Spectrum of Linearly Modulated Signals

$u(t) = \sum_n d(n)\, g(t - nT)$ baseband, linear mod.

$d(n)$ random : $R_d(m) = E\{d(n+m)\, d^*(n)\}$

$R_u(t+\tau, t) = E\{u(t+\tau)\, u^*(t)\}$: autocorrelation

$$R_u(t+\tau, t) = E\left\{ \sum_n \sum_k d(n)\, g(t+\tau-nT)\, d^*(k)\, g^*(t-kT) \right\} =$$

$$= \sum_n \sum_k \underbrace{E\{d(n)\, d^*(k)\}}_{R_d(\underbrace{n-k}_{m})} g(t+\tau-nT)\, g^*(t-kT) =$$

$$= \sum_m \sum_k R_d(m)\, g(t+\tau-kT-mT)\, g^*(t-kT)$$: periodic in t, with period T (T = symbol interval)

$\Rightarrow u(t)$ is not WSS.

$u(t)$ is cyclostationary : $R_u(t_1, t_2) = R_u(t_1+T, t_2+T)$

Def. $$\overline{R_u}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} R_u(t+\tau, t)\, dt$$

$$\Rightarrow \overline{R_u}(\tau) = \frac{1}{T}\cdot \sum_m R_d(m)\cdot \underbrace{\sum_k \int_{-T/2}^{T/2} g(t+\tau-kT-mT)\, g^*(t-kT)\, dt}_{\underbrace{\int_{-\infty}^{+\infty} g(t+\tau-mT)\, g^*(t)\, dt}_{R_g(\tau-mT)}}$$

$$\Rightarrow \overline{R_u}(\tau) = \frac{1}{T}\sum_m R_d(m)\, R_g(\tau-mT) \ ; \quad R_g(\tau) = \int_{-\infty}^{+\infty} g(t+\tau)\, g^*(t)\, dt$$

now it is possible to def. spectrum:

$$S_u(f) = \mathcal{F}\left[\overline{R_u}(\tau)\right] = \int_{-\infty}^{+\infty} \overline{R_u}(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

$$\Rightarrow S_u(f) = \frac{1}{T} S_d(f) \underbrace{S_g(f)}_{|G^2(f)|} \ ; \quad S_d(f) = \sum_m R_d(m)\, e^{-jm2\pi fT}$$

$$\Rightarrow S_u(f) = \frac{1}{T} S_d(f)\, |G^2(f)|$$

So, the spectrum of a lin. modulated signal is

$$S_u(f) = \frac{1}{T} S_d(f)\,|G^2(f)|$$

T: symbol duration

$g(t)$: basic pulse

$S_d(f)$: spectrum of data sequence

ex. $d(n) \in \{\pm 1\}$ iid

$\downarrow$

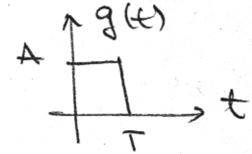
$$R_d(m) = E\{d(n+m)\,d^*(n)\} =$$

$$= \begin{cases} 1, & m = 0 \\ 0, & m \neq 0 \end{cases}$$

$$R_d(m) = \delta(m)$$

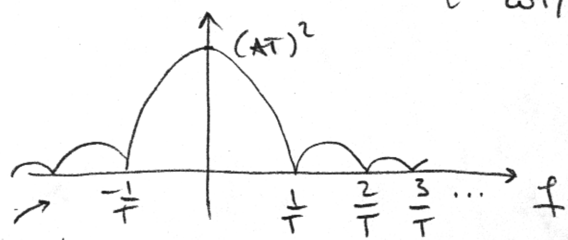
$$S_d(m) = 1$$

$g(t)$: rectangular



$$G(f) = AT \frac{\sin \frac{\omega T}{2}}{\frac{\omega T}{2}} \cdot e^{-j\frac{\omega T}{2}}$$

