\varepsilon_D-1}{\varepsilon_D}\frac{X_t}{P_{t+j}}-\frac{C_{t+j}}{P_{t+j}}\right]=0\\&E_t\sum_{j=0}^{\infty}(\alpha\beta)^j D\left(\frac{X_t}{P_{t+j}}\right)Y_{t+j}\left[\left(\frac{\varepsilon_D-1}{\varepsilon_D}\right)\frac{X_t}{P_{t+j}}\right]=E_t\sum_{j=0}^{\infty}(\alpha\beta)^j D\left(\frac{X_t}{P_{t+j}}\right)Y_{t+j}\left[\frac{C_{t+j}}{P_{t+j}}\right]\end{aligned} \tag{9}$$

Now, note the following:

$$\frac{X_t}{P_{t+j}}=\frac{X_t}{P_t}\frac{P_t}{P_{t+1}}\frac{P_{t+1}}{P_{t+2}}\cdots\frac{P_{t+j-1}}{P_{t+j}} \tag{10}$$

Thus, if we log-linearize equation (10), we will get:

$$1+\hat{x}_t-\sum_{k=1}^{j}\pi_{t+k} \tag{11}$$

Additionally, notice what happens if we log-linearize $D\left(\frac{X_t}{P_{t+j}}\right)Y_{t+j}$.

We will get:

$$D\left(\frac{X}{P}\right)Y\left[1+\varepsilon_D\left(\hat{x}_t-\sum_{k=1}^{j}\pi_{t+k}\right)\right] \tag{12}$$

where X, Y, P represent their respected steady state values.

Keeping this in mind, now let's log-linear the RHS of equation (9) recognizing that we can conveniently use ***variations*** of our results shown in equations (11) and (12). Doing this, we find:

$$\sum_{j=0}^{\infty}(\alpha\beta)^j D\left(\frac{X}{P}\right)Y\left(\frac{C}{P}\right)\left[1+\varepsilon_D\left(\hat{x}_t-\sum_{k=1}^{j}\hat{\pi}_{t+k}\right)+\hat{c}_{t+j}\right] \tag{13}$$

Now let's log-linearize the LHS of equation (9):

$$\sum_{j=0}^{\infty}(\alpha\beta)^j\left(\frac{\varepsilon_D-1}{\varepsilon_D}\right)D\left(\frac{X}{P}\right)Y\left(\frac{X}{P}\right)\left[1+\varepsilon_D\left(\hat{x}_t-\sum_{k=1}^{j}\hat{\pi}_{t+k}\right)+\left(\hat{x}_t-\sum_{k=1}^{j}\hat{\pi}_{t+k}\right)\right] \tag{14}$$

Now, we can equate our RHS in equation (13) with the LHS in equation (14) to get our final result:

$$\sum_{j=0}^{\infty}(\alpha\beta)^j\left(\frac{\varepsilon_D-1}{\varepsilon_D}\right)D\left(\frac{X}{P}\right)Y\left(\frac{X}{P}\right)\left[1+\varepsilon_D\left(\hat{x}_t-\sum_{k=1}^{j}\hat{\pi}_{t+k}\right)+\left(\hat{x}_t-\sum_{k=1}^{j}\hat{\pi}_{t+k}\right)\right]=\sum_{j=0}^{\infty}(\alpha\beta)^j D\left(\frac{X}{P}\right)Y\left(\frac{C}{P}\right)\left[1+\varepsilon_D\left(\hat{x}_t-\sum_{k=1}^{j}\hat{\pi}_{t+k}\right)+\hat{c}_{t+j}\right]$$

$$\sum_{j=0}^{\infty}(\alpha\beta)^j\left(\frac{\varepsilon_D-1}{\varepsilon_D}\right)(X)\left[\left(\hat{x}_t-\sum_{k=1}^{j}\hat{\pi}_{t+k}\right)\right]=\sum_{j=0}^{\infty}(\alpha\beta)^j(C)\left[\hat{c}_{t+j}\right]$$

$$\sum_{j=0}^{\infty}(\alpha\beta)^j\left(\frac{\varepsilon_D-1}{\varepsilon_D}\right)\left(\frac{X}{C}\right)\left[\left(\hat{x}_t-\sum_{k=1}^{j}\hat{\pi}_{t+k}-\hat{c}_{t+j}\right)\right]=0$$

$$\sum_{j=0}^{\infty}(\alpha\beta)^j\left(\frac{1}{\mu^*}\right)\left(\mu^*\right)\left[\left(\hat{x}_t-\sum_{k=1}^{j}\hat{\pi}_{t+k}-\hat{c}_{t+j}\right)\right]=0$$

$$\sum_{j=0}^{\infty}(\alpha\beta)^j\left[\left(\hat{x}_t-\sum_{k=1}^{j}\hat{\pi}_{t+k}-\hat{c}_{t+j}\right)\right]=0$$

