Massachusetts Institute of Technology
Project for Subject 2.29

Formulation and numerical investigation of a new fully nonlinear model for irrotational wave-current interactions

by

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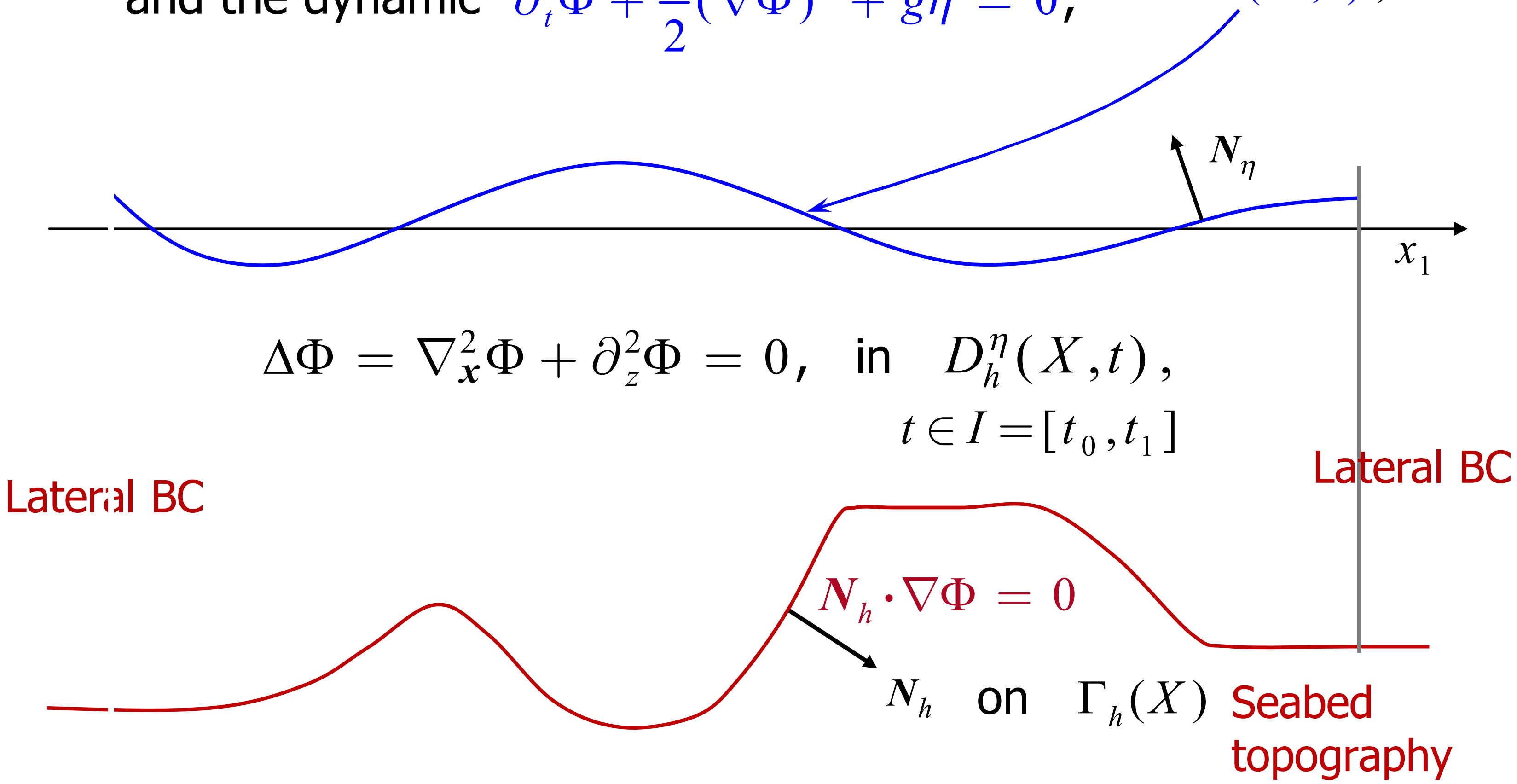
OBJECTIVES OF THIS WORK

