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Massachusetts Institute of Technology

A Finite Volume Model for The Variable-Density Parabolic Equation in Ocean Acoustics: Derivation, Implementation and Validation

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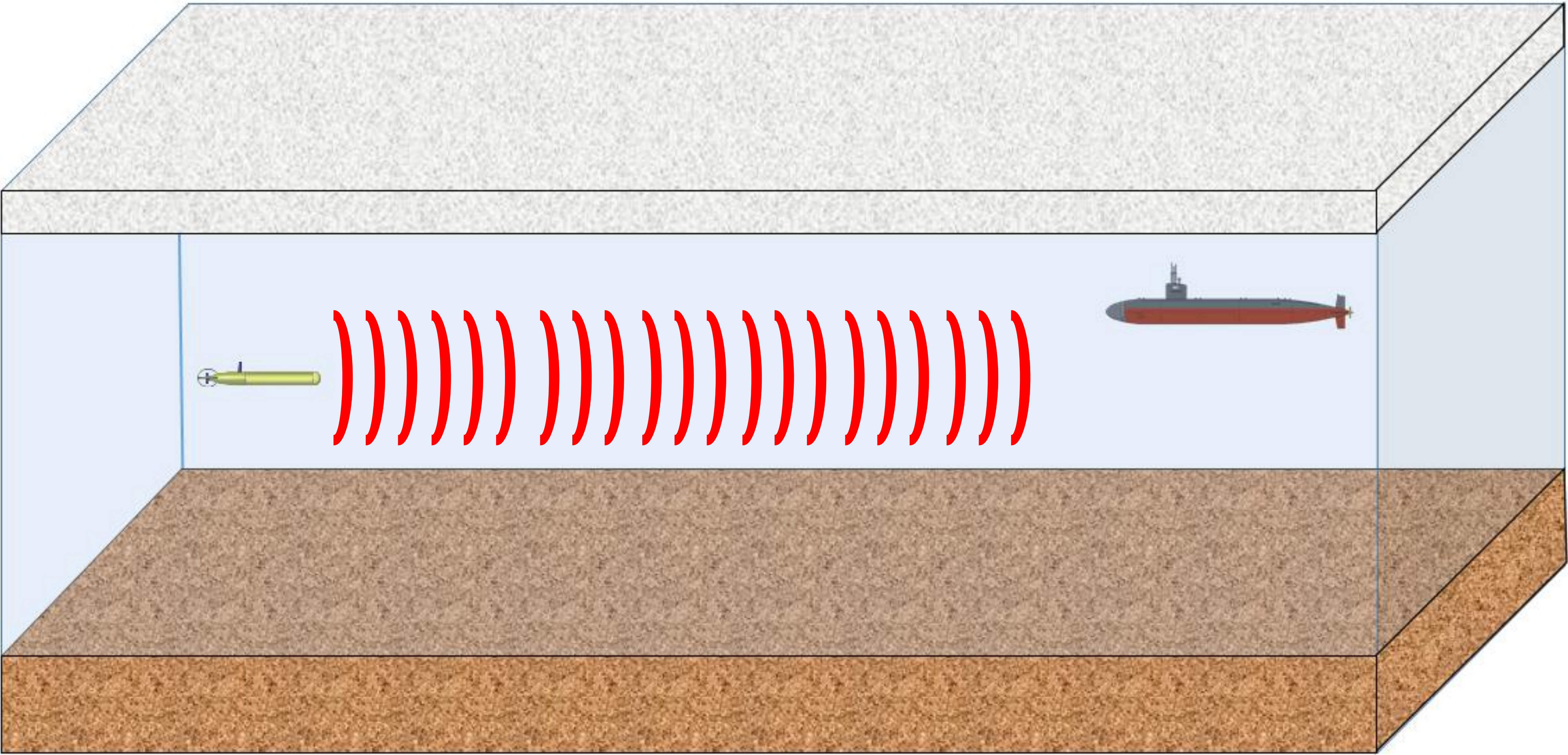
Why ocean acoustics?