(e) Equation (8) can be solved for the relative price $\hat{x}_t$ of firms that have just changed their price as a function of future inflation and real marginal costs. Assuming $\alpha\beta<1$, use a quasi-difference of this relation to prove the following:

$$\hat{x}_t=\alpha\beta E_t\hat{\pi}_{t+1}+(1-\alpha\beta)\hat{c}_t+\alpha\beta E_t\hat{x}_{t+1} \tag{15}$$

Rewriting our result from part (d), we have:

$$\sum_{j=0}^{\infty}(\alpha\beta)^j \hat{x}_t = \sum_{j=0}^{\infty}(\alpha\beta)^j\left(\sum_{k=1}^{j}\hat{\pi}_{t+k} + \hat{c}_{t+j}\right)$$
$$\frac{1}{1-\alpha\beta}\hat{x}_t = \sum_{j=0}^{\infty}(\alpha\beta)^j\left(\sum_{k=1}^{j}\hat{\pi}_{t+k} + \hat{c}_{t+j}\right)$$
$$\hat{x}_t = (1-\alpha\beta)\sum_{j=0}^{\infty}(\alpha\beta)^j\left(\sum_{k=1}^{j}\hat{\pi}_{t+k} + \hat{c}_{t+j}\right)$$
$$\hat{x}_t = (1-\alpha\beta)\hat{c}_t + (1-\alpha\beta)\sum_{j=1}^{\infty}(\alpha\beta)^j\left(\sum_{k=1}^{j}\hat{\pi}_{t+k} + \hat{c}_{t+j}\right) \tag{16}$$

Using this, we also now know that:

$$E_t\hat{x}_{t+1} = (1-\alpha\beta)E_t\sum_{j=0}^{\infty}(\alpha\beta)^j\left(\sum_{k=1}^{j}\hat{\pi}_{t+1+k} + \hat{c}_{t+1+j}\right)$$
$$\alpha\beta E_t\hat{x}_{t+1} = (1-\alpha\beta)E_t\sum_{j=0}^{\infty}(\alpha\beta)^{j+1}\left(\sum_{k=1}^{j}\hat{\pi}_{t+1+k} + \hat{c}_{t+1+j}\right) \tag{17}$$

Subtracting equation (17) from equation (16), we have:

$$\hat{x}_t - \alpha\beta E_t\hat{x}_{t+1} = (1-\alpha\beta)\hat{c}_t + (1-\alpha\beta)\sum_{j=1}^{\infty}(\alpha\beta)^j\left(\sum_{k=1}^{j}\hat{\pi}_{t+k} + \hat{c}_{t+j}\right) - (1-\alpha\beta)E_t\sum_{j=0}^{\infty}(\alpha\beta)^{j+1}\left(\sum_{k=1}^{j}\hat{\pi}_{t+1+k} + \hat{c}_{t+1+j}\right)$$
$$\hat{x}_t - \alpha\beta E_t\hat{x}_{t+1} = (1-\alpha\beta)\hat{c}_t + (1-\alpha\beta)\sum_{j=1}^{\infty}(\alpha\beta)^j\left(\sum_{k=1}^{j}\hat{\pi}_{t+k}\right) - (1-\alpha\beta)E_t\sum_{j=0}^{\infty}(\alpha\beta)^{j+1}\left(\sum_{k=1}^{j}\hat{\pi}_{t+1+k}\right)$$
$$\hat{x}_t - \alpha\beta E_t\hat{x}_{t+1} = (1-\alpha\beta)\hat{c}_t + (1-\alpha\beta)\sum_{j=1}^{\infty}(\alpha\beta)^j\left(\sum_{k=1}^{j}\hat{\pi}_{t+k}\right) - (1-\alpha\beta)E_t\sum_{j=0}^{\infty}(\alpha\beta)^{j+1}\left(\sum_{k=1}^{j}\hat{\pi}_{t+1+k}\right)$$
$$\hat{x}_t - \alpha\beta E_t\hat{x}_{t+1} = (1-\alpha\beta)\hat{c}_t + (1-\alpha\beta)E_t\sum_{j=0}^{\infty}(\alpha\beta)^{j+1}\hat{\pi}_{t+1}$$
$$\hat{x}_t - \alpha\beta E_t\hat{x}_{t+1} = (1-\alpha\beta)\hat{c}_t + \alpha\beta E_t\hat{\pi}_{t+1}$$
$$\hat{x}_t = \alpha\beta E_t\hat{x}_{t+1} + (1-\alpha\beta)\hat{c}_t + \alpha\beta E_t\hat{\pi}_{t+1}$$

