

2.035: Selected Topics in Mathematics with Applications

SOLUTIONS to the FINAL EXAM – Spring 2007

Problem 1: Using first principles (i.e. don't use some formula like $d/dx(\partial F/\partial \phi') - \partial F/\partial \phi = 0$ but rather go through the steps of calculating δF and simplifying it etc.) determine the function $\phi \in \mathbf{A}$ that minimizes the functional

$$F\{\phi\} = \int_0^1 \left(\frac{1}{2}(\phi')^2 + \phi\phi'' + \phi \right) dx$$

over the set

$$\mathbf{A} = \{\phi | \phi : [0, 1] \rightarrow \mathbb{R}, \phi \in C^2[0, 1]\}.$$

Note that the values of ϕ are not specified at either end.

Solution: Taking the first variation of F gives

$$\delta F = \int_0^1 \left(\phi' \delta \phi' + \delta \phi \phi'' + \phi \delta \phi'' + \delta \phi \right) dx$$

Integrating the terms involving $\delta \phi'$ and $\delta \phi''$ by parts yields

$$\delta F = \left[\phi' \delta \phi \right]_0^1 - \int_0^1 \phi'' \delta \phi \, dx + \int_0^1 \phi'' \delta \phi \, dx + \left[\phi \delta \phi' \right]_0^1 - \int_0^1 \phi' \delta \phi' \, dx + \int_0^1 \delta \phi \, dx$$

Two terms in this equation cancel out. We integrate the integral involving $\delta \phi'$ by parts once more to get

$$\delta F = \left[\phi' \delta \phi \right]_0^1 + \left[\phi \delta \phi' \right]_0^1 - \left[\phi' \delta \phi \right]_0^1 + \int_0^1 \phi'' \delta \phi \, dx + \int_0^1 \delta \phi \, dx$$

which simplifies to

$$\delta F = \left[\phi \delta \phi' \right]_0^1 + \int_0^1 (\phi'' + 1) \delta \phi \, dx.$$

Since this has to vanish for all arbitrary variations $\delta \phi(x)$, $0 < x < 1$, and for all arbitrary $\delta \phi'(0)$ and $\delta \phi'(1)$, it follows that the minimizer must satisfy the boundary-value problem

$$\left. \begin{aligned} \phi'' + 1 &= 0 & \text{for } 0 \leq x \leq 1, \\ \phi(0) &= \phi(1) = 0. \end{aligned} \right\}$$

The solution of this is readily found to be

$$\phi(x) = \frac{1}{2}x(1-x) \quad \text{for } 0 \leq x \leq 1.$$

Problem 2: A problem of some importance involves navigation through a network of sensors. Suppose that the sensors are located at fixed positions and that one wishes to navigate in such a way that the navigating observer has minimal exposure to the sensors. Consider the following simple case of such a problem. See figure on last page.

A single sensor is located at the origin of the x, y -plane and one wishes to navigate from the point $A = (a, 0)$ to the point $B = (b \cos \beta, b \sin \beta)$. Let $(x(t), y(t))$ denote the location of the observer at time t so that the travel path is described parametrically by $x = x(t), y = y(t), 0 \leq t \leq T$. The exposure of the observer to the sensor is characterized by

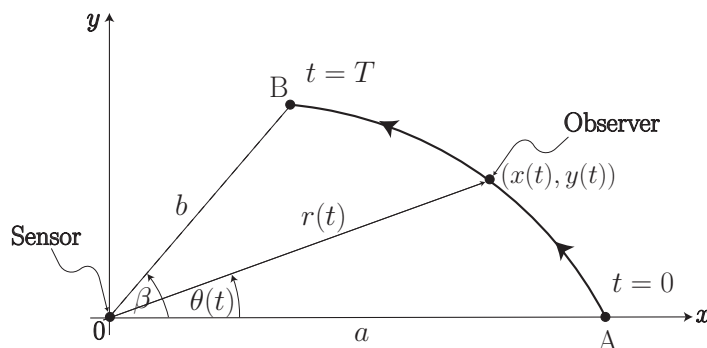
$$E\{x(t), y(t)\} = \int_0^T I(x(t), y(t)) v(t) dt$$

where $v(t)$ is the speed of the observer and the “sensitivity” I is given by

$$I(x, y) = \frac{1}{\sqrt{x^2 + y^2}}.$$

Determine the path from A to B that minimizes the exposure.

