16.682: Communication Systems Engineering

Lecture 18/19: Delay Models for Data Networks

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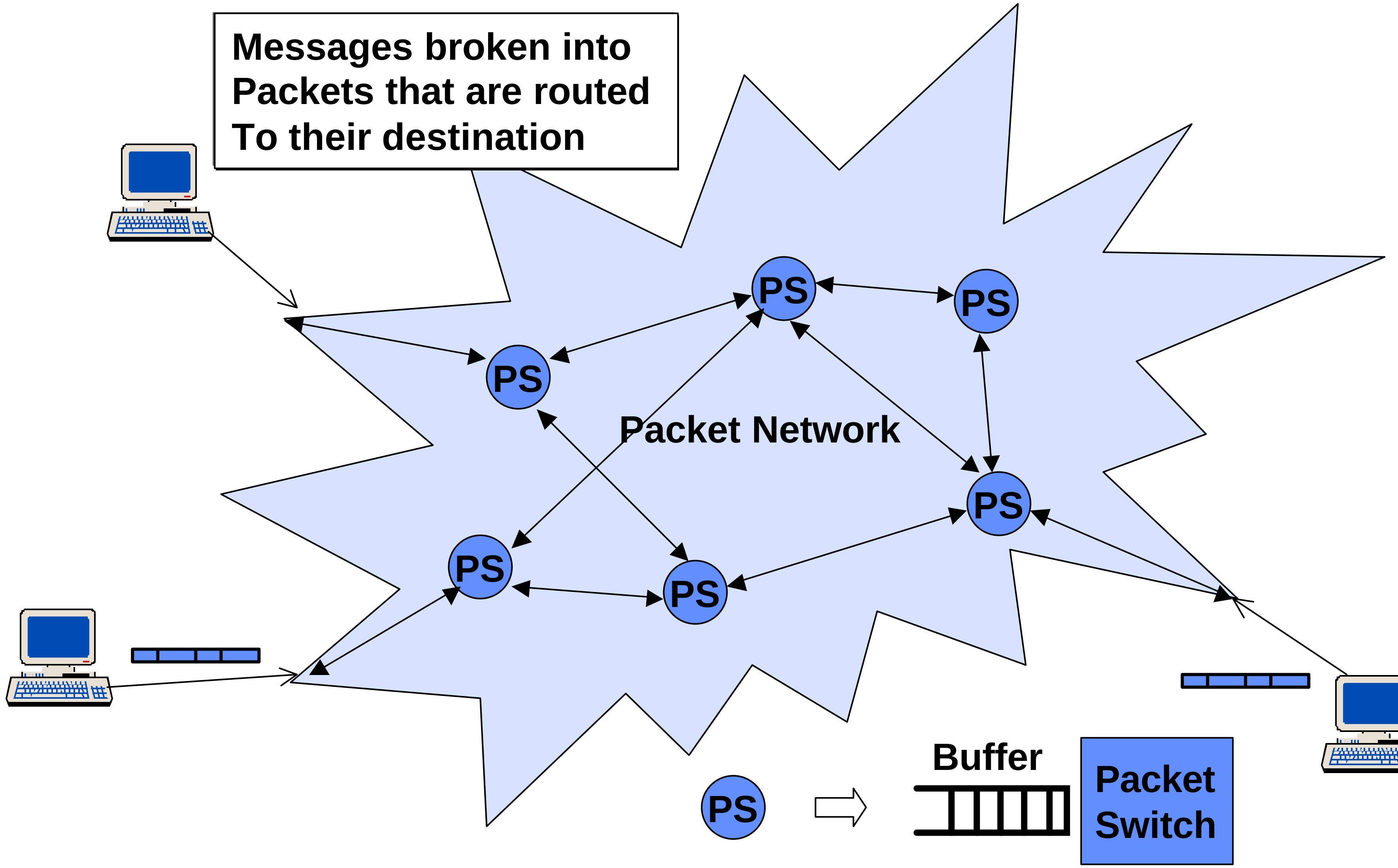
Packet Switched Networks



Queueing Systems

- Used for analyzing network performance

- In packet networks, events are random
  - Random packet arrivals
  - Random packet lengths

- While at the physical layer we were concerned with bit-error-rate, at the network layer we care about delays
  - How long does a packet spend waiting in buffers ?
  - How large are the buffers ?

