LIMITS OF FUNCTIONS

Given $f : \mathbb{R} \to \mathbb{R}$, $\lim_{x\to p} f(x) = q$ means that $|f(x) - q| < \epsilon$ whenever $0 < |x - p| < \delta$.

Theorem 1. $\lim_{x\to p} f(x) = q$ *if and only if* $\lim_{n\to\infty} f(p_n) = q$ *whenever* $\lim_{n\to\infty} p_n = p$ *with* $p_n \neq p$.

Proof. Suppose $\lim_{x\to p} f(x) = q$ and let $\{p_n\}$ be any sequence of real numbers converging to p, but not equal to p. Let $\epsilon > 0$ be arbitrary. Then there exists a $\delta > 0$ such that $|f(x) - q| < \epsilon$ if $0 < |x - p| < \delta$. On the other hand, there is a natural number N such that for all $n > N$, $0 < |p_n - p| < \delta$. Hence for all $n > N$, $|f(p_n) - q| < \epsilon$, and so $\{f(p_n)\}$ converges to q.

Suppose $\lim_{x\to p} f(x) \neq q$. Then there exists some $\epsilon > 0$ such that for every $\delta > 0$ there exists some $x(\delta) \in \mathbb{R}$ for which $0 < |x(\delta) - p| < \delta$ but $|f(x(\delta)) - q| \geq \epsilon$. Hence $\lim_{n\to\infty} f(x(\frac{1}{n})) \neq q$ but $\lim_{n\to\infty} x(\frac{1}{n}) = p$. $\square$