$$|G^2(f)| = (AT)^2 \left[\frac{\sin \omega T/2}{\omega T/2}\right]^2$$



$$S_u(f) = \frac{1}{T}|G^2(f)|$$

Spectrum shaping

$$S_u(f) = \frac{1}{T} S_d(f) \, |G^2(f)|$$

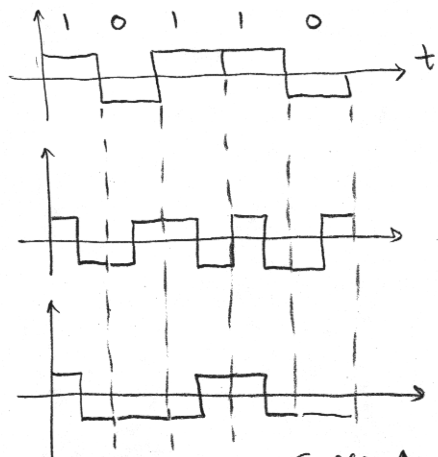
$S_d(f)$ ↓ line coding

$|G^2(f)|$ ↓ pulse shaping

$$R_d(n) = E\{d(n+m)\, d^*(n)\}$$

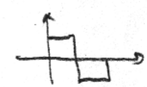
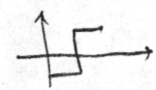
by introducing dependency among $\{d(n)\}$, spectrum is shaped

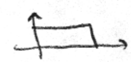
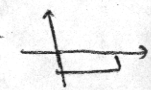
ex. mapping from logical bits "0", "1" into $d(n)$:



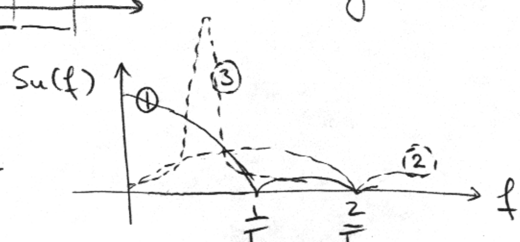
① NRZ — "1" → +1, "0" → −1

② Manchester (biphase) — "1" → + −, "0" → − + — ensures transition in every T

③ Miller (delay mod) — "1" →  or 

"0" →  or 

choose polarity to match previous bit (never T/2 pulse)



- telephone lines
- magnetic recording

Pulse shaping

$$d(n) \rightarrow \boxed{g(t)} \rightarrow u(t)$$

$S_d(f)=1$ $\qquad$ $S_u(f) = \frac{1}{T}|G^2(f)|$

↓ choose $g(t)$ other than rectangular to achieve better spectrum utilization

→ radio channels

Problem: narrower spectrum ↔ longer pulse

↓ reduced rate

↘ intersymbol interference

Nyquist: it is possible to have both narrower spectrum and no ISI with a carefully designed pulse

First Nyquist Criterion: Transmission with no ISI

$$d(n) \rightarrow \boxed{g_T(t)} \rightarrow \boxed{c(t)} \rightarrow \oplus \rightarrow \boxed{g_R(t)} \xrightarrow{t_0+nT} y(nT)$$

$w(t)$

$$y(nT) = \underbrace{\sum_k d(k)\, x(nT-kT)}_{d(n)x(0) \,+\, ISI} + z(nT)$$

want $ISI = 0$:
$$x(mT) = \begin{cases} x(0), & m = 0 \\ 0, & m \neq 0 \end{cases}$$

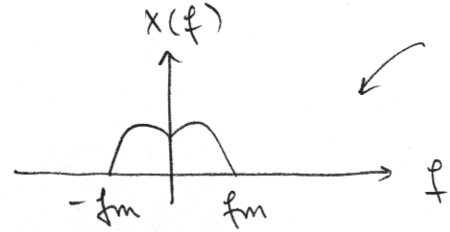
In spectrum :

$$\underbrace{\sum_m x(mT) e^{-jm\omega T}}_{\substack{= x(0) = \text{const.} \\ \text{if no ISI}}} = \underbrace{\frac{1}{T} \sum_m X\left(f + \frac{m}{T}\right)}_{\substack{\text{sum of shifted} \\ \text{spectra } X(f) = \mathcal{F}[x(t)]}}$$