- **Formulate and numerically study a model for wave-current interactions** (shoaling, reflection, focusing) **by using**
- **A fully nonlinear (potential) water-wave solver based on a new efficient Hamiltonian Formulation**

Formulation of the problem:
Differential formulation

The kinematic $\partial_t \eta - \boldsymbol{N}_\eta \cdot \nabla \Phi = 0$, boundary conditions

and the dynamic $\partial_t \Phi + \frac{1}{2}(\nabla \Phi)^2 + g\eta = 0$, on the free surface $\Gamma^\eta(X,t)$, $t \in I = [t_0, t_1]$



$$\Delta \Phi = \nabla_{\boldsymbol{x}}^2 \Phi + \partial_z^2 \Phi = 0, \quad \text{in} \quad D_h^\eta(X,t), \quad t \in I = [t_0, t_1]$$

$$\boldsymbol{N}_h \cdot \nabla \Phi = 0 \quad \text{on} \quad \Gamma_h(X)$$

Formulation of the problem:
Hamilton's equations

Hamilton's equations (**Zakharov 1968, Craig & Sulem 1993**)

$$\partial_t \eta = \delta_\psi \mathcal{H} = G(\eta,h)\psi,$$
$$\partial_t \psi = -\delta_\eta \mathcal{H} = -g\eta - \frac{1}{2}(\nabla_x \psi)^2 + \frac{\left(G(\eta,h)\psi + \nabla_x \eta \cdot \nabla_x \psi\right)^2}{2\left(1+(\nabla_x \eta)^2\right)}$$

where $\mathcal{H}[\psi,\eta] = \frac{1}{2}\int_X \left(\psi\, G(\eta,h)\psi + g\eta^2\right) d\boldsymbol{x}$ is the Hamiltonian

and $G(\eta,h) = \boldsymbol{N}_\eta \cdot [\nabla\Phi]_{z=\eta} = -\nabla_x \eta \cdot [\nabla_x \Phi]_{z=\eta} + [\partial_z \Phi]_{z=\eta}$ is the **Dirichlet to Neumann** (DtN) operator,

defined from the kinematic subproblem

$$\Delta\Phi = 0 \text{ in } D_h^\eta \times [t_0, t_1]$$
$$\boldsymbol{N}_h \cdot \nabla\Phi = 0 \text{ on } \Gamma_h(X)$$
$$\Phi = \psi \text{ on } \Gamma^\eta(X,t)$$

Formulation of the problem:
Hamiltonian coupled-mode formulation

Using the representation

$$\Phi(\boldsymbol{x},z,t) = \sum_{n=-2}^{\infty} \varphi_n(\boldsymbol{x},t)\, Z_n(z;\eta,h) + \Phi_b(\boldsymbol{x},z,t)$$

Similar to the one studied in detail in [Athanassoulis & Papoutsellis PRSLA 2017], in conjunction with

Luke's Variational Formulation (**Luke 1967**)

$$\delta \int_{t_0}^{t_1} \int_X \int_{-h}^{\eta} \left(\partial_t \Phi + \frac{1}{2} |\nabla_{\boldsymbol{x}} \Phi|^2 + \frac{1}{2} (\partial_z \Phi)^2 + g\, z \right) dz\, d\boldsymbol{x}\, dt = 0$$

The two nonlinear Hamiltonian evolution equations take the form

$$\partial_t \eta = -(\nabla_{\boldsymbol{x}} \eta) \cdot (\nabla_{\boldsymbol{x}} \psi + [\nabla_{\boldsymbol{x}} \Phi_b]_\eta) + [\partial_{\boldsymbol{z}} \Phi_b]_\eta + (|\nabla_{\boldsymbol{x}} \eta|^2 + 1) \left(\mathcal{F}_{-2}[\eta,h] \psi / h_0 + \mu_0 \psi \right),$$

