2.035: Midterm Exam - Part 2

Spring 2007

SOLUTION

Problem 1: Consider the set V of all 2×2 skew-symmetric matrices, i.e. matrices of the form

$$\boldsymbol{x} = \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix}, \qquad -\infty < x < \infty. \tag{1.1}$$

a) Define the addition of two vectors by

$$\boldsymbol{x} + \boldsymbol{y} = \begin{pmatrix} 0 & x+y \\ -x-y & 0 \end{pmatrix} \quad \text{where} \quad \boldsymbol{x} = \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} 0 & y \\ -y & 0 \end{pmatrix}, \tag{1.2}$$

and the multiplication of a vector $\boldsymbol{x}$ by the scalar α as

$$\alpha\boldsymbol{x} = \begin{pmatrix} 0 & \alpha x \\ -\alpha x & 0 \end{pmatrix}. \tag{1.3}$$

Then one can readily verify that $\boldsymbol{x} + \boldsymbol{y} \in \mathsf{V}$ whenever $\boldsymbol{x}$ and $\boldsymbol{y} \in \mathsf{V}$, and that $\alpha\boldsymbol{x} \in \mathsf{V}$ for every scalar α whenever $\boldsymbol{x} \in \mathsf{V}$. Thus V is a vector space.

b) Since

$$y \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix} - x \begin{pmatrix} 0 & y \\ -y & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{1.4}$$

it follows that $y\boldsymbol{x} + (-x)\boldsymbol{y} = \boldsymbol{o}$ for any two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. Thus *every* pair of vectors is linearly dependent. There is no pair of vectors that is linearly independent.

c) It follows from item (b) that the dimension of the vector space is 1.

d) If $\boldsymbol{x}$ and $\boldsymbol{y}$ are two vectors in V, and we define

$$\boldsymbol{x} \cdot \boldsymbol{y} = xy + (-x)(-y) = 2xy, \tag{1.5}$$

then we can verify that $\boldsymbol{x} \cdot \boldsymbol{y}$ has the properties

a) $\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{y} \cdot \boldsymbol{x}$,
b) $\boldsymbol{x} \cdot (\alpha\boldsymbol{y} + \beta\boldsymbol{z}) = \alpha\boldsymbol{x} \cdot \boldsymbol{y} + \beta\boldsymbol{x} \cdot \boldsymbol{z}$,
c) $\boldsymbol{x} \cdot \boldsymbol{x} \geq 0$ with $\boldsymbol{x} \cdot \boldsymbol{x} = 0$ if and only if $\boldsymbol{x} = \boldsymbol{o}$,

for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathsf{V}$ and all scalars α, β. Thus this is a proper definition of a scalar product.

e) Since the dimension of V is 1, any **one** (non null) vector provides a basis for it. Pick

$$\boldsymbol{e}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{1.6}$$

Note that the length of $\boldsymbol{e}'$ is

$$|\boldsymbol{e}'| = \sqrt{\boldsymbol{e}' \cdot \boldsymbol{e}'} = \sqrt{1+1} = \sqrt{2}. \tag{1.7}$$

Thus a basis composed of a unit vector is $\{\boldsymbol{e}\}$ where

$$\boldsymbol{e} = \begin{pmatrix} 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 \end{pmatrix}. \tag{1.8}$$

f) Let $\boldsymbol{A}$ be the transformation defined by

$$\boldsymbol{A}\boldsymbol{x} = \begin{pmatrix} 0 & 2x \\ -2x & 0 \end{pmatrix} \qquad \text{for all vectors} \quad \boldsymbol{x} \in \mathsf{V}. \tag{1.9}$$

We can readily verify that (i) the vector $\boldsymbol{A}\boldsymbol{x} \in \mathsf{V}$ for every $\boldsymbol{x} \in \mathsf{V}$, and (ii) that $\boldsymbol{A}(\alpha\boldsymbol{x}+\beta\boldsymbol{y}) = \alpha\boldsymbol{A}\boldsymbol{x} + \beta\boldsymbol{A}\boldsymbol{y}$ for all vectors $\boldsymbol{x}, \boldsymbol{y}$ and all scalars α, β. Therefore $\boldsymbol{A}$ is a linear transformation (tensor).