Remark: For further background on this problem including generalization to n sensors, see the paper Qingfeng Huang, ”Solving an Open Sensor Exposure Problem using Variational Calculus”, Technical Report WUCS-03-1, Washington University, Department of Computer Science and Engineering, St. Louis, Missouri, 2003 that I emailed you earlier in the term.



Problem 2: An observer moves along a path in the x, y -plane such that its rectangular cartesian coordinates at time t are $(x(t), y(t))$ (or its polar coordinates are $(r(t), \theta(t))$). A sensor is located at the origin.

Solution: Let

$$x = r \cos \theta, \quad y = r \sin \theta.$$

Then the speed

$$v = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{\dot{r}^2 + (r\dot{\theta})^2} = \sqrt{(r')^2 + r^2 \dot{\theta}}$$

where

$$r' = r'(\theta) = \frac{dr}{d\theta}.$$

Therefore substituting these into the given integral for the exposure E yields

$$E = \int_0^T \frac{1}{r} \sqrt{(r')^2 + r^2} \dot{\theta} dt = \int_0^\beta \frac{1}{r} \sqrt{(r')^2 + r^2} d\theta$$

where in changing the limits we have used the fact that $\theta = 0$ when $t = 0$ and $\theta = \beta$ when $t = T$. This is an integral of the standard form

$$E = \int_0^\beta f(\theta, r(\theta), r'(\theta)) d\theta \quad \text{where} \quad f(\theta, r, r') = \frac{1}{r} \sqrt{(r')^2 + r^2}.$$

The Euler equation is therefore

$$\frac{d}{d\theta} \left(\frac{\partial f}{\partial r'} \right) - \left(\frac{\partial f}{\partial r} \right) = 0.$$

Since $f(\theta, r, r')$ does not depend explicitly on θ , i.e. since $f(\theta, r, r') = f(r, r')$, the Euler equation can be integrated once (in general, not just in this problem,) to give

$$f - r' \frac{\partial f}{\partial r'} = \text{constant}.$$

Substituting the specific expression for f for this problem and simplifying leads to

$$\frac{r}{\sqrt{(r')^2 + r^2}} = \text{constant}$$

or

$$\frac{r'}{r} = \text{constant}.$$

Therefore we have

$$\frac{1}{r} \frac{dr}{d\theta} = c_1$$

which after integration yields

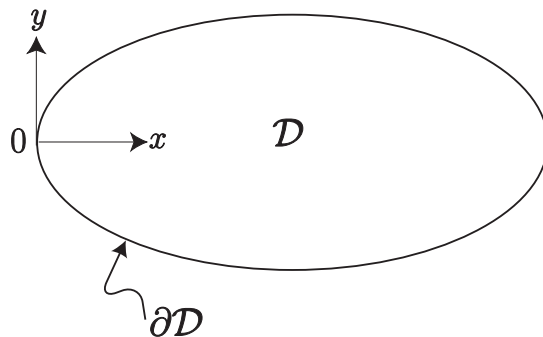
$$r = c_2 \exp(c_1 \theta)$$

where c_1 and c_2 are constants whose values can be found by using the boundary conditions $r = a$ at $\theta = 0$ and $r = b$ at $\theta = \beta$. This leads to

$$r(\theta) = a \exp \left[\theta \frac{\ln(b/a)}{\beta} \right].$$

Problem 3: Consider a domain \mathcal{D} of the x, y -plane whose boundary $\partial\mathcal{D}$ is smooth. Let \mathcal{A} denote the set of all functions $\phi(x, y)$ that are defined and suitably smooth on \mathcal{D} and which vanish on the boundary of \mathcal{D} : $\phi = 0$ for $(x, y) \in \partial\mathcal{D}$. Define the functional

$$F\{\phi\} = \int_{\mathcal{D}} \left(\frac{1}{2} \frac{\partial^2 \phi}{\partial x \partial y} + \frac{1}{2} \phi^2 + q\phi \right) dA$$