- Underwater sound propagation

- Commercial interests:
  - Submerged cables
  - Resources

- Marine Science



Governing equations - I:

- For single frequency continuous wave, the inhomogeneous wave pressure equation:

$$\rho \nabla \cdot \left(\frac{1}{\rho} \nabla P \right) + k^2 P = 0$$

- Propagation from a point source – use cylindrical coordinates:

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\rho \frac{\partial}{\partial \theta}\left(\frac{1}{\rho}\frac{\partial}{\partial \theta} \right) + \rho \frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial}{\partial z} \right) + k^2\left(r, \theta, z\right) \right] P = 0$$

Governing equations - II:

- Postulate solution to be of the form:

$$P\left(r,\theta,z\right)=u\left(r,\theta,z\right)H_0^{(1)}\left(k_0 r\right)$$

- Parabolic equation for the outgoing wave:

$$\frac{\partial\psi}{\partial r}=\left(-ik_0+ik_0\sqrt{1+X^{+}+Y^{+}}\right)\psi$$

Where:

$$X^{+}=n^2\left(r,\theta,z\right)-1+\frac{1}{k_0^2}\rho\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial}{\partial z}\right)$$

$$Y^{+}=\frac{1}{k_0^2 r^2}\rho\frac{\partial}{\partial\theta}\left(\frac{1}{\rho}\frac{\partial}{\partial\theta}\right)$$

Governing equations - III:

- For narrow angle propagation:

$$\sqrt{1+X^{+}+Y^{+}} \approx 1+\frac{1}{2}X^{+}+\frac{1}{2}Y^{+}$$

- Variable-Density Parabolic equation (Tappert, 1977):

$$\frac{\partial \psi}{\partial r} = \frac{\rho i}{2k_0}\left(\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial \psi}{\partial z}\right)+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\rho}\frac{\partial \psi}{\partial \theta}\right)\right)+\frac{ik_0}{2}\left(n^2-1\right)\psi$$

F. D. Tappert. The parabolic approximation method. In J. B. Keller and J. S. Papadakis, editors, Wave Propagation and Underwater Acoustics, volume 70 of Lecture Notes in Physics, Berlin Springer Verlag, page 224, 1977

FV implementation: 2D PE - constant density case

- Governing equation reduces to (with attenuation):

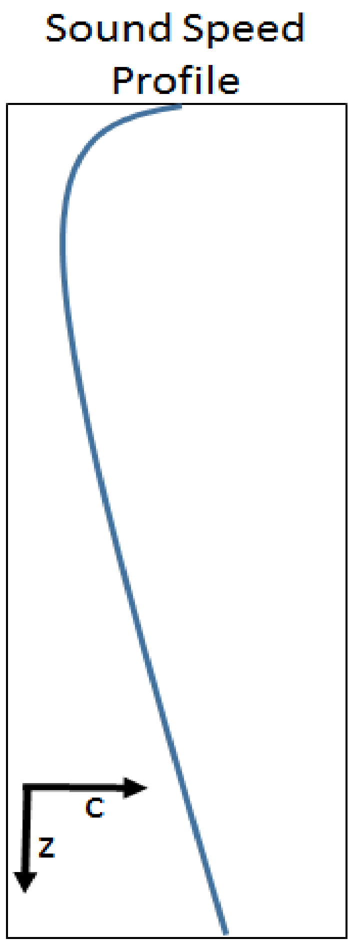
$$\frac{\partial \psi}{\partial r} = \frac{i}{2k_0}\frac{\partial^2 \psi}{\partial z^2} + \frac{ik_0}{2}\left(n^2 - 1 + i\nu\right)\psi$$