g) From item (d), $\boldsymbol{a} \cdot \boldsymbol{b} = 2(a)(b)$ for any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Thus it follows that

$$\boldsymbol{A}\boldsymbol{x} \cdot \boldsymbol{y} = 2(2x)(y) = 4xy$$

while

$$\boldsymbol{x} \cdot \boldsymbol{A}\boldsymbol{y} = 2(x)(2y) = 4xy.$$

Thus $\boldsymbol{A}\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{x} \cdot \boldsymbol{A}\boldsymbol{y}$ and so $\boldsymbol{A}$ is symmetric.

h) If $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{o}$ then $2x = 0$ and so $x = 0$ from which it follows that $\boldsymbol{x} = \boldsymbol{o}$. Thus $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{o}$ if and only if $\boldsymbol{x} = \boldsymbol{o}$. Thus $\boldsymbol{A}$ is non-singular.

i) To find the eigenvalues of $\boldsymbol{A}$ we must find λ such that $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$, i.e.

$$\begin{pmatrix} 0 & 2x \\ -2x & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda x \\ -\lambda x & 0 \end{pmatrix}. \tag{1.10}$$

Thus $\lambda = 2$.

Problem 2: Consider the 3-dimensional Euclidean vector space V which is comprised of all polynomials of degree ≤ 2; a typical vector in V has the form

$$\boldsymbol{x} = x(t) = c_o + c_1 t + c_2 t^2. \tag{2.1}$$

Addition and multiplication by a scalar are defined in the "natural way". The scalar product between two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined as

$$\boldsymbol{x} \cdot \boldsymbol{y} = \int_{-1}^{1} x(t)y(t)dt. \tag{2.2}$$

Clearly,

$$\boldsymbol{f}_1 = 1, \quad \boldsymbol{f}_2 = t, \quad \boldsymbol{f}_3 = t^2, \tag{2.3}$$

is a basis for V (since, as one can readily verify, they form a set of three linearly independent vectors). We now consider the following three vectors,

$$\boldsymbol{e}_1 = \boldsymbol{f}_1, \quad \boldsymbol{e}_2 = \boldsymbol{f}_2 + \alpha \boldsymbol{f}_1, \quad \boldsymbol{e}_3 = \boldsymbol{f}_3 + \beta \boldsymbol{f}_2 + \gamma \boldsymbol{f}_1, \tag{2.4}$$

and choose α, β and γ such that the three $\boldsymbol{e}$'s are orthogonal to each other.

Note that

$$\boldsymbol{e}_1 = 1. \tag{2.5}$$

First, enforcing $\boldsymbol{e}_2 \cdot \boldsymbol{e}_1 = 0$ leads to

$$\boldsymbol{e}_2 \cdot \boldsymbol{e}_1 = \int_{-1}^{1} (t + \alpha)(1)\, dt = 2\alpha = 0 \tag{2.6}$$

whence $\alpha = 0$ and therefore

$$\boldsymbol{e}_2 = t. \tag{2.7}$$

Next enforcing $\boldsymbol{e}_3 \cdot \boldsymbol{e}_1 = 0$ leads to

$$\boldsymbol{e}_3 \cdot \boldsymbol{e}_1 = \int_{-1}^{1} (t^2 + \beta t + \gamma)(1)\, dt = 2(\gamma + 1/3) = 0 \tag{2.8}$$

whence $\gamma = -1/3$. Finally enforcing $\boldsymbol{e}_3 \cdot \boldsymbol{e}_2 = 0$ leads to

$$\boldsymbol{e}_3 \cdot \boldsymbol{e}_2 = \int_{-1}^{1} (t^2 + \beta t + \gamma)(t)\, dt = 2\beta/3 = 0 \tag{2.9}$$

whence $\beta = 0$. Thus

$$\boldsymbol{e}_3 = t^3 - \frac{1}{3}. \tag{2.10}$$

The triplet of vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ are mutually orthogonal. In order to generate an orthonormal basis, we have to make each of these vectors a unit vector. To this end we calculate