$$\partial_t \psi = -g\eta - [\partial_{\boldsymbol{t}} \Phi_b]_\eta - \frac{1}{2} (\nabla_{\boldsymbol{x}} \psi + [\nabla_{\boldsymbol{x}} \Phi_b]_\eta)^2 - \frac{1}{2} [\partial_{\boldsymbol{z}} \Phi_b]_\eta^2 + \frac{1}{2} (|\nabla_{\boldsymbol{x}} \eta|^2 + 1) \left(\mathcal{F}_{-2}[\eta,h] \psi / h_0 + \mu_0 \psi \right)^2 ,$$

where $\mathcal{F}_{-2}[\eta,h]$ is **a nonlocal operator**, corresponding to $\varphi_{-2}(\boldsymbol{x};t)$.

Formulation of the problem:
Hamiltonian coupled-mode formulation

Nonlocal operator $\mathcal{F}_{-2}[\eta, h]$ is obtained by solving the **substrate kinematic problem** (in **modal form**)

$$\sum_{n=-2}^{\infty} \mathrm{L}_{mn}\varphi_n = -(\nabla_{\boldsymbol{x}} h \cdot [\nabla_{\boldsymbol{x}}\Phi_b]_{-h} + [\partial_z \Phi_b]_{-h})[\mathrm{Z}_m]_{-h}, \quad m \geq -2, \quad \boldsymbol{x} \in X$$

$$\sum_{n=-2}^{\infty} \varphi_n = \psi(\boldsymbol{x}, t), \quad \boldsymbol{x} \in X$$

$$\sum_{n=-2}^{\infty} \left[N_{\partial \boldsymbol{x}} \cdot \nabla_{\boldsymbol{x}} \varphi_n \mathrm{A}_{mn} + \frac{1}{2}\varphi_n (\mathrm{B}^1_{mn}, \mathrm{B}^2_{mn}) \cdot N_{\partial \boldsymbol{x}} \right]_{\partial x} = \int_{-h}^{\eta} V_{\partial x} [Z_m]_{\partial x} \, dz, \quad m \geq -2$$

The coefficients of the **substrate problem** are given by

$$\mathrm{L}_{mn}[\eta, h] = \mathrm{A}_{mn} \nabla^2_{\boldsymbol{x}} + (\mathrm{B}^1_{mn}, \mathrm{B}^2_{mn}) \cdot \nabla_{\boldsymbol{x}} + \mathrm{C}_{mn}$$

$$\mathrm{A}_{mn} = \int_{-h}^{\eta} Z_n Z_m \, dz, \quad \mathrm{B}^i_{mn} = 2\int_{-h}^{\eta} \partial_{x_i} Z_n Z_m \, dz + \partial_{x_i} h \left[Z_m Z_n \right]_{z=-h} \quad i = 1, 2$$

$$\mathrm{C}_{mn} = \int_{-h}^{\eta} \Delta Z_n Z_m \, dz - \boldsymbol{N}_h \cdot \left[\nabla Z_n Z_m \right]_{z=-h}$$

Formulation of the problem:
Hamiltonian coupled-mode formulation

$$\partial_t \eta = -(\nabla_x \eta)\cdot(\nabla_x \psi) + (|\nabla_x \eta|^2 + 1)\left(\mathcal{F}_{-2}[\eta, h]\psi / h_0 + \mu_0 \psi\right),$$

$$\partial_t \psi = -g\eta - \frac{1}{2}(\nabla_x \psi)^2 + \frac{1}{2}(|\nabla_x \eta|^2 + 1)\left(\mathcal{F}_{-2}[\eta, h]\psi / h_0 + \mu_0 \psi\right)^2,$$

Remark: The evolution equations presented before reduce to **Zakharov-Craig-Sulem formulation** after the identification (**modal form of the DtN operator**)

$$G(\eta, h)\psi = -\nabla_x \eta \cdot \nabla_x \psi + (|\nabla_x \eta|^2 + 1)(\mathcal{F}_{-2}[\eta, h]\psi / h_0 + \mu_0 \psi)$$