Random events

- **Arrival process**
  - **Packets arrive according to a random process**
  - **Typically the arrival process is modeled as Poisson**

- **The Poisson process**
  - **Arrival rate of** l **packets per second**
  - **Over a small interval** d**,**

    P(exactly one arrival) = $l d$
    P(0 arrivals) = 1 - $l d$
    P(more than one arrival) = 0

  - **It can be shown that:**

$$\text{P(n arrivals in interval T)} = \frac{(l T)^n e^{-l T}}{n!}$$

The Poisson Process

$$\text{P(n arrivals in interval T)} = \frac{(l T)^n e^{-l T}}{n!}$$

n = number of arrivals in T

It can be shown that,

$E[n] = l T$

$E[n^2] = l T + (l T)^2$

$s^2 = E[(n - E[n])^2] = E[n^2] - E[n]^2 = l T$

Inter-arrival times

- **Time that elapses between arrivals (IA)**

$P(IA <= t) = 1 - P(IA > t)$

$= 1 - P(0 \text{ arrivals in time } t)$

$= 1 - e^{-\lambda t}$

- **This is known as the exponential distribution**
  - **Inter-arrival CDF = $F_{IA}(t) = 1 - e^{-\lambda t}$**
  - **Inter-arrival PDF = $d/dt\, F_{IA}(t) = \lambda e^{-\lambda t}$**

- **The exponential distribution is often used to model the service times (I.e., the packet length distribution)**

Markov property (Memoryless)

$$P(T \le t_0 + t \mid T > t_0) = P(T \le t)$$

$\mathrm{Pr}\,oof:$

$$P(T \le t_0 + t \mid T > t_0) = \frac{P(t_0 < T \le t_0 + t)}{P(T > t_0)}$$

$$= \frac{\int_{t_0}^{t_0+t} l e^{-lt} dt}{\int_{t_0}^{\infty} l e^{-lt} dt} = \frac{-e^{-lt} \big|_{t_0}^{t_0+t}}{-e^{-lt} \big|_{t_0}^{\infty}} = \frac{-e^{-l(t+t_0)} + e^{-l(t_0)}}{e^{-l(t_0)}}$$

$$= 1 - e^{-lt} = P(T \le t)$$

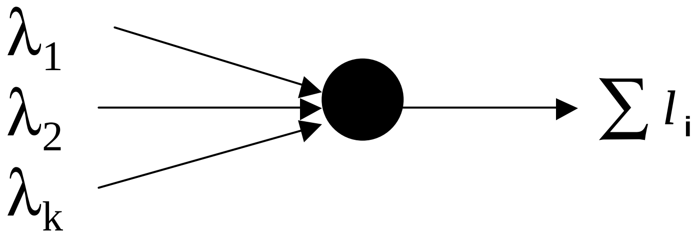
- **Previous history does not help in predicting the future!**
- **Distribution of the time until the next arrival is independent of when the last arrival occurred!**

Example

- Suppose a train arrives at a station according to a Poisson process with average inter-arrival time of 20 minutes
- When a customer arrives at the station the average amount of time until the next arrival is 20 minutes
  - Regardless of when the previous train arrived
- The average amount of time since the last departure is 20 minutes!
- Paradox: If an average of 20 minutes passed since the last train arrived and an average of 20 minutes until the next train, then an average of 40 minutes will elapse between trains
  - But we assumed an average inter-arrival time of 20 minutes!
  - What happened?
- Answer: You tend to arrive during long inter-arrival times
  - If you don't believe me you have not taken the T

