An introduction to log-linearizations

Fall 2000

One method to solve and analyze nonlinear dynamic stochastic models is to approximate the nonlinear equations characterizing the equilibrium with log-linear ones. The strategy is to use a first order Taylor approximation around the steady state to replace the equations with approximations, which are linear in the log-deviations of the variables.

Let X_t be a strictly positive variable, X its steady state and

$$x_t \equiv \log X_t - \log X \tag{1}$$

the logarithmic deviation.

First notice that, for X small, $\log(1+X) \simeq X$, thus:

$$x_t \equiv \log(X_t) - \log(X) = \log(\frac{X_t}{X}) = \log(1 + \%change) \simeq \%change.$$

1 The standard method

Suppose that we have an equation of the following form:

$$f(X_t, Y_t) = g(Z_t). \tag{2}$$

where X_t, Y_t and Z_t are strictly positive variables.

This equation is clearly also valid at the steady state:

$$f(X, Y) = g(Z). \tag{3}$$

To find the log-linearized version of (2), rewrite the variables using the identity $X_t = \exp(\log(X_t))$[1] and then take logs on both sides:

$$\log(f(e^{\log(X_t)}, e^{\log(Y_t)})) = \log(g(e^{\log(Z_t)})). \tag{4}$$

Now take a first order Taylor approximation around the steady state ($\log(X)$, $\log(Y)$, $\log(Z)$). After some calculations, we can write the left hand side as

$$\log(f(X,Y)) + \frac{1}{f(X,Y)} \left[f_1(X,Y)X(\log(X_t) - \log(X)) + f_2(X,Y)Y(\log(Y_t) - \log(Y))\right]. \tag{5}$$

[1] This procedure allows us to obtain an equation in the log-deviations.

Similarly, the right hand side can be written as

$$\log(g(Z)) + \frac{1}{g(Z)}\,[g'(Z)Z(\log(Z_t) - \log(Z))]. \tag{6}$$

Equating (5) and (6), and using (3) and (1), yields the following log-linearized equation:

$$[f_1(X,Y)Xx_t + f_2(X,Y)Yy_t] \simeq [g'(Z)Zz_t]. \tag{7}$$

Notice that this is a linear equation in the deviations!

Generalizing, the log-linearization of an equation of the form

$$f(x_t^1, ..., x_t^n) = g(y_t^1, ..., y_t^n)$$

is:

$$\sum_{i=1}^{n} f_i(x^1, ..., x^n)x^i x_t^i \simeq \sum_{j=1}^{m} g_j(y^1, ..., y^m)y^j y_t^j.$$

2 A simpler method

However, in the large majority of cases, there is no need for explicit differentiation of the function f and g. Instead, the log-linearized equation can usually be obtained with a simpler method. Let's see.

Notice first that you can write

$$X_t = X(\frac{X_t}{X}) = Xe^{\log(X_t/X)} = Xe^{x_t}$$

Taking a first order Taylor approximation around the steady state yields

$$\begin{aligned} Xe^{x_t} &\simeq Xe^0 + Xe^0(x_t - 0) \\ &\simeq X(1 + x_t) \end{aligned}$$

By the same logic, you can write

$$\begin{aligned} X_t Y_t &\simeq X(1 + x_t)Y(1 + y_t) \\ &\simeq XY(1 + x_t + y_t + x_t y_t) \end{aligned}$$

where $x_t y_t \simeq 0$, since x_t and y_t are numbers close to zero.

Second, notice that

$$\begin{aligned} f(X_t) &\simeq f(X) + f'(X)(X_t - X) \\ &\simeq f(X) + f'(X)X(X_t/X - 1) \\ &\simeq f(X) + f(X)\eta(1 + x_t - 1) \\ &\simeq f(X)(1 + \eta x_t) \end{aligned}$$

where $\eta \equiv \frac{\partial f(X)}{\partial X}\frac{X}{f(X)}$.