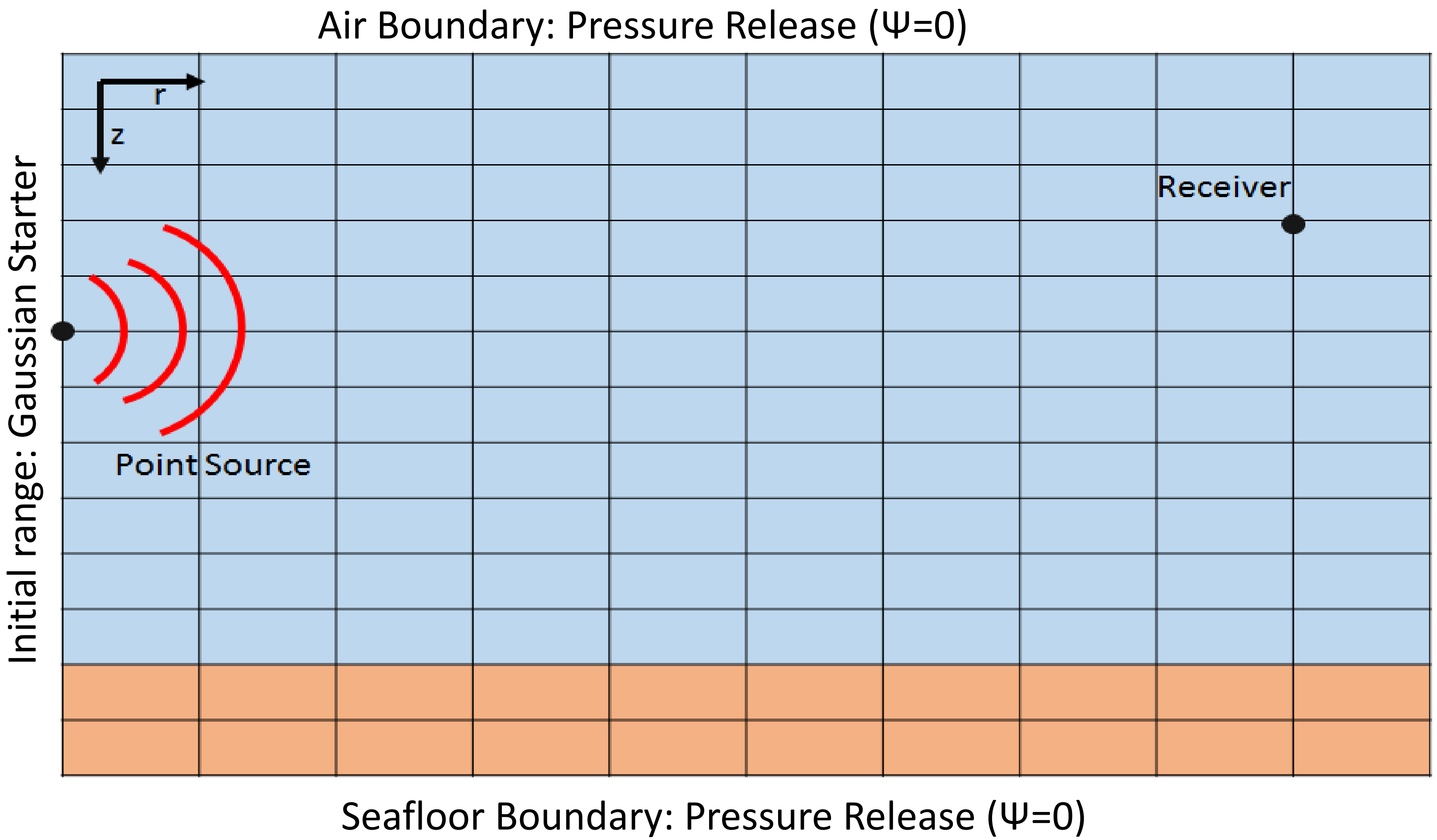
"Time"marching

Diffusion term

Forcing term

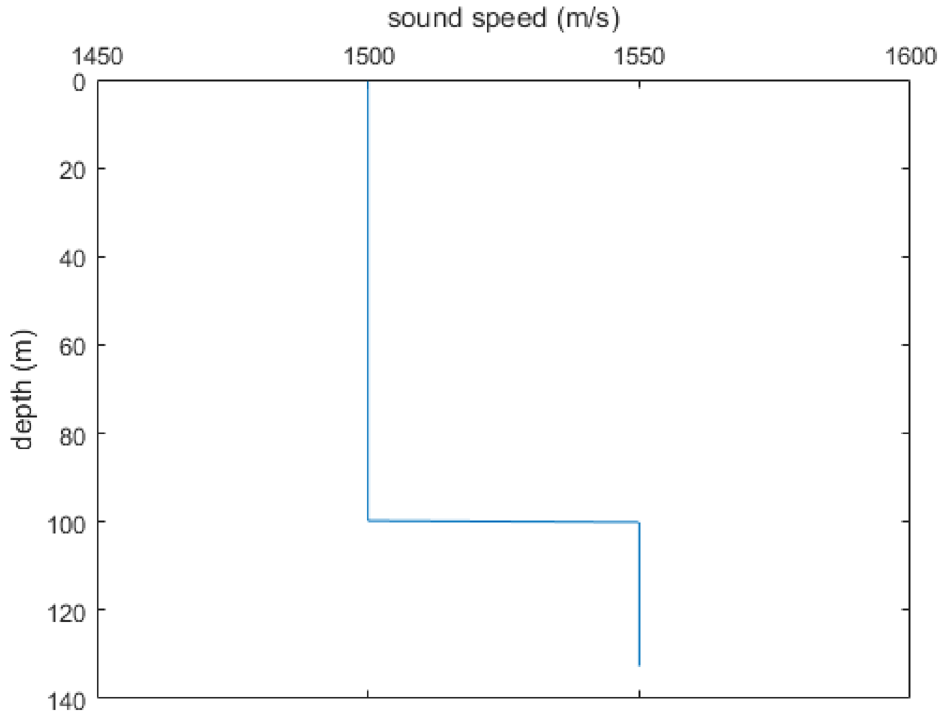
FV implementation: 2D PE - constant density case