Properties of the Poisson process

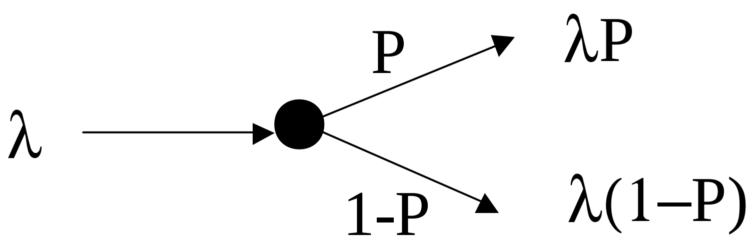
- **Merging Property**



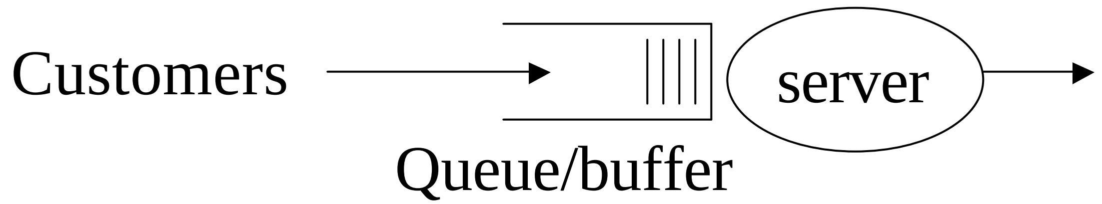
Let A1, A2, … Ak be independent Poisson Processes of rate l 1, l 2, …l k

$$A = \sum A_i \text{ is also Poisson of rate} = \sum l_i$$

- **Splitting property**
  - **Suppose that every arrival is randomly routed with probability P to stream 1 and (1-P) to stream 2**
  - **Streams 1 and 2 are Poisson of rates Pl and (1-P)l respectively**



Queueing Models



- **Model for**
  - **Customers waiting in line**
  - **Assembly line**
  - **Packets in a network (transmission line)**

- **Want to know**
  - **Average number of customers in the system**
  - **Average delay experienced by a customer**

- **Quantities obtained in terms of**
  - **Arrival rate of customers (average number of customers per unit time)**
  - **Service rate (average number of customers that the server can serve per unit time)**

Analyzing delay in networks (queueing theory)

- **Little's theorem**
  - **Relates delay to number of users in the system**
  - **Can be applied to any system**

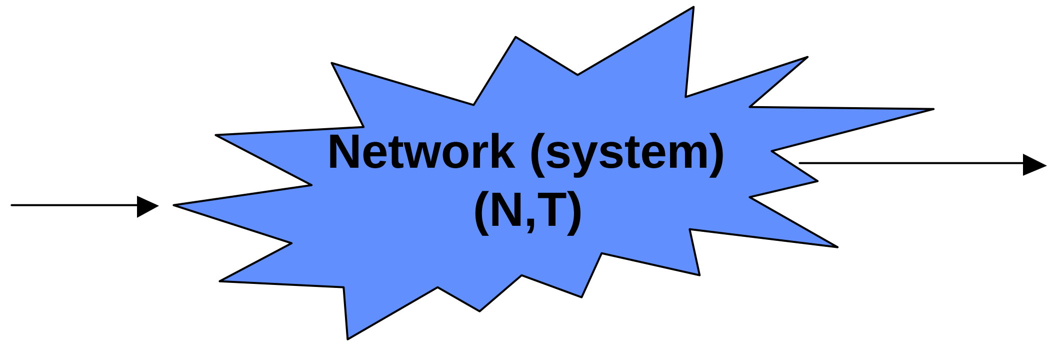
- **Simple queueing systems (single server)**
  - **M/M/1, M/G/1, M/D/1**
  - **M/M/m/m**