$$|\boldsymbol{e}_1|^2 = \int_{-1}^{1} (1)^2\, dt = 2, \tag{2.11}$$

$$|\boldsymbol{e}_2|^2 = \int_{-1}^{1} (t)^2\, dt = 2/3, \tag{2.12}$$

$$|\boldsymbol{e}_3|^2 = \int_{-1}^{1} (t^2 - 1/3)^2\, dt = 8/45, \tag{2.13}$$

Thus an orthonormal basis is given by the three vectors

$$\frac{1}{\sqrt{2}}, \qquad \sqrt{\frac{3}{2}}t, \qquad \sqrt{\frac{45}{8}}(t^2 - 1/3). \tag{2.14}$$

Problem 3: Since $\boldsymbol{A}$ is symmetric it has n real eigenvalues $\alpha_1, \alpha_2, \ldots, \alpha_n$ and a corresponding set of orthonormal eigenvectors $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots \boldsymbol{a}_n\}$. The eigenvalues can be ordered as $\alpha_1 \leq \alpha_2 \ldots \leq \alpha_n$.

Throughout this problem we use $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots \boldsymbol{a}_n\}$ as the basis with respect to which we calculate all components of vectors and tensors. In particular, we know that the matrix of components of $\boldsymbol{A}$ in this basis is diagonal and that $A_{ii} = \alpha_i$. Let x_i be the ith component of a unit vector $\boldsymbol{x}$. Then

$$\boldsymbol{x} \cdot \boldsymbol{A}\boldsymbol{x} = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j \tag{3.1}$$

which in this principal basis of $\boldsymbol{A}$ simplifies to

$$\boldsymbol{x} \cdot \boldsymbol{A}\boldsymbol{x} = \alpha_1 x_1^2 + \alpha_2 x_2^2 + \ldots + \alpha_n x_n^2. \tag{3.2}$$

Since $\alpha_n \geq \alpha_{n-1} \ldots \geq \alpha_2 \geq \alpha_1$ it follows that

$$\alpha_1 x_1^2 + \alpha_2 x_2^2 + \ldots + \alpha_n x_n^2 \geq \alpha_1 x_1^2 + \alpha_1 x_2^2 + \ldots + \alpha_1 x_n^2 = \alpha_1(x_1^2 + x_2^2 + \ldots + x_n^2) = \alpha_1. \tag{3.3}$$

Therefore

$$\alpha_1 \leq \boldsymbol{x} \cdot \boldsymbol{A}\boldsymbol{x} \tag{3.4}$$

for all unit vectors $\boldsymbol{x}$. Since $\alpha_1 = \boldsymbol{x} \cdot \boldsymbol{A}\boldsymbol{x}$ when $\boldsymbol{x} = \boldsymbol{a}_1$ it follows that

$$\alpha_1 = \min\Big(\boldsymbol{x} \cdot \boldsymbol{A}\boldsymbol{x}\Big) \tag{3.5}$$

where the minimization is taken over all unit vectors $\boldsymbol{x} \in \mathsf{V}$.

Similarly, since $\alpha_1 \leq \alpha_2 \ldots \leq \alpha_n$ it follows that

$$\alpha_1 x_1^2 + \alpha_2 x_2^2 + \ldots + \alpha_n x_n^2 \leq \alpha_n x_1^2 + \alpha_n x_2^2 + \ldots + \alpha_n x_n^2 = \alpha_n(x_1^2 + x_2^2 + \ldots + x_n^2) = \alpha_n. \tag{3.6}$$

Therefore

$$\alpha_n \geq \boldsymbol{x} \cdot \boldsymbol{A}\boldsymbol{x} \tag{3.7}$$

for all unit vectors $\boldsymbol{x}$. Since $\alpha_n = \boldsymbol{x} \cdot \boldsymbol{A}\boldsymbol{x}$ when $\boldsymbol{x} = \boldsymbol{a}_n$ it follows that

$$\alpha_n = \max\Big(\boldsymbol{x} \cdot \boldsymbol{A}\boldsymbol{x}\Big) \tag{3.8}$$

where the maximization is taken over all unit vectors $\boldsymbol{x} \in \mathsf{V}$.