FV implementation: 2D PE - constant density case

- Isovelocity shallow water Case (ISWC) (Jensen and Kuperman, 1979)



F. Jensen and W. A. Kuperman, "Environmental Acoustical Modeling at Saclant Cen", Saclant Cen Rep. SR – 34 (1979)

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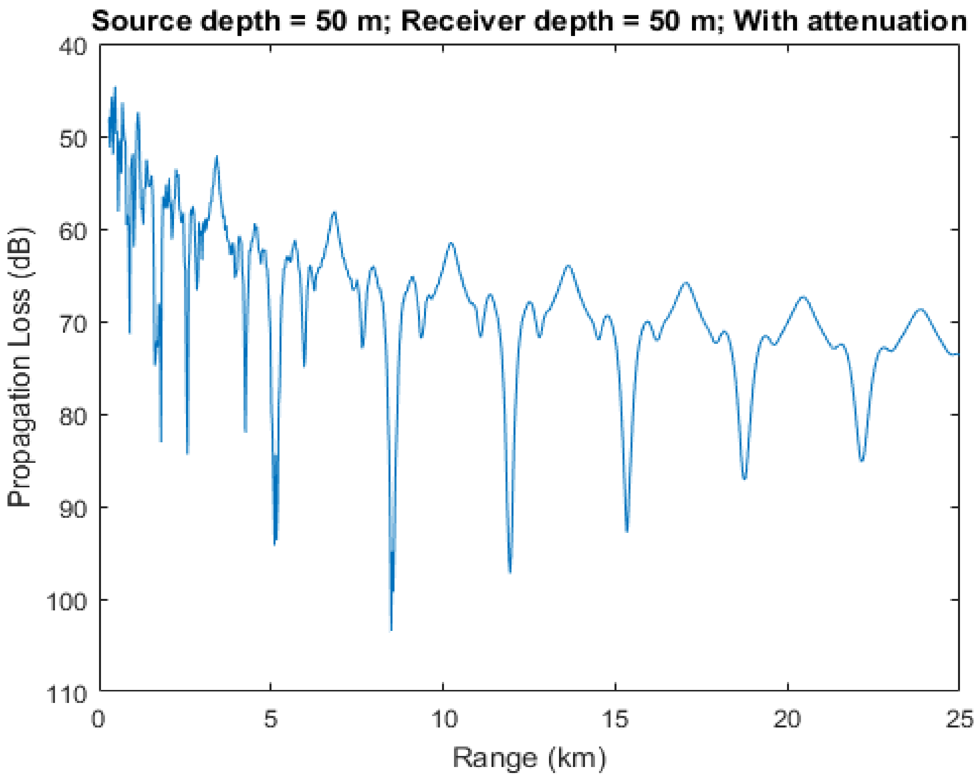