- **Poisson Arrivals =>** $P(\text{n arrivals in interval T}) = \dfrac{(\lambda T)^n e^{-\lambda T}}{n!}$
  - λ **= arrival rate in packets/second**

- **Exponential service time =>** $P(\text{service time} < T) = 1 - e^{-\mu T}$
  - μ **= service rate in packets/second**

Little's theorem

λ **packet per second**

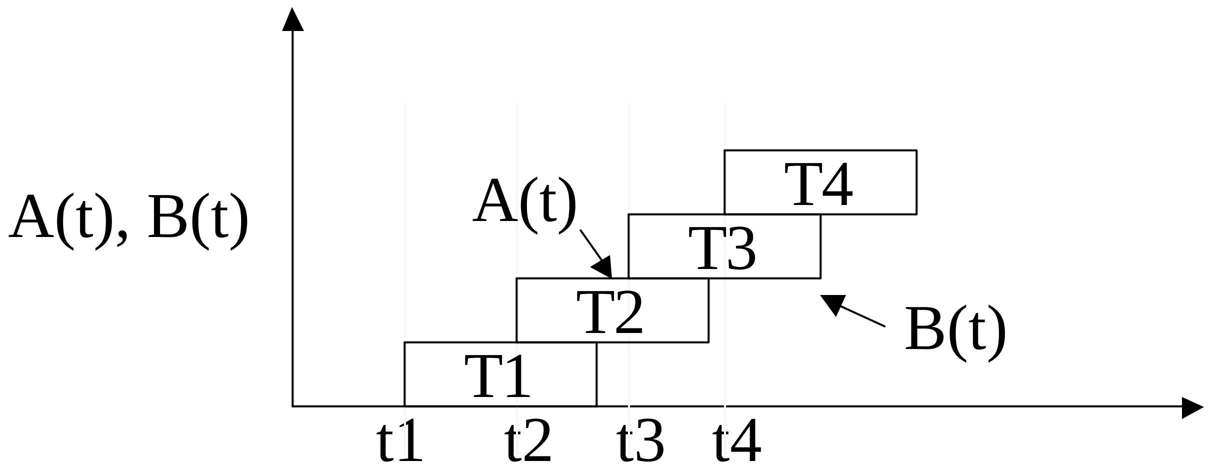


- **N = average number of packets in system**
- **T = average amount of time a packet spends in the system**
- **λ = arrival rate of packets into the system**
  (not necessarily Poisson)

- **Little's theorem: N = λT**
  - **Can be applied to entire system or any part of it**
  - **Crowded system $\Rightarrow$ long delays**

    On a rainy day people drive slowly and roads are more congested!

Proof of Little's Theorem



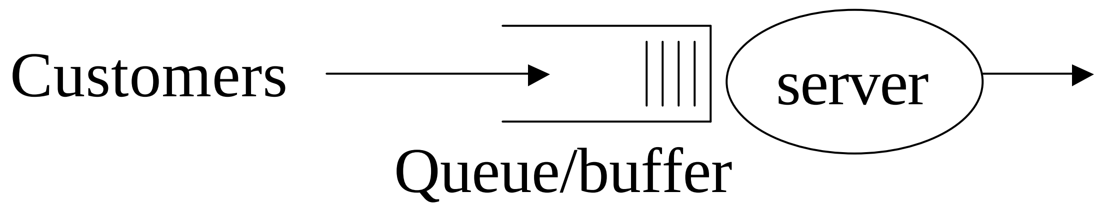
- $A(t)$ = number of arrivals by time t
- $B(t)$ = number of departures by time t
- t_i = arrival time of i^{th} customer
- T_i = amount of time i^{th} customer spends in the system
- $N(t)$ = number of customers in system at time t = $A(t) - B(t)$

$$N = \lim_{t \to \infty} \frac{\sum_{i=1}^{A(t)} T_i}{t}, \quad T = \lim_{t \to \infty} \frac{\sum_{i=1}^{A(t)} T_i}{A(t)} \Rightarrow \sum_{i=1}^{A(t)} T_i = A(t)T$$

$$N = \frac{\sum_{i=1}^{A(t)} T_i}{t} = \left(\frac{A(t)}{t}\right) \frac{\sum_{i=1}^{A(t)} T_i}{A(t)} = l\, T$$