Problem 4: For any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, by the definition of the transpose of a tensor $\boldsymbol{A}$ we have

$$\boldsymbol{A}\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{A}^T \boldsymbol{b}. \tag{4.1}$$

Let

$$\boldsymbol{x} = \boldsymbol{A}\boldsymbol{a}, \qquad \boldsymbol{y} = \boldsymbol{A}^T \boldsymbol{b}. \tag{4.2}$$

Since $\boldsymbol{A}$ (and therefore $\boldsymbol{A}^T$) are non-singular (4.2) gives

$$\boldsymbol{a} = \boldsymbol{A}^{-1}\boldsymbol{x}, \qquad \boldsymbol{b} = \left(\boldsymbol{A}^T\right)^{-1} \boldsymbol{y}. \tag{4.3}$$

Substituting (4.3) into (4.1) gives

$$\boldsymbol{x} \cdot \left(\boldsymbol{A}^T\right)^{-1} \boldsymbol{y} = \boldsymbol{A}^{-1}\boldsymbol{x} \cdot \boldsymbol{y}. \tag{4.4}$$

Next, applying the identity (4.1) to the tensor $\boldsymbol{A}^{-1}$ allows us to write the last term in (4.4) as

$$\boldsymbol{A}^{-1}\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{x} \cdot \left(\boldsymbol{A}^{-1}\right)^T \boldsymbol{y}. \tag{4.5}$$

Combining, (4.4) and (4.5) with gives

$$\boldsymbol{x} \cdot \left(\boldsymbol{A}^T\right)^{-1} \boldsymbol{y} = \boldsymbol{x} \cdot \left(\boldsymbol{A}^{-1}\right)^T \boldsymbol{y}. \tag{4.6}$$

Since this must hold for all vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ it follows that

$$\left(\boldsymbol{A}^{-1}\right)^T = \left(\boldsymbol{A}^T\right)^{-1}. \tag{4.7}$$

Problem 5: Motivated by geometry, consider the projection tensor

$$\mathbf{\Pi} = \boldsymbol{I} - \boldsymbol{n} \otimes \boldsymbol{n} \tag{5.1}$$

where $\boldsymbol{n}$ is a unit vector. Then

$$\mathbf{\Pi}^2 = (\boldsymbol{I} - \boldsymbol{n} \otimes \boldsymbol{n})(\boldsymbol{I} - \boldsymbol{n} \otimes \boldsymbol{n}) = \boldsymbol{I} - \boldsymbol{n} \otimes \boldsymbol{n} - \boldsymbol{n} \otimes \boldsymbol{n} + (\boldsymbol{n} \otimes \boldsymbol{n})(\boldsymbol{n} \otimes \boldsymbol{n}). \tag{5.2}$$

On using the identity $(\boldsymbol{a} \otimes \boldsymbol{b})(\boldsymbol{c} \otimes \boldsymbol{d}) = (\boldsymbol{b} \cdot \boldsymbol{c})(\boldsymbol{a} \otimes \boldsymbol{d})$ and the fact that $\boldsymbol{n} \cdot \boldsymbol{n} = 1$ this simplifies to

$$\mathbf{\Pi}^2 = \boldsymbol{I} - \boldsymbol{n} \otimes \boldsymbol{n} - \boldsymbol{n} \otimes \boldsymbol{n} + \boldsymbol{n} \otimes \boldsymbol{n} = \boldsymbol{I} - \boldsymbol{n} \otimes \boldsymbol{n} = \mathbf{\Pi}. \tag{5.3}$$

Thus

$$\mathbf{\Pi} = \mathbf{\Pi}^2. \tag{5.4}$$

Operating on this by $\mathbf{\Pi}$ gives $\mathbf{\Pi}^2 = \mathbf{\Pi}^3$ and operating on that by $\mathbf{\Pi}$ gives $\mathbf{\Pi}^3 = \mathbf{\Pi}^4$ and so on. Thus, for the projection tensor $\mathbf{\Pi}$

$$\mathbf{\Pi} = \mathbf{\Pi}^2 = \mathbf{\Pi}^3 = \mathbf{\Pi}^4 = \dots \tag{5.5}$$