(f) Now, given a few assumptions, we can show that the rate of increase of the price index satisfies the following relation in our log-linear approximation:

$$\hat{\pi}_t = \left(\frac{1-\alpha}{\alpha}\right)\hat{x}_t \tag{18}$$

(Please see page 1115-6 if you want more details of where this equation comes from.) Substitute (18) into (15) and use the fact that $\hat{\mu}_t = -\hat{c}_t$, where $\hat{\mu}_t$ denotes the percentage deviation of the average markup $\mu_t = P_t / C_t$ from its steady state value of μ^*, to show that:

$$\hat{\pi}_t = \beta E_t\hat{\pi}_{t+1} - \kappa\hat{\mu}_t \tag{19}$$

where $\kappa \equiv (1-\alpha\beta)(1-\alpha)/\alpha$

From equation (18), we have:

$$E_t\hat{\pi}_{t+1} = \left(\frac{1-\alpha}{\alpha}\right)E_t\hat{x}_{t+1}$$

Plugging this into our result from part (e) and using equation (18), we have:

$$\left(\frac{\alpha}{1-\alpha}\right)\hat{\pi}_t = \alpha\beta\left(\frac{\alpha}{1-\alpha}\right)E_t\hat{\pi}_{t+1} + (1-\alpha\beta)\hat{c}_t + \alpha\beta E_t\hat{\pi}_{t+1}$$
$$\hat{\pi}_t = \alpha\beta E_t\hat{\pi}_{t+1} + \frac{(1-\alpha\beta)(1-\alpha)}{\alpha}\hat{c}_t + (1-\alpha)\beta E_t\hat{\pi}_{t+1}$$
$$\hat{\pi}_t = \beta E_t\hat{\pi}_{t+1} + \frac{(1-\alpha\beta)(1-\alpha)}{\alpha}\hat{c}_t$$
$$\hat{\pi}_t = \beta E_t\hat{\pi}_{t+1} - \kappa\hat{\mu}_t$$

PART 3 – Interpreting the impact of Sticky Prices on Markups

Okay, now the math part of this problem set is over. Now, that you have an idea where these results come from, you will need to analyze the intuition and implications of these results. The rest of this problem set does not need any math calculations.

(g) Holding future expectations of inflation constant, consider a positive shock to aggregate demand. Using equation (19), how will this shock to aggregate demand affect the average markup? Are markups pro- or counter-cyclical in this model? What is the intuition for this result? [Hint: You should first consider how a positive shock to aggregate demand will affect average prices and output. Then, holding future expectations of prices constant, you can analyze the impact on the average markups using equation (19)]

Going back to the standard aggregate demand and supply framework of Keynes (with an upward sloping short-run aggregate supply), an increase in aggregate demand should result in an immediate increase in output and prices. However, if we hold our expectations of future prices constant, then we see from our equation (19) that it must be the case that the average markup falls. Thus, we conclude that our sticky price model will yield us counter-cyclical movements in the markup just as we had hoped. Using our result from part (a), we know that this fall in the markup of firms will increase aggregate output. Thus, the initial shock to aggregate demand will be amplified via the effect of the rise in prices on the markup.

[We could also have assumed a vertical aggregate supply curve such that the aggregate demand shock only affects prices and not output initially. However, because the average markup will still fall, we will still get a positive movement in aggregate output!]

What is the intuition of this result? Well, firms that can adjust their prices will raise them upward to meet the new demand and to maintain their markup. However, some firms find their prices fixed at the time of the aggregate demand shock and are not be able to adjust their prices upward to maintain their optimal markup. Their markup will fall because their input prices are increasing in their level of production. Given the fall in the average markup, output will be pushed even higher.

If we had accounted for changes of future inflation, we will still get the same results but the analysis is more complicated.

(h) How will an increase in α affect the magnitude of movements in the markup in response to a shock to the economy? (Again, hold future expectations constant to do your analysis). Please interpret this result.

An increase in α will decrease the multiplier κ in equation (19). This implies that for any given shock to current prices, the movement in the markup will have to be greater than before (again, holding our expectations of future inflation constant). Thus, the counter-cyclical movements in the markup are amplified as will be the amplification effect of markups. What is the intuition of this result? Well, a higher α implies that that a smaller fraction of firms are able to adjust their prices each period. Thus, more firms will be unable to increase their prices in response to the shock and the average markup will fall even further than before!

(i) Now suppose that the elasticity of demand isn't constant. Rather, assume that elasticity of demand is an INCREASING function of the firm's relative price.

i. How does an average firm's relative price move immediately following a positive shock to aggregate demand?

Again, a positive aggregate demand shock increases aggregate prices. Firms that are unable to adjust their price immediately will see their relative price fall, but firms that are able to adjust the price upwards will see an increase in their relative price.

ii. What will happen to an average firm's desired markup during an economic boom? [Hint: Use your answer to part (b) above.]

In the economic boom, the increase in aggregate demand increases the demand for inputs and thus pushes up the marginal cost of firms. If a firm can adjust its price, it would usually like to increase its price to maintain its usual *desired* markup. However, since other firms' prices are being held down because they are unable to adjust immediately, a firm that can adjust its price upward faces a relatively more elastic demand when increasing its price relative to those with fixed prices. This relatively more elastic demand helps dampen the *desired* markup for firms able to increase their prices, and prices will adjust upwards more slowly.

iii. Would allowing the elasticity of demand to vary as described above reduce or increase amplification in this model?

Clearly, the fall in the average desired markup during an economic boom will cause the average markup of the economy to be even more counter-cyclical than before. This increases the amplification of the model! (This is what Rotemberg-Woodford discuss in section 4.3 of their chapter handbook).