Application of little's Theorem

- **Little's Theorem can be applied to almost any system or part of it**
- **Example:**



1) The transmitter: D_{TP} = packet transmission time

- **Average number of packets at transmitter = $\lambda D_{TP} = \rho$ = link utilization**

2) The transmission line: D_p = propagation delay

- **Average number of packets in flight = λD_p**

3) The buffer: D_q = average queueing delay

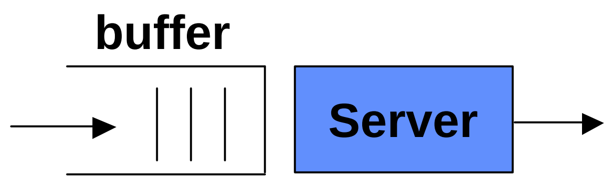
- **Average number of packets in buffer = $N_q = \lambda D_q$**

4) Transmitter + buffer

- **Average number of packets = $\rho + N_q$**

Single server queues

λ packet per second

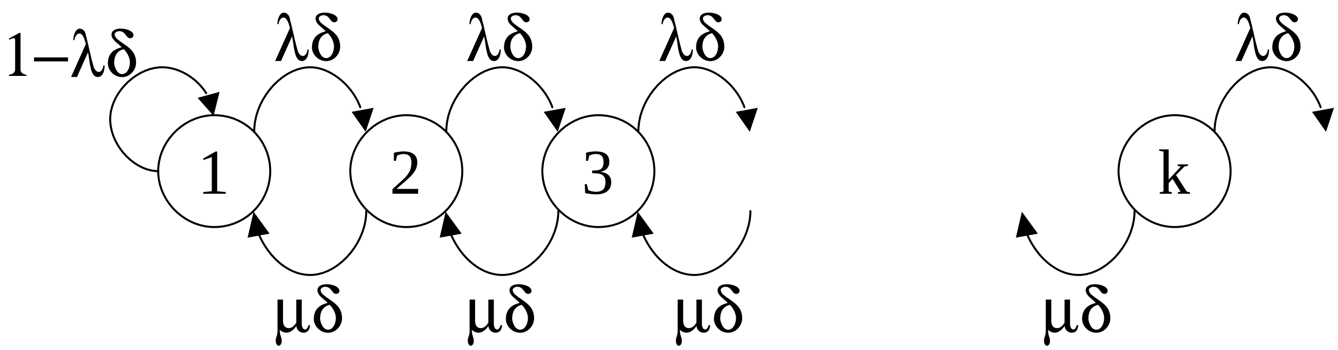


μ packet per second

➔ Service time = $1/\mu$

- **M/M/1**
  - Poisson arrivals, exponential service times
- **M/G/1**
  - Poisson arrivals, general service times
- **M/D/1**
  - Poisson arrivals, deterministic service times (fixed)

Markov Chain for M/M/1 system



- **State k => k customers in the system**

- **P(I,j) = probability of transition from state I to state j**
  - **As** d **=> 0, we get:**

    P(0,0) = 1 - l d**,** **P(j,j+1) =** l d

    P(j,j) = 1 - l d -md **P(j,j-1) =** md

    P(I,j) = 0 for all other values of I,j.