Problem 3: A domain \mathcal{D} in the x, y -plane with a smooth boundary $\partial\mathcal{D}$.

for all $\phi \in \mathbf{A}$ where $q = q(x, y)$ is a given function on \mathcal{D} . You are asked to minimize the functional F over the set \mathbf{A} . Determine the boundary-value problem that the minimizer must satisfy. (You do NOT need to solve it.)

Remark: Unfortunately I had a typo in the problem. I had MEANT to ask you to minimize

$$\int_{\mathcal{D}} \left[\frac{1}{2} \left(\frac{\partial^2 \phi}{\partial x \partial y} \right)^2 + \frac{1}{2} \phi^2 + q\phi \right] dA .$$

The problem as I stated it, with the square missing, is a badly-posed problem in general (as you will see below). The “formal calculation” below pertains to the problem as stated in the exam. Questions: What does this formal calculation actually mean? How does the result relate to a minimizer (if at all)?

Solution: Calculating the first variation of F gives

$$\delta F = \int_{\mathcal{D}} \left(\frac{1}{2} \delta \phi_{,xy} + \phi \delta \phi + q \delta \phi \right) dA = 0.$$

where $\phi_{,x} = \partial\phi/\partial x$, $\phi_{,xy} = \partial^2\phi/\partial x\partial y$ etc. To simplify this we write the first term in the form $\partial(\delta\phi_{,x}/2)/\partial y$ which allows us to use the divergence theorem, and thereby convert that term from being an area integral on \mathcal{D} into a boundary integral on $\partial\mathcal{D}$. We now carry out the calculation below where in going from the second equality to the third equality we have used the divergence theorem, and in going from the third equality to the fourth equality we have used the fact that $\delta\phi_{,x} = \delta\phi_{,n} n_x + \delta\phi_{,s} s_x$ (which follows from $\nabla\delta\phi = \delta\phi_{,x}\mathbf{i} + \delta\phi_{,y}\mathbf{j} =$

$\delta\phi_{,n}\mathbf{n} + \delta\phi_{,s}\mathbf{s}$):

$$\begin{aligned}
\delta F &= \int_{\mathcal{D}} \left(\frac{1}{2} \delta\phi_{,xy} + \phi\delta\phi + q\delta\phi \right) dA, \\
&= \int_{\mathcal{D}} \left[\left(\frac{1}{2} \delta\phi_{,x} \right)_{,y} + (\phi + q)\delta\phi \right] dA \\
&= \frac{1}{2} \int_{\partial\mathcal{D}} \delta\phi_{,x} n_y ds + \int_{\mathcal{D}} (\phi + q)\delta\phi dA \\
&= \frac{1}{2} \int_{\partial\mathcal{D}} (\delta\phi_{,n} n_x + \delta\phi_{,s} s_x) n_y ds + \int_{\mathcal{D}} (\phi + q)\delta\phi dA \\
&= \frac{1}{2} \int_{\partial\mathcal{D}} (\delta\phi_{,n} n_x n_y + \delta\phi_{,s} s_x n_y) ds + \int_{\mathcal{D}} (\phi + q)\delta\phi dA \\
&= \frac{1}{2} \int_{\partial\mathcal{D}} \delta\phi_{,n} n_x n_y ds + \frac{1}{2} \int_{\partial\mathcal{D}} \left(\frac{\partial}{\partial s} (\delta\phi s_x n_y) - \delta\phi \frac{\partial}{\partial s} (s_x n_y) \right) ds + \int_{\mathcal{D}} (\phi + q)\delta\phi dA
\end{aligned}$$

