18.099b Problem Set 5b

Due: Thursday, April 1st (in class or before).

This assignment is to write a short paper on the *completion* of a metric space. It should be typeset in Tex. Arrange and express the material discussed below in whatever way you think best. Be sure to define all the terms I have italicised (even if I have not defined them here).

Recall that a binary relation $\sim$ on a set S is called an *equivalence relation* if it satisfies the following properties:

Symmetry: $x \sim x$, for all $x \in S$.
Reflexivity: $x \sim y$ implies $y \sim x$, for all $x, y \in S$.
Transitivity: $x \sim y$ and $y \sim z$ implies $x \sim z$, for all $x, y, z \in S$.

A subset $A \subset S$ is called an *equivalence class* if the following two conditions are satisfied:

For all $a, a' \in A$, $a \sim a'$.
For all $x \in S$, if $x \sim a$ for some $a \in A$, then $x \in A$.

Show that every element $x \in S$ belongs to a unique equivalence class.

Suppose X is a metric space. We say two *Cauchy sequences* $\{p_n\}$ and $\{q_n\}$ in X are *equivalent*, denoted by $\{p_n\} \sim \{q_n\}$, if the sequence of real numbers $\{d(p_n, q_n)\}$ converges to zero. Show that $\sim$ is an equivalence relation.

Let X^* be the set of equivalence classes of Cauchy sequences in X. Given $P, Q \in X^*$ and $\{p_n\} \in P$ and $\{q_n\} \in Q$, define $\Delta(P, Q) := \lim_{n\to\infty} d(p_n, q_n)$. Show that Δ is well defined. (This entails proving two things: that the above limit always exists; and that $\Delta(P, Q)$ depends only on P and Q, and not on the particular choice of $\{p_n\} \in P$ and $\{q_n\} \in Q$.)

Show that Δ makes X^* into a *complete* metric space.

Let $\phi : X \to X^*$ be the mapping which assigns to every $p \in X$ the element in X^* which contains the Cauchy sequence all of whose terms are p. (Note that as $\sim$ is an equivalence relation, $\phi(p)$ is well defined). Show that ϕ is an *isometry* from X to X^*; that is, $\Delta(\phi(p), \phi(q)) = d(p, q)$ for all $p, q \in X$. In particular, ϕ is injective.

Show that $\phi(X)$ is *dense* in X^*. Show that $\phi(X) = X$ if and only if X is complete.

We call X^* the *completion* of X.

Consider the case when $X = \mathbb{Q}$ is the rational numbers with distance function $d(x, y) = |x - y|$. Show that there is a bijective isometry $\alpha : \mathbb{R} \to X^*$ such that $\alpha(p) = \phi(p)$ for all $p \in \mathbb{Q}$. That is, $\mathbb{R}$ is (isometric to) the completion of $\mathbb{Q}$.

Suppose we do not know what the real numbers are. The above construction might tempt us to define $\mathbb{R}$ as the completion of $\mathbb{Q}$ with respect to the distance function $|x-y|$. The problem with doing this is that our definitions of metric space, Cauchy sequence, and convergent sequence, all use the real numbers! Describe how we can modify things so that the above construction does give us a way of defining the reals from the rationals. (This is the construction of the reals given by Cantor – you don't need to carry this out, just sketch the idea).