Problem 6: Since $\boldsymbol{c}$ and $\boldsymbol{d}$ are distinct non-zero vectors belonging to a 3-dimensional vector space, they define a plane in that space. The figure shows this plane; the figure on the left corresponds to the case when the angle between these two vectors is obtuse while the figure on the right corresponds to the case when the angle is acute. The unit vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4$ have been selected such that the scalar products $\boldsymbol{c} \cdot \boldsymbol{x}_k$ and $\boldsymbol{d} \cdot \boldsymbol{x}_k$ have definite signs, either positive or negative as shown in the figure. In the case depicted on the left, $(\boldsymbol{c} \cdot \boldsymbol{x}_3)(\boldsymbol{d} \cdot \boldsymbol{x}_3) > 0$ while $(\boldsymbol{c} \cdot \boldsymbol{x}_4)(\boldsymbol{d} \cdot \boldsymbol{x}_4) < 0$. Similarly in the case depicted on the right, $(\boldsymbol{c} \cdot \boldsymbol{x}_1)(\boldsymbol{d} \cdot \boldsymbol{x}_1) > 0$ while $(\boldsymbol{c} \cdot \boldsymbol{x}_2)(\boldsymbol{d} \cdot \boldsymbol{x}_2) < 0$. Thus in either case, one can always find a unit vector $\boldsymbol{x}$ that makes $(\boldsymbol{c} \cdot \boldsymbol{x})(\boldsymbol{d} \cdot \boldsymbol{x}) > 0$, and some other unit vector $\boldsymbol{x}$ that makes $(\boldsymbol{c} \cdot \boldsymbol{x})(\boldsymbol{d} \cdot \boldsymbol{x}) < 0$.

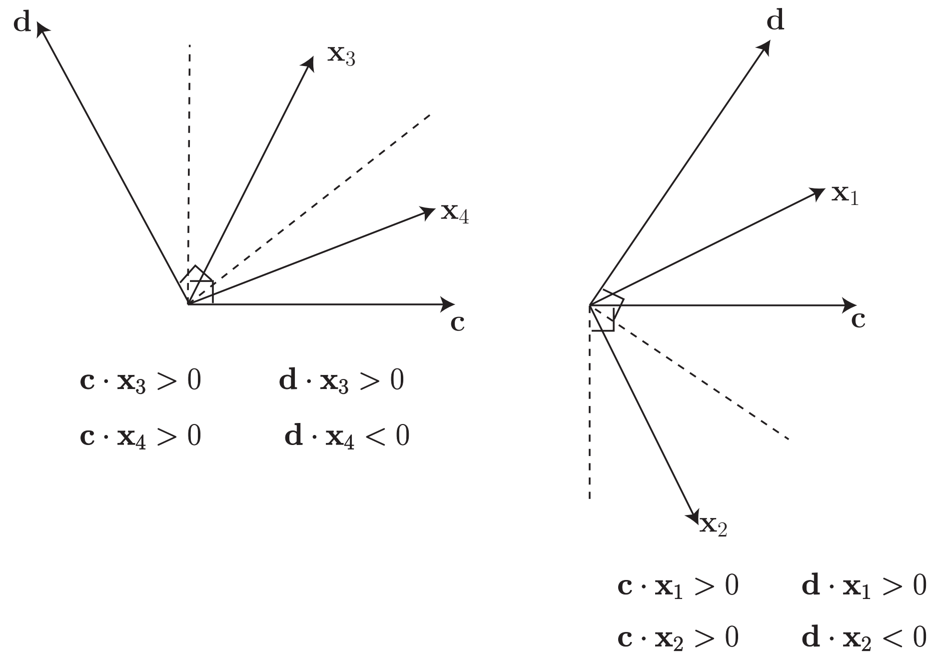

Figure 6.1: The plane defined by the vectors $\boldsymbol{c}$ and $\boldsymbol{d}$ with four unit vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4$ for which the scalar products $\boldsymbol{c} \cdot \boldsymbol{x}_k$ and $\boldsymbol{d} \cdot \boldsymbol{x}_k$ have definite signs, either positive or negative.