- **Birth-death chain: Transitions exist only between adjacent states**
  - l d , md **are flow rates between states**

Equilibrium analysis

- **We want to obtain P(n) = the probability of being in state n**
- **At equilibrium** l**P(n) =** m**P(n+1) for all n**
  - **P(n+1) = (**l/m**)P(n) =** r **P(n),** $r = l/m$
- **It follows: P(n) =** r^n **P(0)**
- **Now by axiom of probability:**

$$\sum_{i=0}^{\infty} P(n) = 1$$

$$\Rightarrow \sum_{i=0}^{\infty} r^n P(0) = \frac{P(0)}{1-r} = 1$$

$$\Rightarrow P(0) = 1 - r$$

$$P(n) = r^n(1-r)$$

Average queue size

$$N = \sum_{n=0}^{\infty} nP(n) = \sum_{n=0}^{\infty} n r^{n}(1 - r) = \frac{r}{1 - r}$$

$$N = \frac{r}{1 - r} = \frac{l / m}{1 - l / m} = \frac{l}{m - l}$$

- **N = Average number of customers in the system**
- **The average amount of time that a customer spends in the system can be obtained from Little's formula (N=lT => T = N/l)**

$$T = \frac{1}{m - l}$$

- **T includes the queueing delay plus the service time (Service time = D_{TP} = 1/m)**
  - **W = amount of time spent in queue = T - 1/m =>** $W = \frac{1}{m - l} - \frac{1}{m}$

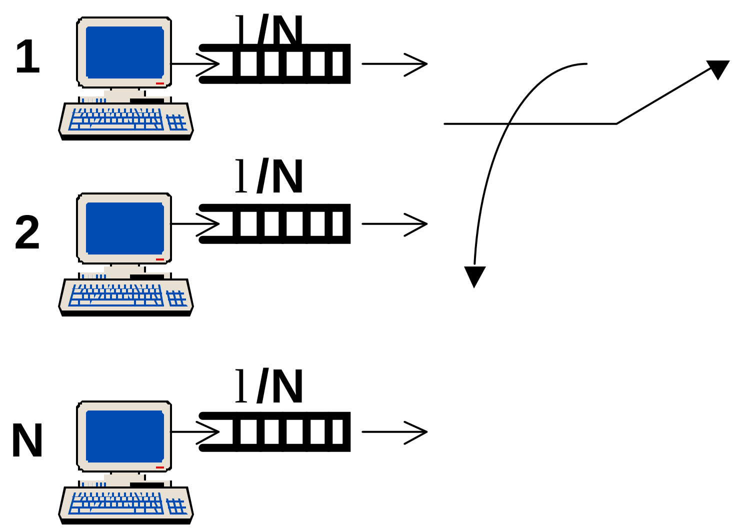
- **Finally, the average number of customers in the buffer can be obtained from little's formula**

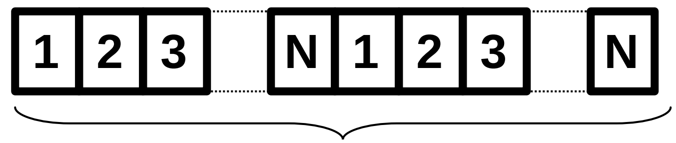
$$N_Q = lW = \frac{l}{m - l} - \frac{l}{m} = N - r$$

Example (fast food restaurant)

- **Customers arrive at a fast food restaurant at a rate of 100 per hour and take 30 seconds to be served.**
- **How much time do they spend in the restaurant?**
  - **Service rate = μ = 60/0.5=120 customers per hour**
  - **T = 1/$\mu-\lambda$ = 1/(120-100) = 1/20 hrs = 3 minutes**
- **How much time waiting in line?**
  - **W = T - 1/μ = 2.5 minutes**
- **How many customers in the restaurant?**
  - **N = λT = 5**
- **What is the server utilization?**
  - **$\rho = \lambda/\mu$ = 5/6**