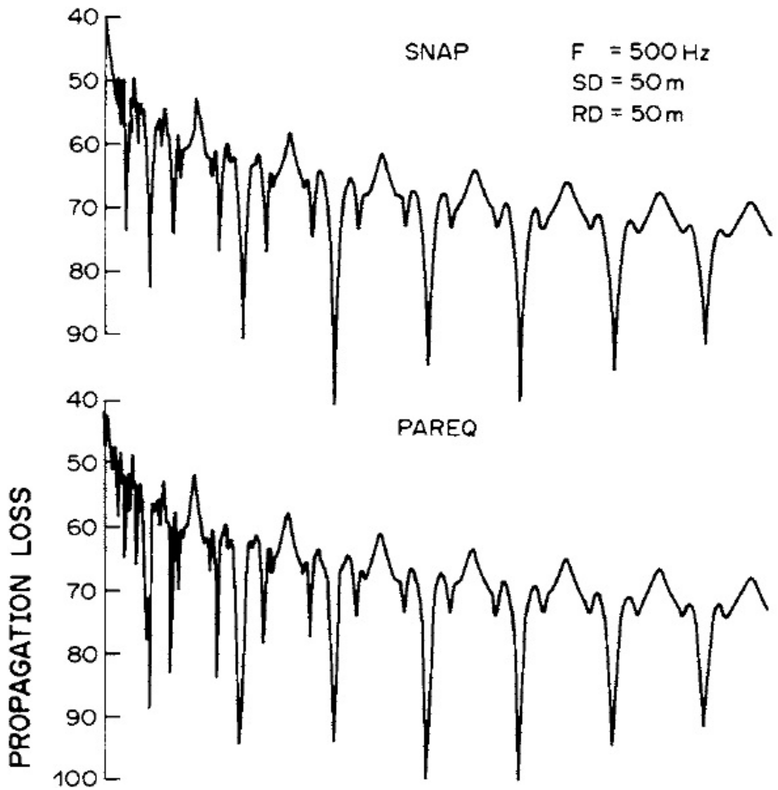

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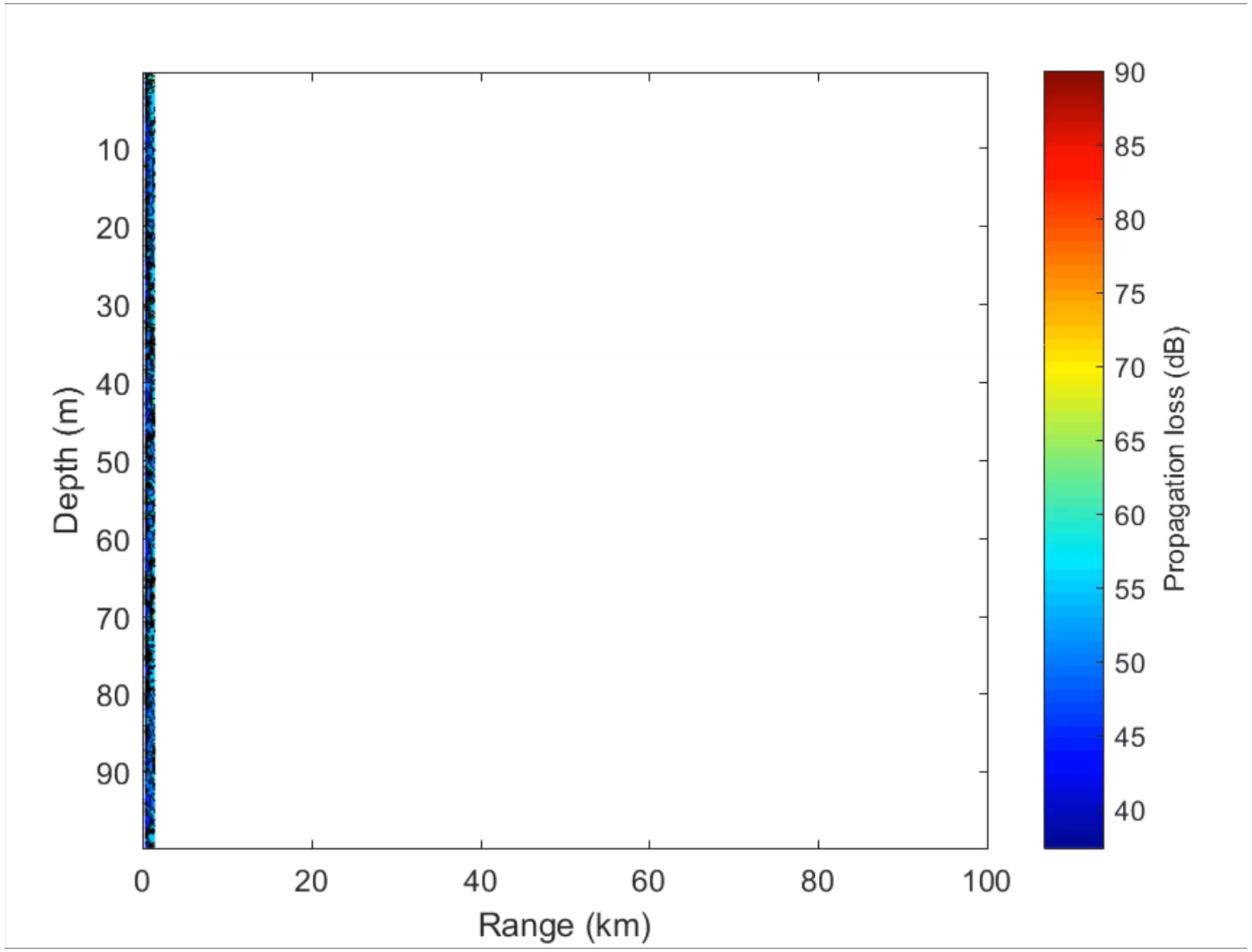