Now, the log-linearized equation can be obtained as follows. After having multiplied out everything in the original equation, simply use the following approximations:

$$X_t \simeq X(1 + x_t) \tag{8}$$

$$X_t Y_t \simeq XY(1 + x_t + y_t) \tag{9}$$

$$f(X_t) \simeq f(X)(1 + \eta x_t) \tag{10}$$

2.1 Some examples

2.1.1 The economy resource constraint

Consider the economy resource constraint

$$Y_t = C_t + I_t.$$

and rewrite it as

$$1 = \frac{C_t}{Y_t} + \frac{I_t}{Y_t}.$$

Using (9) we obtain

$$1 \simeq \frac{C}{Y}(1 + c_t - y_t) + \frac{I}{Y}(1 + i_t - y_t)$$

where i_t is the log-deviation of investment.

Since at the steady state

$$Y = C + I,$$

we can cancel out (some) constants and rearrange to obtain

$$y_t \simeq \frac{C}{Y}c_t + \frac{I}{Y}i_t.$$

2.1.2 The marginal propensity to consume out of wealth

Assume that the marginal propensity to consume out of wealth is governed by the following first order difference equation:

$$R_{t+1}^{\sigma-1}\beta^{\sigma}\frac{\Pi_t}{\Pi_{t+1}} = 1 - \Pi_t.$$

Notice that at the steady state

$$R^{\sigma-1}\beta^{\sigma} = 1 - \Pi.$$

and

$$\Pi = 1 - R^{\sigma-1}\beta^{\sigma}.$$

Using (8) and (9) we can write the nonlinear difference equation as

$$R^{\sigma-1}\beta^{\sigma}(1 + (\sigma - 1)r_{t+1} + \pi_t - \pi_{t+1}) \simeq 1 - (1 - R^{\sigma-1}\beta^{\sigma})(1 + \pi_t).$$

Canceling out constants yields

$$R^{\sigma-1}\beta^{\sigma}[(\sigma - 1)r_{t+1} + \pi_t - \pi_{t+1}] \simeq -(1 - R^{\sigma-1}\beta^{\sigma})\pi_t.$$

Rearranging, we obtain

$$\frac{R^{\sigma-1}\beta^{\sigma} - 1}{R^{\sigma-1}\beta^{\sigma}}\pi_t \simeq (\sigma - 1)r_{t+1} + \pi_t - \pi_{t+1}$$

and, finally,

$$\pi_t \simeq R^{\sigma-1}\beta^{\sigma}\left[(1 - \sigma)r_{t+1} + \pi_{t+1}\right].$$

2.1.3 The Euler equation

The consumption Euler equation is

$$1 = R_{t+1}\beta(C_{t+1}/C_t)^{-\gamma}.$$

Using (9) and (10) we can write it as

$$1 \simeq R\beta(1 + r_{t+1} - \gamma(c_{t+1} - c_t)).$$

Canceling out constants yields

$$0 \simeq r_{t+1} - \gamma(c_{t+1} - c_t)$$

and, rearranging,

$$c_t \simeq -\sigma r_{t+1} + c_{t+1}$$

where $\sigma = 1/\gamma$ is the intertemporal elasticity of substitution.

2.1.4 Multiplicative equations

If the equation to log-linearize contains only multiplicative terms, there is a faster procedure. Suppose we have the following equation:

$$\frac{X_t Y_t}{Z_t} = \alpha$$

where α is a constant. To log-linearize divide first by the steady state variables:

$$\frac{(\frac{X_t}{X})(\frac{Y_t}{Y})}{(\frac{Z_t}{Z})} = \frac{\alpha}{\alpha} = 1.$$

Now take logs:

$$\log(\frac{X_t}{X}) + \log(\frac{Y_t}{Y}) - \log(\frac{Z_t}{Z}) = \log(1) = 0.$$

Using (1) we arrive then easily to the log-linearized equation:

$$x_t + y_t - z_t = 0.$$

Notice that in this case the log-linearized equation is not an approximation!