Packet switching vs. Circuit switching



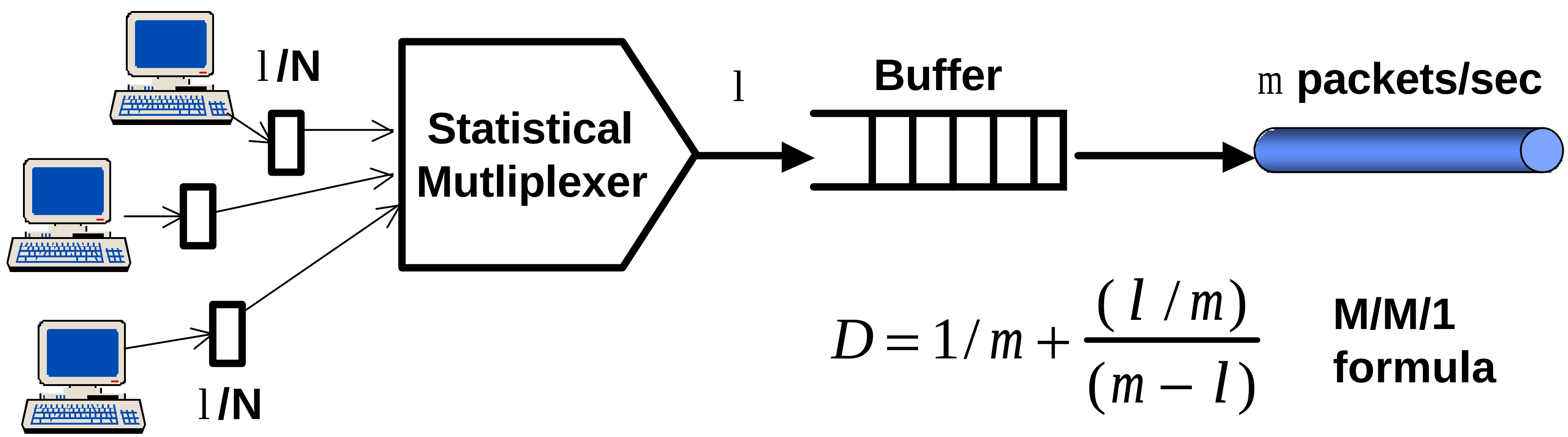


TDM, Time Division Multiplexing
Each user can send m/N packets/sec and has packet arriving at rate l/N packets/sec

