18.099b Problem Set 5a
Due: Thursday, April 1st (in class or before).

This assignment is to write a short paper on the *oscillation* of a function. It should be typeset in Tex. Arrange and express the material discussed below in whatever way you think best. Be sure to define all the terms I have italicised (even if I have not defined them here).

Given a function $f(x)$ and a real number c, we say that f *is bounded about* c if there exists some $\epsilon > 0$ such that

(i) f is defined on $[c-\epsilon, c+\epsilon]$ except possibly at c itself, and
(ii) the set $\{f(x) : x \in [c-\epsilon, c+\epsilon], x \neq c\}$ is bounded from above and below.

For any such ϵ let $A_\epsilon := \{f(x) : x \in [c-\epsilon, c+\epsilon], x \neq c\}$. The *oscillation of* f *at* c is defined to be the number

$$\operatorname{osc}_f(c) := \lim_{\epsilon \to 0} (\sup A_\epsilon - \inf A_\epsilon)$$

Show that this limit exists and hence $\operatorname{osc}_f(c)$ is well defined.

Show that for functions f bounded about a real number c, $\operatorname{osc}_f(c) = 0$ if and only if f is continuous or has a *removable singularity* at c. Give two examples of non-zero oscillation, one where c is a *jump singularity* and one where c is an *essential singularity*.

An intuitive way to describe what it means for a function to be continuous at a point c would be to say: "$f(x)$ can be made arbitrarily close to $f(c)$ by making x arbitrarily close to c". Fixing $\tau \geq 0$, give an analogous intuitive description of what it means for f to have oscillation τ at a point c. The idea is that oscillation measures the "amount" of discontinuity.

Suppose f is monotonically increasing and defined on an interval $[a, b]$. Note that f cannot have any removable singularities in $[a, b]$, and that that for any $c \in (a, b)$, f is bounded about c. Moreover, $\operatorname{osc}_f(c) = \lim_{x \to c^+} f(x) - \lim_{x \to c^-} f(x)$. Conclude that for any $\tau > 0$, the number of points in (a, b) at which f has oscillation greater than or equal to τ is at most $\frac{f(b) - f(a)}{\tau}$. Use this to show that f has at most countably many discontinuities in (a, b). (Recall that a set S is *countable* if there is a bijection between $\mathbb{N}$ and S).