Variable-Density PE

- Recall:

$$\frac{\partial \psi}{\partial r} = \frac{\rho i}{2k_0}\left(\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial \psi}{\partial z}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\rho}\frac{\partial \psi}{\partial \theta}\right)\right) + \frac{ik_0}{2}\left(n^2 - 1\right)\psi$$

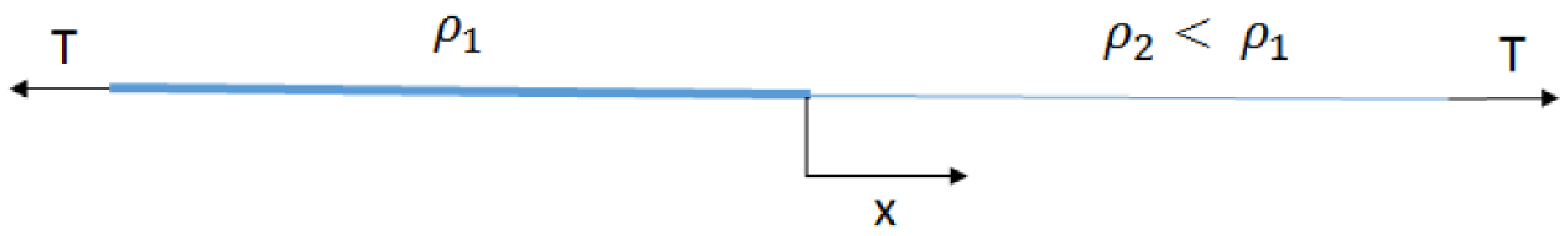
- Can be rewritten as:

$$\frac{\partial \psi}{\partial r} = \frac{i}{2k_0}\left(\frac{\partial^2 \psi}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2}\right) + \frac{ik_0}{2}\left(n_{eff}^2 - 1\right)\psi$$

Where:

$$n_{eff}^2 = n^2 + \frac{1}{2k_0^2}\rho^{1/2}\nabla \cdot \left(\rho^{-3/2}\nabla \rho\right)$$

Variable-Density PE - Analogy to wave motion in taut string with stepped density



- Approach 1: Solve 2 wave equations with constant density and impose interface conditions,i.e.:

$$\frac{\rho_1}{T}\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial x^2} \text{for } x < 0$$

$$\frac{\rho_2}{T}\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial x^2} \text{for } x > 0$$

With

$$\begin{cases} y_{0+} = y_{0-} \\ T\left[\frac{\partial y}{\partial x}\right]_{0+} = T\left[\frac{\partial y}{\partial x}\right]_{0-} \end{cases}$$

- Approach 2: Solve the wave equation on $-\infty < x < +\infty$ with variable density, i.e.:

