Theorem 1. *For all $n \geq 1$,*

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}. \tag{1}$$

Proof. (Version #1)

$n = 1$ is clear.

$$\begin{aligned} 1 + 2 + \cdots + n + (n+1) &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1)}{2} + \frac{2n+2}{2} \\ &= \frac{n^2+3n+2}{2} \\ &= \frac{(n+1)((n+1)+1)}{2} \end{aligned}$$

□

Proof. (Version #2)

We use induction on $n \geq 1$. For the base case, when $n = 1$, we need to show that $1 = \frac{1(1+1)}{2}$. Since $\frac{1(1+1)}{2} = \frac{1 \cdot 2}{2} = \frac{2}{2} = 1$, we have verified the base case. Now suppose, as our inductive hypothesis, that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for some fixed n. We need to show

$$1 + 2 + \cdots + n + (n+1) = \frac{(n+1)((n+1)+1)}{2}. \tag{2}$$

Computing the right hand side of (**??**) we see that

$$\begin{aligned} \frac{(n+1)((n+1)+1)}{2} &= \frac{n^2+3n+2}{2} \\ &= \frac{(n^2+n)+(2n+2)}{2} \\ &= \frac{n(n+1)+2(n+1)}{2} \\ &= \frac{n(n+1)}{2} + (n+1) \end{aligned}$$

But by the inductive hypothesis, $\frac{n(n+1)}{2} = 1 + 2 + \cdots + n$, and so we have

$$\frac{(n+1)((n+1)+1)}{2} = 1 + 2 + \cdots + n + (n+1).$$

This proves (**??**) and hence the theorem. □

Proof. (Version #3)

The identity (**??**) is clearly true when $n = 1$. Assuming it holds for a fixed n, we prove it for $n+1$ as follows:

$$\begin{aligned}
1 + 2 + \cdots + n + (n+1) &= \frac{n(n+1)}{2} + (n+1) \\
&= \frac{n(n+1)}{2} + \frac{2n+2}{2} \\
&= \frac{n^2 + 3n + 2}{2} \\
&= \frac{(n+1)((n+1)+1)}{2}.
\end{aligned}$$

Hence, by induction, identity (**??**) is true for all $n \geq 1$. □