Numerical solution of the substrate problem. The two-dimensional (2D) case

Starting from initial conditions $\left(\eta(x,t_0),\psi(x,t_0)\right)=(\eta_0,\psi_0)$, the **numerical scheme** is implemented in **two steps**

1. The truncated **substrate problem** is solved at each time t

$$\sum_{n=-2}^{M} \mathrm{L}_{mn}[\eta,h]\varphi_n = -(\nabla_{\boldsymbol{x}} h\cdot[\nabla_{\boldsymbol{x}}\Phi_b]_{-h}+[\partial_z\Phi_b]_{-h})[\mathrm{Z}_m]_{-h}, \quad m=-2\,(1)\,(M-1),$$

$$\sum_{n=-2}^{M} \varphi_n = \psi, \quad x\in X, \quad [N_{tot}=M+3 \text{ equations for } M+3 \ \ \varphi_n\text{'s}]$$

$$\sum_{n=-2}^{M}\left[N_{\partial\boldsymbol{x}}\cdot\nabla_{\boldsymbol{x}}\varphi_n\,\mathrm{A}_{mn}+\frac{1}{2}\varphi_n(\mathrm{B}^1_{mn},\mathrm{B}^2_{mn})\cdot N_{\partial\boldsymbol{x}}\right]_{\partial x} = \int_{-h}^{\eta} V_{\partial x}[Z_m]_{\partial x}\,dz,\ m=-2\,(1)\,(M-1)$$

2. Then, the evolution equations are marched in time using $\varphi_{-2} = \mathcal{F}_{-2}^{(N_{tot})}[\eta,h]\psi$

$$\partial_t\eta=-(\nabla_{\boldsymbol{x}}\eta)\cdot(\nabla_{\boldsymbol{x}}\psi+[\nabla_{\boldsymbol{x}}\Phi_b]_\eta)+[\partial_{\boldsymbol{z}}\Phi_b]_\eta+(|\nabla_{\boldsymbol{x}}\eta|^2+1)\left(\mathcal{F}_{-2}^{(N_{tot})}[\eta,h]\psi/h_0+\mu_0\psi\right),$$

$$\partial_t\psi=-g\eta-[\partial_{\boldsymbol{t}}\Phi_b]_\eta-\frac{1}{2}(\nabla_{\boldsymbol{x}}\psi+[\nabla_{\boldsymbol{x}}\Phi_b]_\eta)^2-\frac{1}{2}[\partial_{\boldsymbol{z}}\Phi_b]^2_\eta$$
$$+\frac{1}{2}(|\nabla_{\boldsymbol{x}}\eta|^2+1)\left(\mathcal{F}_{-2}^{(N_{tot})}[\eta,h]\psi/h_0+\mu_0\psi\right)^2,$$

Numerical solution of the two evolution equations

Set $U = (\eta, \psi)^{\mathrm{T}}$, $U = U(x,t)$ and write the evolution equations as

$$\boxed{\partial_t U = \mathcal{N}[t, U]}, \quad U(x,0) = U_0 = (\eta_0, \psi_0), \; x \in X = [a,b].$$