Let $\boldsymbol{B}$ be the symmetric tensor defined by

$$\boldsymbol{B} = \boldsymbol{c} \otimes \boldsymbol{d} + \boldsymbol{d} \otimes \boldsymbol{c}. \tag{6.1}$$

Since $\boldsymbol{B}$ is symmetric it has three real eigenvalues which we can order as $\beta_1 \leq \beta_2 \leq \beta_3$. First: for any unit vector $\boldsymbol{x}$

$$\boldsymbol{B}\boldsymbol{x} \cdot \boldsymbol{x} = (\boldsymbol{d} \cdot \boldsymbol{x})(\boldsymbol{c} \cdot \boldsymbol{x}). \tag{6.2}$$

From Problem 3 we know that

$$\beta_1 = \min\Big(\boldsymbol{x} \cdot \boldsymbol{B}\boldsymbol{x}\Big) = \min\Big((\boldsymbol{d} \cdot \boldsymbol{x})(\boldsymbol{c} \cdot \boldsymbol{x})\Big) \tag{6.3}$$

where the minimization is taken over all unit vectors $\boldsymbol{x} \in \mathsf{V}$. However from the first part of this problem we know that there is some unit vector $\boldsymbol{x}$ for which $(\boldsymbol{d} \cdot \boldsymbol{x})(\boldsymbol{c} \cdot \boldsymbol{x}) < 0$. Therefore $(\boldsymbol{d} \cdot \boldsymbol{x})(\boldsymbol{c} \cdot \boldsymbol{x})$ is negative for some value of $\boldsymbol{x}$ and so its minimum value is necessarily negative as well. Thus

$$\beta_1 < 0. \tag{6.4}$$

In an entirely analogous way, from Problem 3 we know that

$$\beta_3 = \max\Big(\boldsymbol{x} \cdot \boldsymbol{B}\boldsymbol{x}\Big) \tag{6.5}$$

where the maximization is taken over all unit vectors $\boldsymbol{x} \in \mathsf{V}$. However from the first part of this problem we know that there is some unit vector $\boldsymbol{x}$ for which $(\boldsymbol{d}\cdot\boldsymbol{x})(\boldsymbol{c}\cdot\boldsymbol{x}) > 0$. Therefore $(\boldsymbol{d}\cdot\boldsymbol{x})(\boldsymbol{c}\cdot\boldsymbol{x})$ is positive for some value of $\boldsymbol{x}$ and so its maximum value is necessarily positive as well. Thus

$$\beta_3 > 0. \tag{6.6}$$

Finally, as noted previously the vectors $\boldsymbol{c}$ and $\boldsymbol{d}$ define a plane. Let $\boldsymbol{e}$ be a vector that is normal to that plane so that $\boldsymbol{e} \cdot \boldsymbol{c} = \boldsymbol{e} \cdot \boldsymbol{d} = 0$. Then

$$\boldsymbol{B}\boldsymbol{e} = (\boldsymbol{d} \cdot \boldsymbol{e})\boldsymbol{c} + (\boldsymbol{c} \cdot \boldsymbol{e})\boldsymbol{d} = \boldsymbol{o} \tag{6.7}$$

(which we can write as $\boldsymbol{B}\boldsymbol{e} = 0\boldsymbol{e}$) and so 0 is an eigenvalue of $\boldsymbol{B}$. Thus

$$\beta_2 = 0. \tag{6.8}$$

The eigenvalues λ_i of the symmetric tensor $\boldsymbol{C}$ defined by

$$\boldsymbol{C} = \boldsymbol{I} + \boldsymbol{c} \otimes \boldsymbol{d} + \boldsymbol{d} \otimes \boldsymbol{c} = \boldsymbol{I} + \boldsymbol{B}$$

are related to the eigenvalues β_i of $\boldsymbol{B}$ by $\lambda_i = 1 + \beta_i$. Thus the eigenvalues of $\boldsymbol{C}$ have the property that

$$\lambda_1 < 1, \qquad \lambda_2 = 1, \qquad \lambda_3 > 1. \tag{6.9}$$