$$D = N/m + \frac{N(l/m)}{(m-l)}$$

M/M/1 formula

Packets generated at random times



$$D = 1/m + \frac{(l/m)}{(m-l)}$$

M/M/1 formula

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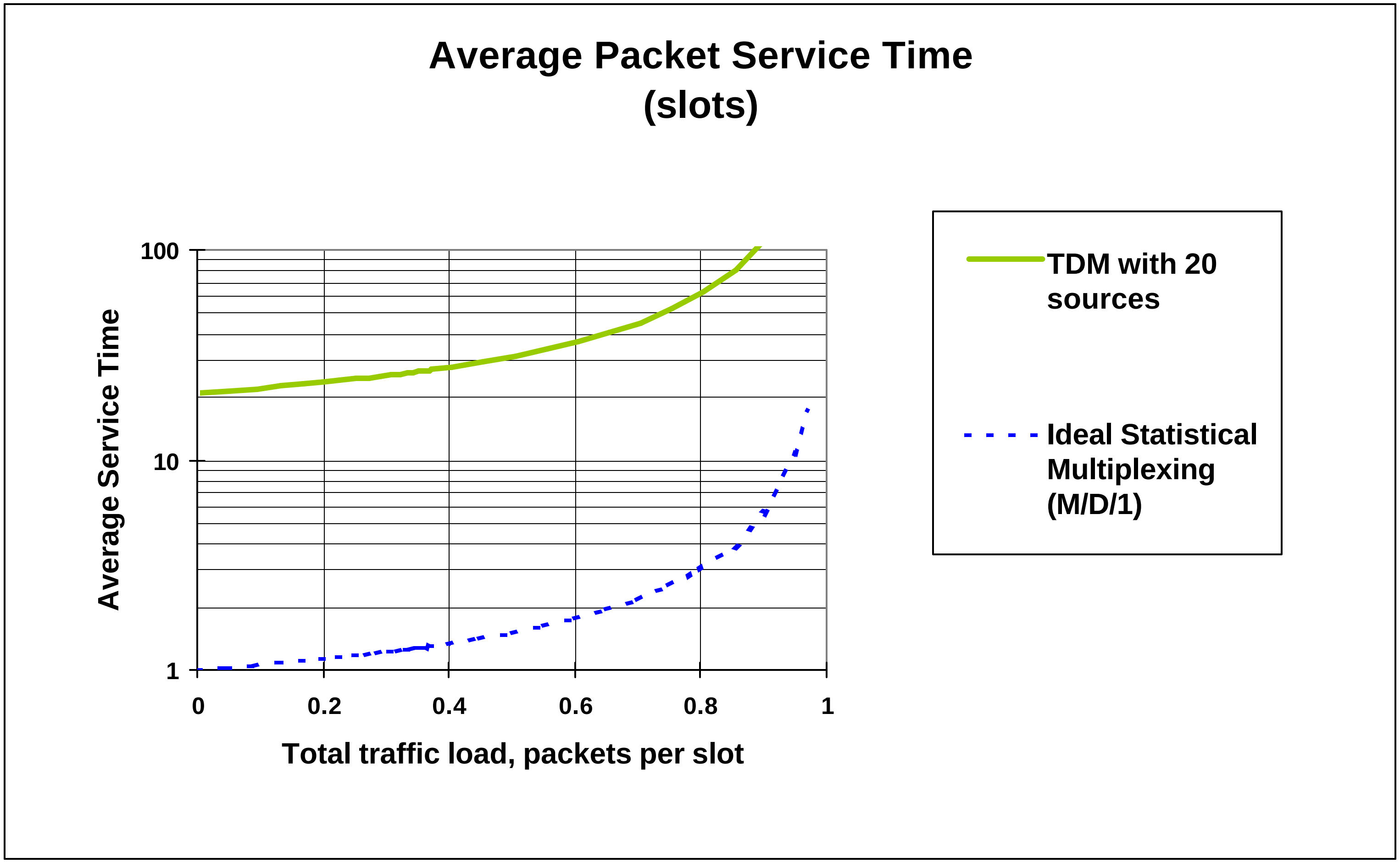
Circuit (tdm/fdm) vs. Packet switching



Delay formulas

- **M/G/1**

$$D = \overline{X} + \frac{l\,\overline{X}^2}{2(1 - l / m)}$$

- **M/M/1**

$$D = \overline{X} + \frac{l / m}{m - l}$$

- **M/D/1**

$$D = \overline{X} + \frac{l / m}{2(m - l)}$$

<u>Delay components:</u>

Service (transmission) time (LHS)

Queueing delay (RHS)

Use Little's Theorem to compute N, the average number of customers in the system

Blocking Probability

- A circuit switched network can be viewed as a Multi-server queueing system
  - Calls are blocked when no servers available - “busy signal”
  - For circuit switched network we are interested in the call blocking probability

- M/G/m/m system
  - Poisson call arrivals and General call duration distribution
  - m servers => m circuits
  - Last m indicated that the system can hold no more than m users

- Erlang B formula
  - Gives the probability that a caller finds all circuits busy

$$P_B = \frac{(l/m)^m / m!}{\sum_{n=0}^{m} (l/m)^n / n!}$$

Erlang B formula

- Used for sizing transmission line
  - How many circuits does the satellite need to support?
  - The number of circuits is a function of the blocking probability that we can tolerate
    - Systems are designed for a given load predictions and blocking probabilities (typically small)
- Example
  - Arrival rate = 4 calls per minute, average 3 minutes per call
  - How many circuits do we need to provision?
    - Depends on the blocking probability that we can tolerate

Circuits	P_B
20	1%
15	8%
7	30%