18.099b Problem Set 6 (Corrected version)
Due: Thursday, April 15th.

This assignment is to write a short paper on *convex functions*. It should be typeset in Tex. Arrange and express the material discussed below in whatever way you think best. The paper should flow smoothly and be well motivated.

A real valued function f on an open (possibly unbounded) interval $I \subseteq \mathbb{R}$ is *convex* if for all $x, y \in I$

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

for all $0 < \lambda < 1$. Explain the geometric meaning of this definition; namely that the part of the graph of the function between any two points $x, y \in I$ lies below the chord joining $(x, f(x))$ and $(y, f(y))$.

Make precise and prove the statement that an increasing convex function applied to a convex function is again convex.

Show that if f is continuous and $f(\frac{x+y}{2}) \leq \frac{f(x)+f(y)}{2}$ for all $x, y \in I$. then f is convex. *Hint: Use the fact that if g and h are continuous functions with $g(a) > h(a)$ for some a, then $g > h$ on $(a-\delta, a+\delta)$ for some $\delta > 0$.*

Prove that f is convex on I if and only if for every $x < t < y$ in I

$$(y-x)f(t) \quad \leq \quad (y-t)f(x) + (t-x)f(y). \tag{1}$$

In particular if f is convex and $a \in I$, then the function $\dfrac{f(t)-f(a)}{t-a}$ is increasing on $I \setminus \{a\}$. Conclude that every convex function is continuous and has both right-hand and left-hand derivatives at every point in I.

Suppose f is convex and differentiable on I. Show that for every $a \in I$, the graph of f lies above the tangent line to the graph at $(a, f(a))$.

Suppose f is differentiable on I. Show that f is convex if and only if f' is increasing. *Hint: Use the mean value theorem and the inequality* (1).

Give (interesting) examples of convex elementary functions.

Prove Jensen's inequality: If f is convex on I, $x_1, \ldots, x_n \in I$, and $\lambda_1, \ldots, \lambda_n$ are non-negative real numbers such that $\sum_{i=1}^{n} \lambda_i = 1$, then

$$f(\sum_{i=1}^{n} \lambda_i x_i) \leq \sum_{i=1}^{n} \lambda_i f(x_i).$$

(Notice that in order for this to even make sene, $\sum_{i=1}^{n} \lambda_i x_i$ must also be in I. Why is this the case?).

As an application of Jensen's inequality, deduce the "inequality of the arithmetic and geometric means": if $x_1, \ldots, x_n$ are positive real numbers, then

$$(x_1 x_2 \cdots x_n)^{\frac{1}{n}} \leq \frac{1}{n} \sum_{i=1}^{n} x_i.$$