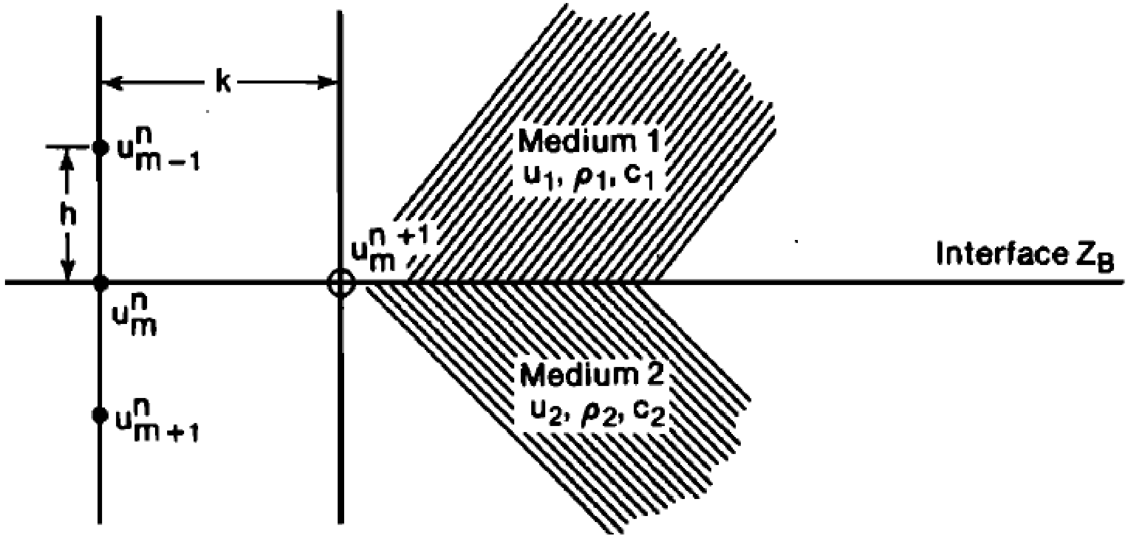
$$\frac{\rho(x)}{T}\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial x^2}$$

With

$$\rho(x) = \begin{cases} \rho_1 & \text{if } x < 0 \\ \rho_2 & \text{if } x > 0 \end{cases}$$

Variable-Density PE – Approach 1: implementation in FD

- Interface conditions:

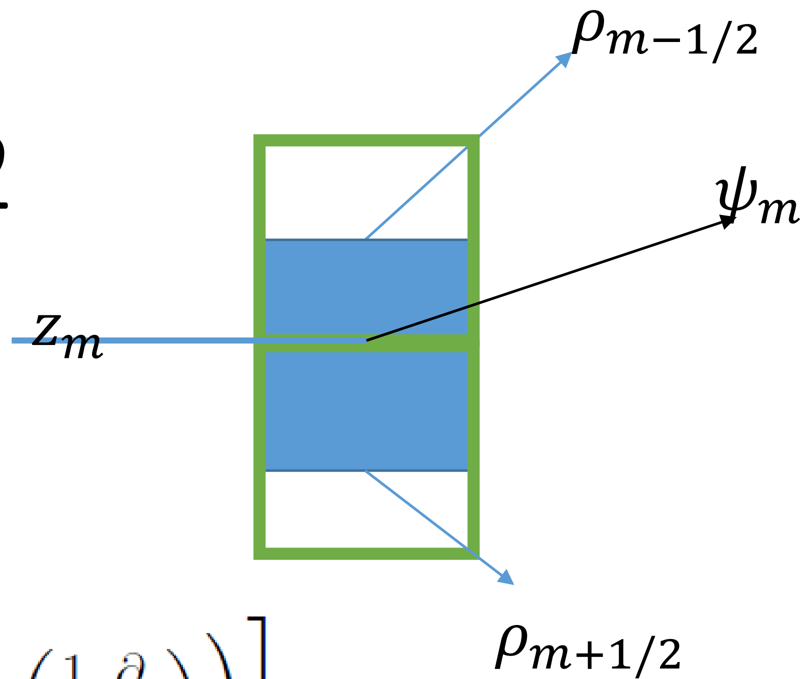


$$(\psi_m^n)_{z=zb-} = (\psi_m^n)_{z=zb+}$$

$$\frac{1}{\rho_1}\left(\frac{\partial \psi_m^n}{\partial z}\right)_{z=zb-} = \frac{1}{\rho_2}\left(\frac{\partial \psi_m^n}{\partial z}\right)_{z=zb+}$$