The second term in the middle integral vanishes because $\delta\phi = 0$ on $\partial\mathcal{D}$. Also, recall that if some field $f(x, y)$ varies smoothly along $\partial\mathcal{D}$, and if the curve $\partial\mathcal{D}$ itself is smooth, then

$$\int_{\partial\mathcal{D}} \frac{\partial f}{\partial s} ds = 0$$

since this is an integral over a closed curve. Therefore if $\partial\mathcal{D}$ is a smooth curve and $\delta\phi$ varies smoothly along it the first term in the middle integral also vanishes. Thus

$$\delta F = \frac{1}{2} \int_{\partial\mathcal{D}} \delta\phi_{,n} n_x n_y ds + \int_{\mathcal{D}} (\phi + q)\delta\phi dA.$$

Since this has to vanish for all admissible variations $\delta\phi$; and since $\delta\phi(x, y)$ is arbitrary on \mathcal{D} ; and since $\partial(\delta\phi)/\partial n$ is arbitrary on $\partial\mathcal{D}$ it follows that we must have

$$\left. \begin{aligned} \phi &= -q \quad \text{for} \quad (x, y) \in \mathcal{D} \\ n_x n_y &= 0 \quad \text{for} \quad (x, y) \in \partial\mathcal{D} \end{aligned} \right\}$$

Remarks: Note that the second condition here requires that either n_x or n_y vanish on the boundary which is a condition on the shape of the boundary $\partial\mathcal{D}$ (and not a condition on ϕ); specifically it says that the boundary must be composed of straight line segments that are parallel to the x - and y -axes (e.g. a rectangle). Of course this is not a smooth boundary and so the assumptions underlying the calculation above have to be re-examined. Also, note that the first requirement $\phi(x, y) = -q(x, y)$ on \mathcal{D} is inconsistent with the boundary condition $\phi = 0$ on $\partial\mathcal{D}$ unless the prescribed function $q(x, y)$ happens to vanish on the boundary.

Problem 4: Derive the two corner conditions

$$\left. \frac{\partial f}{\partial \phi'} \right|_{x=s-} = \left. \frac{\partial f}{\partial \phi'} \right|_{x=s+}, \quad \left(f - \phi' \frac{\partial f}{\partial \phi'} \right) \Big|_{x=s-} = \left(f - \phi' \frac{\partial f}{\partial \phi'} \right) \Big|_{x=s+},$$

that a minimizer of the functional

$$F\{\phi\} = \int_0^1 f(x, \phi, \phi') dx$$

must satisfy if the minimizer is continuous and has piecewise continuous derivatives; here $x = s$ denotes the location of a discontinuity in ϕ' and the value of s is not known a priori.

Remark: This question simply asks you to derive the results that I quoted in class without proof on Thursday May 3.

Solution: See any standard textbook on Calculus of Variations, e.g see Section 15 (pp. 61 - 63) of Calculus of Variations by I.M. Gelfand and S.V. Fomin, Prentice Hall, 1963

Problem 5: Consider the eigenvalue problem (1), (2) for finding the eigenfunctions $\phi(x)$ and eigenvalues λ of the boundary-value problem

$$a \phi''(x) - b \phi(x) + \lambda c \phi(x) = 0 \quad \text{for } 0 \leq x \leq 1, \quad (1)$$

$$\phi(0) = 0, \quad \phi(1) = 0, \quad (2)$$

where a, b, c and λ are *constants*. (Remark: There is NO typo in the equations above, just in case you expected it to be $b\phi'(x)$ instead of $b\phi(x)$.)

a) Show that minimizing the functional

$$F\{\phi\} = \int_0^1 \left[a (\phi')^2 + (b - \lambda c) (\phi)^2 \right] dx$$

over the set $A = \{\phi | \phi : [0, 1] \rightarrow \mathbb{R}, \phi \in C^2[0, 1], \phi(0) = \phi(1) = 0\}$ leads to equation (1) as its Euler equation.