Runge-Kutta 4th order: 4 inversions of the substrate problem per time step

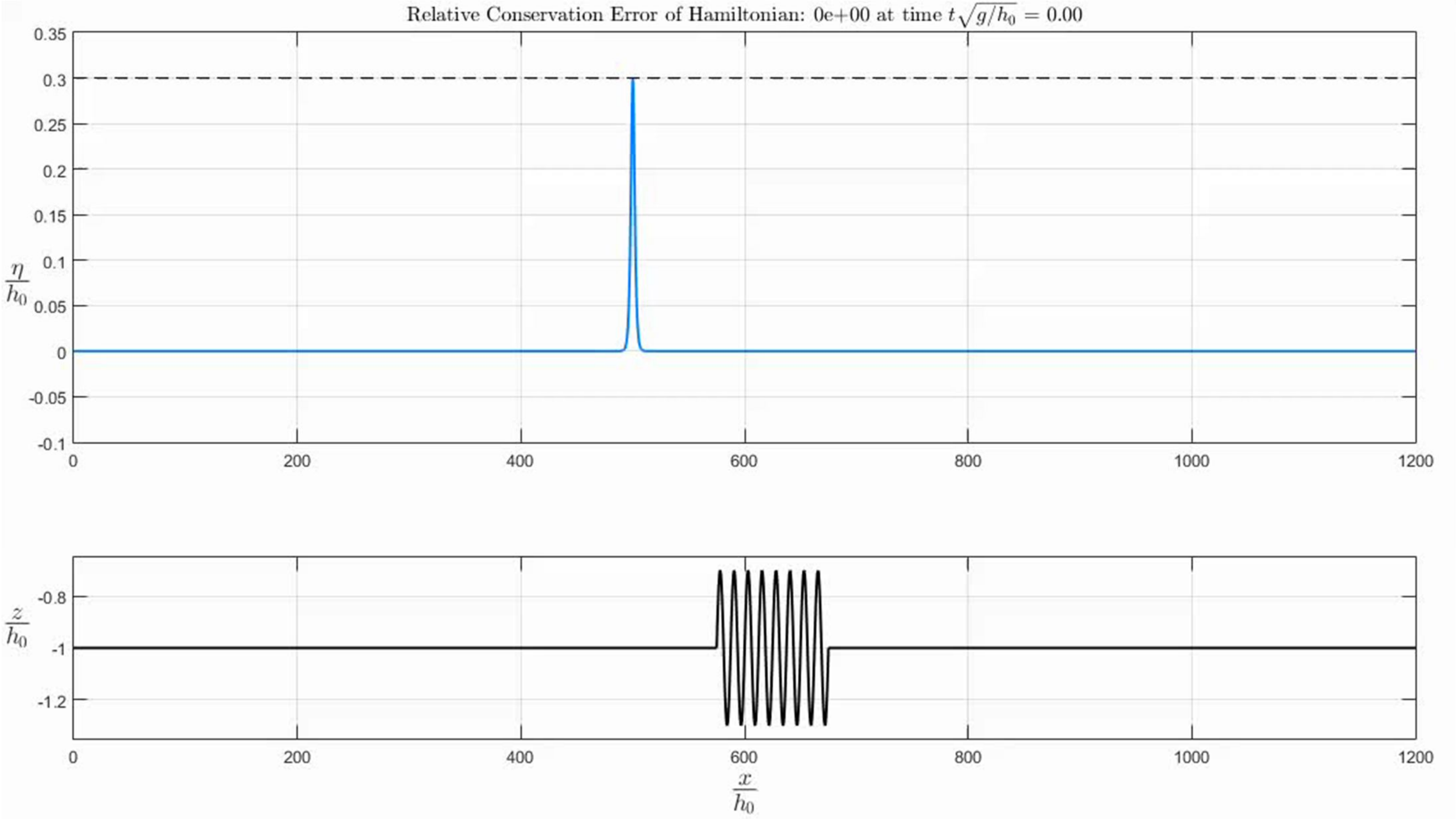
$$U_{n+1} = U_n + (\delta t / 6)(\mathcal{K}_1 + 2\mathcal{K}_2 + 2\mathcal{K}_3 + \mathcal{K}_4).$$

$$\mathcal{K}_1 = \mathcal{N}[t_n, U_n], \quad \mathcal{K}_2 = \mathcal{N}\left[t_n + \frac{\delta t}{2}, U_n + \frac{\delta t}{2}\mathcal{K}_1\right],$$