$$\frac{d\psi_m^n}{dr} = \left(\frac{1}{b_1} + \frac{\rho_1}{\rho_2}\frac{1}{b_2}\right)^{-1}\left[\left(\frac{a_1}{b_1} + \frac{\rho_1}{\rho_2}\frac{a_2}{b_2}\right)\psi_m^n + \frac{2}{\Delta z^2}\left(\frac{\rho_1}{\rho_2}\left(\psi_{m+1}^n - \psi_m^n\right) - \left(\psi_m^n - \psi_{m-1}^n\right)\right]$$

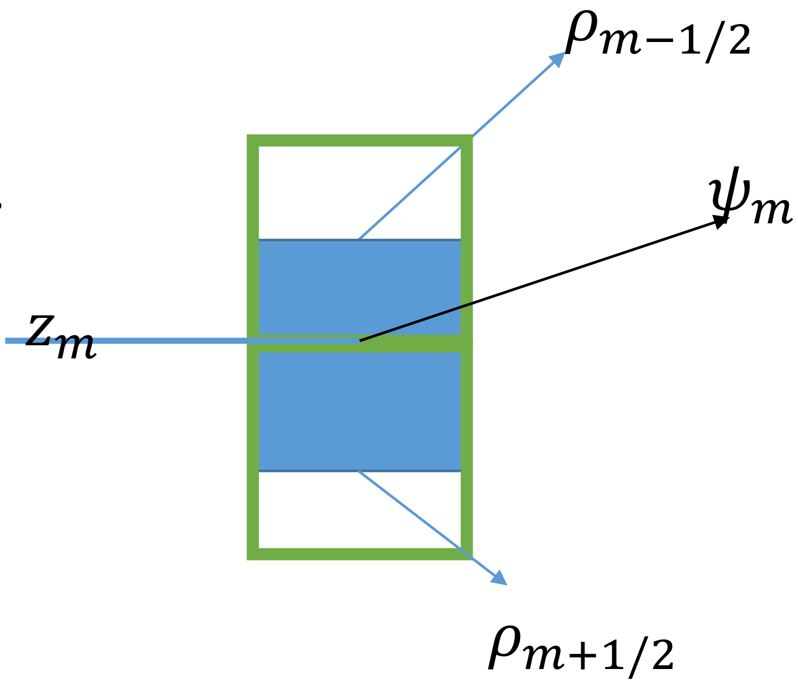
Variable-Density PE – Approach 2



$$\left[\frac{1}{\rho}-\frac{\delta}{4}\left(\frac{n^2-1}{\rho}+\frac{1}{k_0^2}\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial}{\partial z}\right)\right)\right]\psi_{k+1}=\left[\frac{1}{\rho}+\frac{\delta}{4}\left(\frac{n^2-1}{\rho}+\frac{1}{k_0^2}\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial}{\partial z}\right)\right)\right]\psi_k \quad (17)$$

$$LHS=\int_{zm-1/2}^{zm+1/2}\frac{1}{\rho}\psi^{k+1}dz-\frac{\delta}{4}\left(\int_{zm-1/2}^{zm+1/2}\frac{n^2-1}{\rho}\psi^{k+1}dz+\frac{1}{k_0^2}\int_{zm-1/2}^{zm+1/2}\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial\psi^{k+1}}{\partial z}dz\right)\right)$$

Variable-Density PE – Approach 2



$$Term1 \approx \frac{1}{2}\left(\frac{1}{\rho_{m+1/2}} + \frac{1}{\rho_{m-1/2}}\right)\psi_m^{k+1}$$

$$Term2 \approx \frac{1}{2}\left(\frac{n_{m+1/2}^2 - 1}{\rho_{m+1/2}} + \frac{n_{m-1/2}^2 - 1}{\rho_{m-1/2}}\right)\psi_m^{k+1}$$

$$Term3 = \frac{1}{\rho_{m+1/2}}\frac{1}{\Delta z}\left(\psi_{m+1}^{k+1} - \psi_m^{k+1}\right) - \frac{1}{\rho_{m-1/2}}\frac{1}{\Delta z}\left(\psi_m^{k+1} - \psi_{m-1}^{k+1}\right)$$

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