b) Since the functional F involves the unknown eigenvalues λ , the preceding is not a particularly useful variational statement, which motivates us to look for other variational formulations of the eigenvalue problem. Show that minimizing the functional

$$G\{\phi\} = \int_0^1 \left[a (\phi')^2 + b (\phi)^2 \right] dx$$

over the set A , *subject to the constraint*

$$H\{\phi\} = \int_0^1 c (\phi)^2 dx = \text{constant}$$

leads to equation (1) as its Euler equation with λ as a Lagrange multiplier.

c) Show that minimizing the functional

$$I\{\phi\} = \frac{\int_0^1 \left[a(\phi')^2 + b(\phi)^2 \right] dx}{\int_0^1 c\phi^2 dx}$$

over the set **A** leads to equation (1) as its Euler equation, and that the values of I at the minimizers are the eigenvalues.

Solution:

a): Taking the first variation of F and setting it equal to zero gives

$$\delta F = \int_0^1 (2a\phi'\delta\phi' + 2(b - \lambda c)\phi\delta\phi) dx = 0$$

which on integrating the first term by parts gives

$$\left[2a\phi'\delta\phi \right]_0^1 + \int_0^1 (-2a\phi''\delta\phi + 2(b - \lambda c)\phi\delta\phi) dx = 0.$$

Since $\delta\phi(0) = \delta\phi(1) = 0$ the first term drops out and we have that

$$\int_0^1 (a\phi'' - (b - \lambda c)\phi) \delta\phi dx = 0$$

for all admissible variations $\delta\phi(x)$. Thus we have the Euler equation

$$a\phi'' - (b - \lambda c)\phi = 0 \quad \text{for } 0 \leq x \leq 1$$

which together with the boundary conditions

$$\phi(0) = \phi(1) = 0$$

is the original eigenvalue problem.

Remark: Recall that if λ_1 and $\phi_1(x)$ are an eigenvalue and eigenfunction respectively, then λ_1 and $\alpha\phi_1(x)$ is also an eigenvalue and eigenfunction for any scalar value α .

b): Taking the first variation of $G - \lambda H$ and setting it equal to zero gives

$$\delta G - \lambda\delta H = \int_0^1 (2a\phi'\delta\phi' + 2b\phi\delta\phi - \lambda 2c\phi\delta\phi) dx = 0.$$

Integrating the first term by parts, and using the fact that $\delta\phi(0) = \delta\phi(1) = 0$ to cancel out the boundary term, leads to

$$\int_0^1 (a\phi'' - b\phi + \lambda c\phi) \delta\phi dx = 0$$

for all admissible variations $\delta\phi(x)$. Thus we have the Euler equation

$$a\phi'' - b\phi - \lambda c\phi = 0 \quad \text{for } 0 \leq x \leq 1$$

which together with the boundary conditions

$$\phi(0) = \phi(1) = 0$$

and the side constraint

$$\int_0^1 c\phi^2 dx = \text{constant} \quad (*)$$

gives the minimizer. Note that this is the original eigenvalue problem PLUS the constraint condition (*). As noted in the previous remark, an eigenfunction of the original eigenvalue problem has an arbitrary amplitude and this side constraint serves simply as a way to determine the amplitude in some specific way (i.e. a normalization of the eigenfunctions).

c): Dividing G by H , and working with the functional I , is equivalent to working with a set of normalized functions ϕ from the start.

Problem 6: Suppose that an airplane flies in the x, y -plane at a constant speed v_o relative to the wind. The flight path begins and ends at the same location so that the path is a loop. The flight takes a given duration T . Assume that the wind velocity has a constant magnitude $c (< v_o)$ and a constant direction (which, without loss of generality, you can take to be the x -direction).

Along what closed curve should the plane fly if the flight path is to enclose the greatest area?

Solution: Since no one solved this problem, I shall keep it to use in a future year!! If you'd like to solve it in your spare time, the formulation of the boat crossing a river problem in the homework should help. The answer is an elliptical path whose major and minor axes are parallel to the x - and y -axes.