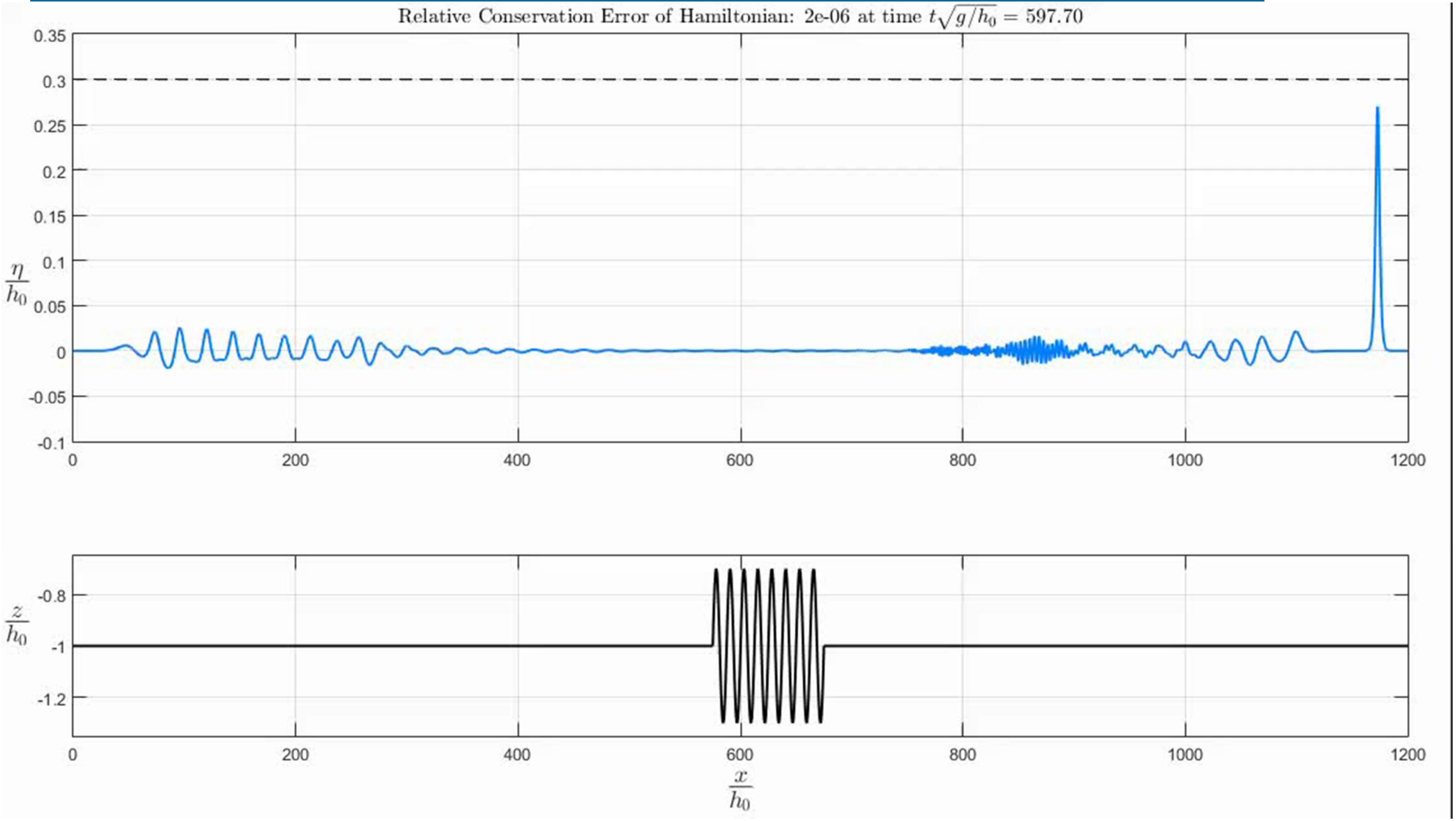
$$\mathcal{K}_3 = \mathcal{N}\left[t_n + \frac{\delta t}{2}, U_n + \frac{\delta t}{2}\mathcal{K}_2\right], \qquad \mathcal{K}_4 = \mathcal{N}\left[t_n + \delta t, U_n + \delta t\, \mathcal{K}_3\right].$$

Remark: For the case of solitary waves, initial conditions (η_0, ψ_0) are derived from [Clamond & Dutykh 2013] as highly accurate solitary wave solution