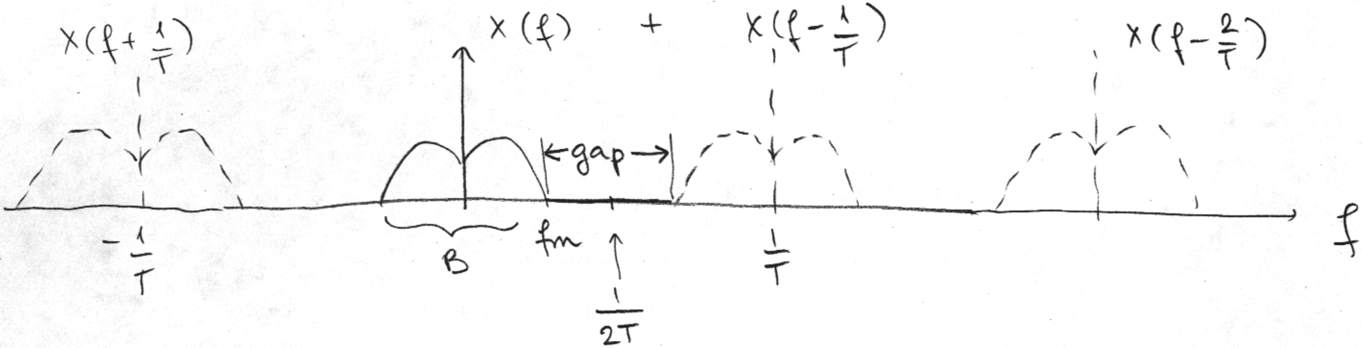
So, sum of shifted spectra must be equal to a constant if we want no ISI at sampling times nT.

There are three cases, depending upon the width of $X(f)$, that can be distinguished.

Case 1.



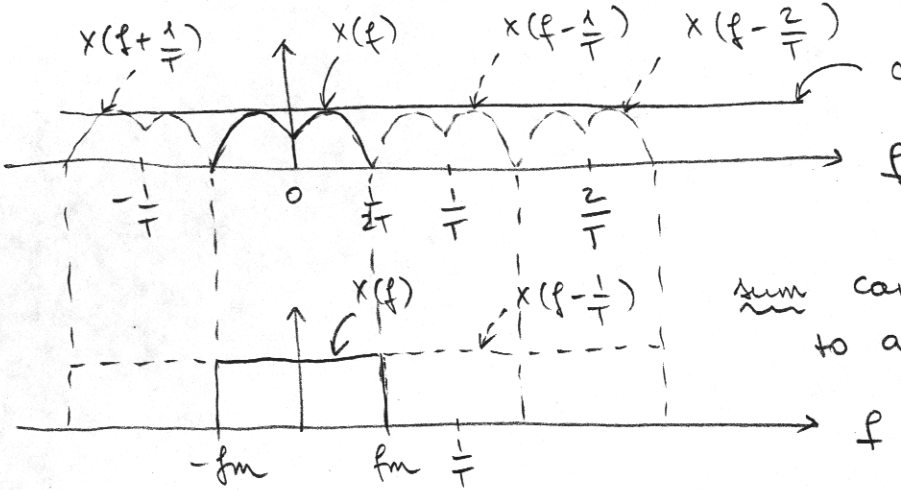
X(f) is bandlimited



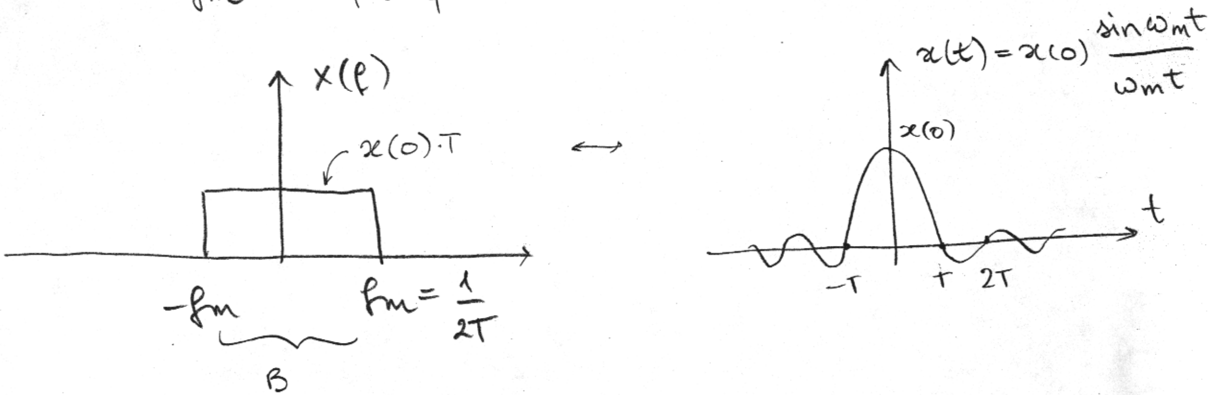
Sum of shifted spectra cannot be equal to a const a constant because there is a gap, i.e. $f_m < \frac{1}{2T}$

So, as long as $\underbrace{\frac{1}{T}}_{R} > \underbrace{2f_m}_{B}$, it is not possible to avoid ISI.

