LIMITS OF FUNCTIONS

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In this short note we discuss the notion of the limit of a function.

Given a function $f : \mathbb{R} \to \mathbb{R}$ and points $p, q \in \mathbb{R}$, we say that the *limit of f at p is q*, denoted by $\lim_{x \to p} f(x) = q$, to mean that for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - q| < \epsilon$ whenever $0 < |x - p| < \delta$. It is important here that we take $|x - p| \neq 0$ in the above defiition. That is, whether or not q is the limit of f at p does *not* depend on the value of f at p.

As the following theorem shows, this can be rephrased in terms of sequences.

Theorem 0.1. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function and $p, q \in \mathbb{R}$. Then the following are equivalent:*

(i) $\lim_{x \to p} f(x) = q$

(ii) *For any sequence $\{p_n\}$ with $\lim_{n \to \infty} p_n = p$ but $p_n \neq p$, $\lim_{n \to \infty} f(p_n) = q$.*

Proof. Suppose (i) holds and let $\{p_n\}$ be any sequence of real numbers converging to p, but not equal to p. Let $\epsilon > 0$ be arbitrary. Then there exists a $\delta > 0$ such that $|f(x) - q| < \epsilon$ if $0 < |x - p| < \delta$. On the other hand, there is a natural number N such that for all $n > N$, $0 < |p_n - p| < \delta$. Hence for all $n > N$, $|f(p_n) - q| < \epsilon$, and so $\{f(p_n)\}$ converges to q. That is, (ii) holds.

Conversely, suppose (i) fails. Then there exists some $\epsilon > 0$ such that for every $\delta > 0$ there exists some $x(\delta) \in \mathbb{R}$ for which $0 < |x(\delta) - p| < \delta$ but $|f(x(\delta)) - q| \geq \epsilon$. Hence the sequence $\{x(\frac{1}{n})\}$ witnesses the failure of (ii). □