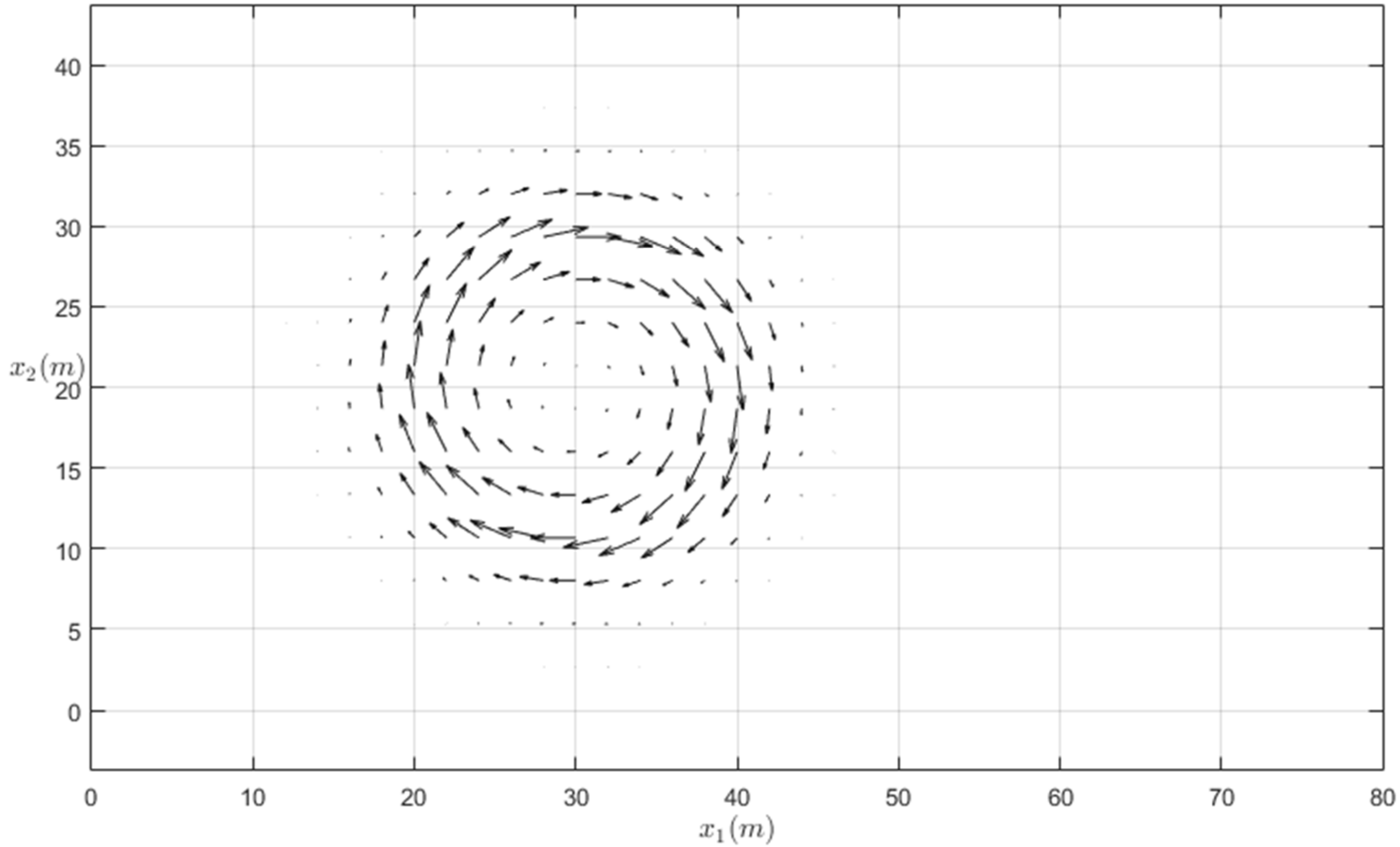
Solitary wave – propagating over an undulating bottom