Case 2. In which the shifted spectra touch.

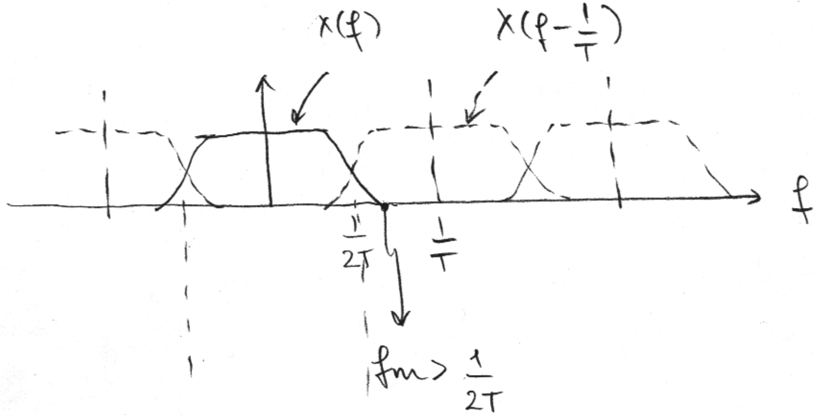


sum can be equal to a constant



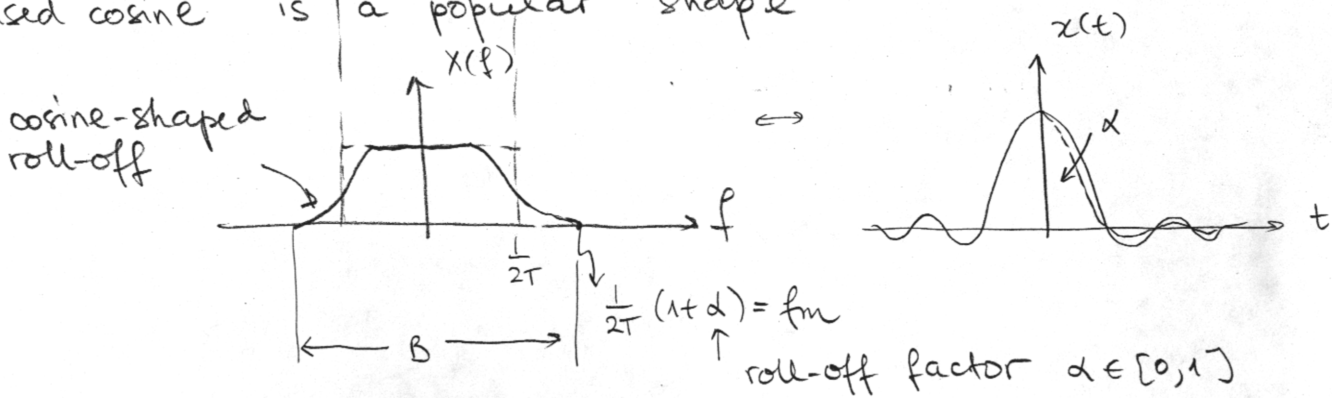
If $f_m = \frac{1}{2T}$, i.e. $R = B$, transmission with no ISI is possible by having $X(f)$ as above.

Case 3. In which the shifted spectra overlap.



If $f_m > \frac{1}{2T}$, there are (infinitely) many shapes $X(f)$ that can sum to a constant after shifting.

"Raised cosine" is a popular shape



Spectral raised cosine is often used in practice.

It provides:

- limited bandwidth $\quad B = R(1+\alpha)$
- no ISI $\quad R = 1/T$
- well-behaved pulse

$\alpha \in [0, 1]$

$f_m = \frac{1}{2T}$

minimum bandwidth needed for no ISI transmission (actually, this pulse is not realizable)

by allowing a small excess bandwidth, $\alpha \cdot \frac{1}{2T}$, transmission at $R = \frac{B}{1+\alpha}$ is possible with no ISI

Implementation on an ideal AWGN channel