

2.035: Midterm Exam - Part 2 (Take home)

Spring 2004

“Examinations are formidable even to the best prepared, for the greatest fool may ask more than the wisest person may answer.”

Charles Caleb Colton (1780-1832)

INSTRUCTIONS:

- Do not spend more than 3 hours.
 - Please give reasons justifying each (nontrivial) step in your calculations.
 - You may use the notes you took in class, Chapters 1, 2 and 3 of the text, and any handouts originating from me.
 - No other sources are to be used (not even the appendices of the text).
 - Your completed solutions are due no later than 9:30 AM on Wednesday April 7.
 - Please include, on the first page of your solutions, a signed statement confirming that you adhered to the time limit and the permitted resources.
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Problem 1: (Knowles 3.24) A tensor \mathbf{T} is symmetric, orthogonal and positive definite. Determine \mathbf{T} .

Problem 2: (Essentially Knowles 1.18) Let \mathbf{R} be the 3-dimensional Euclidean vector space of polynomials of degree not exceeding two, where the scalar product between two vectors $\mathbf{f} = f(t)$ and $\mathbf{g} = g(t)$ is defined by

$$\mathbf{f} \cdot \mathbf{g} = \int_{-1}^1 f(t)g(t) dt.$$

- i) Show that $\mathbf{f}_1 = 1, \mathbf{f}_2 = t, \mathbf{f}_3 = t^2$ is a basis for \mathbf{R} .
 - ii) Find an orthonormal basis for \mathbf{R} .
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Problem 3: (Based on Knowles 3.17) Let \mathbf{A} and \mathbf{B} be two symmetric tensors whose matrices of components in an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ are

$$[\mathbf{A}] = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \quad \text{and} \quad [\mathbf{B}] = \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix} \quad \text{respectively.}$$

- i) Do \mathbf{A} and \mathbf{B} have a common principal basis?
 - ii) Determine a principal basis for \mathbf{A} .
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Problem 4: (Essentially Knowles 3.18) If a tensor \mathbf{P} has the property $\mathbf{P}\mathbf{P}^T = \mathbf{I}$ show that

- i) \mathbf{P} is nonsingular,
 - ii) $\mathbf{P}^T\mathbf{P} = \mathbf{I}$, and
 - iii) \mathbf{P} is orthogonal, i.e. that \mathbf{P} preserves length ($\Leftrightarrow |\mathbf{P}\mathbf{x}| = |\mathbf{x}|$ for all vectors \mathbf{x}).
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Problem 5: (Essentially Knowles 3.10, 3.26). Let \mathbf{A} be a skew-symmetric tensor on a finite dimensional Euclidean vector space.

- i) If \mathbf{A} has a real eigenvalue α , show that $\alpha = 0$.
 - ii) Show that $\mathbf{I} + \mathbf{A}$ and $\mathbf{I} - \mathbf{A}$ are both nonsingular tensors.
 - iii) Show that $\mathbf{I} + \mathbf{A}$ and $\mathbf{I} - \mathbf{A}$ commute, i.e. that $(\mathbf{I} + \mathbf{A})(\mathbf{I} - \mathbf{A}) = (\mathbf{I} - \mathbf{A})(\mathbf{I} + \mathbf{A})$.
 - iv) Show that $(\mathbf{I} - \mathbf{A})(\mathbf{I} + \mathbf{A})^{-1}$ is an orthogonal tensor.
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Problem 6: (Essentially Knowles 2.18) Let \mathbf{A} be a symmetric tensor on a n -dimensional Euclidean vector space. Suppose that \mathbf{A} has distinct eigenvalues $\alpha_1, \alpha_2, \dots, \alpha_n$ and a corresponding set of orthonormal eigenvectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$.

- i) For any positive integer m , show that \mathbf{A}^m (defined as $\underbrace{\mathbf{A}\mathbf{A} \dots \mathbf{A}}_{m \text{ times}}$) has eigenvalues $\alpha_1^m, \alpha_2^m, \dots, \alpha_n^m$ and corresponding eigenvectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$.
 - ii) Let $p(x) = \sum_{k=0}^n c_k x^k$ be an arbitrary polynomial of degree n where the c 's are real numbers. Let \mathbf{P} be the tensor defined by $\mathbf{P} = \mathbf{P}(\mathbf{A}) = \sum_{k=0}^n c_k \mathbf{A}^k$ where $\mathbf{A}^0 = \mathbf{I}$. Show that the eigenvalues of \mathbf{P} are $p(\alpha_1), p(\alpha_2), \dots, p(\alpha_n)$ and that the corresponding eigenvectors are $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$.
 - iii) Consider the special case where $p(x)$ is the characteristic polynomial of \mathbf{A} , i.e. $p(x) = \det[\mathbf{A} - x\mathbf{I}]$. Show that the corresponding tensor $\mathbf{P}(\mathbf{A})$ is the null tensor \mathbf{O} .
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