2.035: Midterm Exam - Part 1 (In class)
Spring 2004

1.5 hours

You may use the notes you took in class but no other sources.

Please give reasons justifying each (nontrivial) step in your calculations.

Problem 1: Let R be a 3-dimensional Euclidean vector space and let $\{\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3\}$ be an arbitrary (not necessarily orthonormal) basis for R . Define a set of vectors $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ by

$$\left.\begin{aligned} \boldsymbol{e}_1 &= \boldsymbol{f}_1, \\ \boldsymbol{e}_2 &= \boldsymbol{f}_2 + c_{21}\boldsymbol{e}_1, \\ \boldsymbol{e}_3 &= \boldsymbol{f}_3 + c_{31}\boldsymbol{e}_1 + c_{32}\boldsymbol{e}_2. \end{aligned}\right\}$$

i) Calculate the values of the scalars c_{21}, c_{31} and c_{32} that makes $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ a mutually orthogonal set of vectors.

ii) Is the set $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ linearly independent?

iii) Does $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ form an orthonormal basis for R?

Problem 2: Let R be the Euclidean vector space consisting of "trigonometric polynomials", a typical vector $\boldsymbol{p}$ having the form

$$\boldsymbol{p} = p(t) = \sum_{n=0}^{2} \alpha_n \cos nt \qquad \text{where the } \alpha' s \text{ span all real numbers.}$$

The natural operations of addition and multiplication by a scalar are in force. The scalar product between two vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ is taken to be

$$\boldsymbol{p} \cdot \boldsymbol{q} = \int_0^{2\pi} p(t)q(t)\, dt.$$

Let $\boldsymbol{A}$ be the tensor that carries a vector $\boldsymbol{p} = p(t)$ into its second derivative:

$$\boldsymbol{A}\boldsymbol{p} = p''(t).$$

i) Is $\boldsymbol{A}$ singular or nonsingular?

ii) Is $\boldsymbol{A}$ symmetric?

iii) Determine the eigenvalues of $\boldsymbol{A}$.

Problem 3: Let R be an arbitrary 3-dimensional vector space and let $\boldsymbol{A}$ be a linear transformation on R. (Note that R might not be Euclidean and $\boldsymbol{A}$ might not be symmetric.) Suppose that $\boldsymbol{A}$ has three real eigenvalues $\alpha_1, \alpha_2, \alpha_3$, and suppose that they are distinct: $\alpha_1 \neq \alpha_2 \neq \alpha_3 \neq \alpha_1$. Let $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ be the corresponding eigenvectors.

i) Show that any pair of these eigenvectors, e.g. $\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$, is a linearly independent pair of vectors.

ii) Next show that $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$ is a linearly independent set of vectors.