18.099b Problem Set 7a (Corrected)
Due: Thursday, April 29th.

This assignment is to write an essay on functions of *bounded variation*. It should be typeset in Tex. Arrange and express the material discussed below in whatever way you think best. The paper should flow smoothly and be well motivated.

Suppose $a \leq b$ are real numbers. Let $\mathcal{P}[a,b]$ denote the set of all *partitions* of the interval $[a,b]$. Suppose $f(x)$ is a function defined on an interval $[a,b]$. For any $P \in \mathcal{P}[a,b]$ denote by $S_f(P)$ the sum

$$S_f(P) := \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|$$

where P is the partition $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$. The *total variation* of $f(x)$ on $[a,b]$ is defined to be

$$V_f(a,b) \equiv \sup\{S_f(P) : P \in \mathcal{P}[a,b]\}.$$

That is, $V_f(a,b)$ is the supremum of the set of all $S_f(P)$ for all possible partitions P of $[a,b]$. If $V_f(a,b) < \infty$, then f is said to be of *bounded variation* on $[a,b]$.

Show that all monotonic functions are of bounded variation. In fact, the total variation of a monotonic function on $[a,b]$ is $|f(b) - f(a)|$.

Note that functions of bounded variation are bounded. However, there are functions that are bounded, and even continuous on a closed interval, but are not of bounded variation. *Hint: Consider* $x\sin(\frac{1}{x})$ *on* $[0,1]$.

Let $\mathrm{BV}[a,b]$ denote the set of functions of bounded variation on $[a,b]$. Show that $\mathrm{BV}[a,b]$ is a *linear space*: If $f, g \in \mathrm{BV}[a,b]$ and $c \in \mathbb{R}$, then $f + g \in \mathrm{BV}[a,b]$ and $cf \in \mathrm{BV}[a,b]$.

A function f on a closed interval $[a,b]$ is said to *satisfy a Lipschitz condition* if there exists a constant K such that

$$|f(x) - f(y)| \leq K|x - y|$$

for all $x, y \in [a,b]$. Show that if f satisfies a Lipschitz condition then it is of bounded variation. Use this to conclude that if $f(x)$ has a derivative at every point on $[a,b]$ and if $f'(x)$ is bounded on $[a,b]$, then $f \in \mathrm{BV}[a,b]$. *Hint: Mean value theorem.*

As an aside, discuss *contractions*, which are a special case of functions satisfying a Lipschitz condition. A function f on a closed interval $[a,b]$ is a *contraction* if there exists a constant $\lambda < 1$ such that

$$|f(x) - f(y)| \leq \lambda|x - y|$$

for all $x, y \in [a,b]$. Prove the fixed-point theorem which states that if f is a contraction on $[a,b]$ with values in $[a,b]$, then it has a unique *fixed-point* in $[a,b]$; that is, there exists a unique number $x \in [a,b]$ such that $f(x) = x$.

Getting back to bounded variation, prove the following striking fact: A function $f(x)$ defined on $[a,b]$ is of bounded variation if and only if it is the difference of two increasing functions on $[a,b]$. *Hint: Suppose* $f \in \mathrm{BV}[a,b]$. *Show that for any* $x \leq x'$ *from* $[a,b]$,

$$V_f(a,x') \geq V_f(a,x) + |f(x) - f(x')|.$$

Conclude that the function $g(x) = V_f(a,x)$ *and the function* $h(x) = f(x) + V_f(a,x)$ *are both increasing on* $[a,b]$.

Is a representation of $f \in \mathrm{BV}[a, b]$ as a difference of two increasing functions unique? Give several examples.

Prove that functions of bounded variation are integrable.