Solitary wave – propagating over an undulating bottom, reverse simulation



Regular waves interacting with a vortex ring



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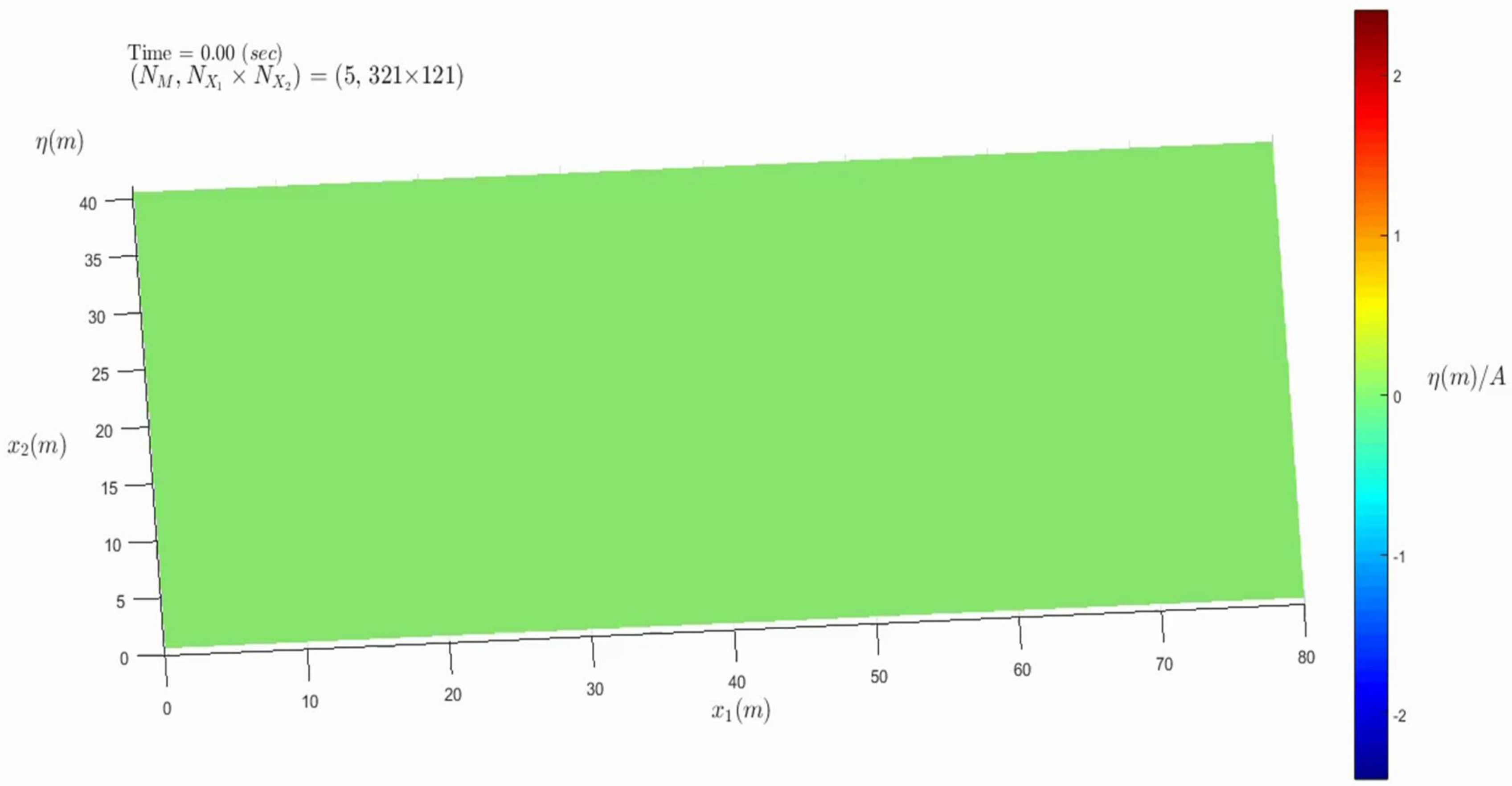
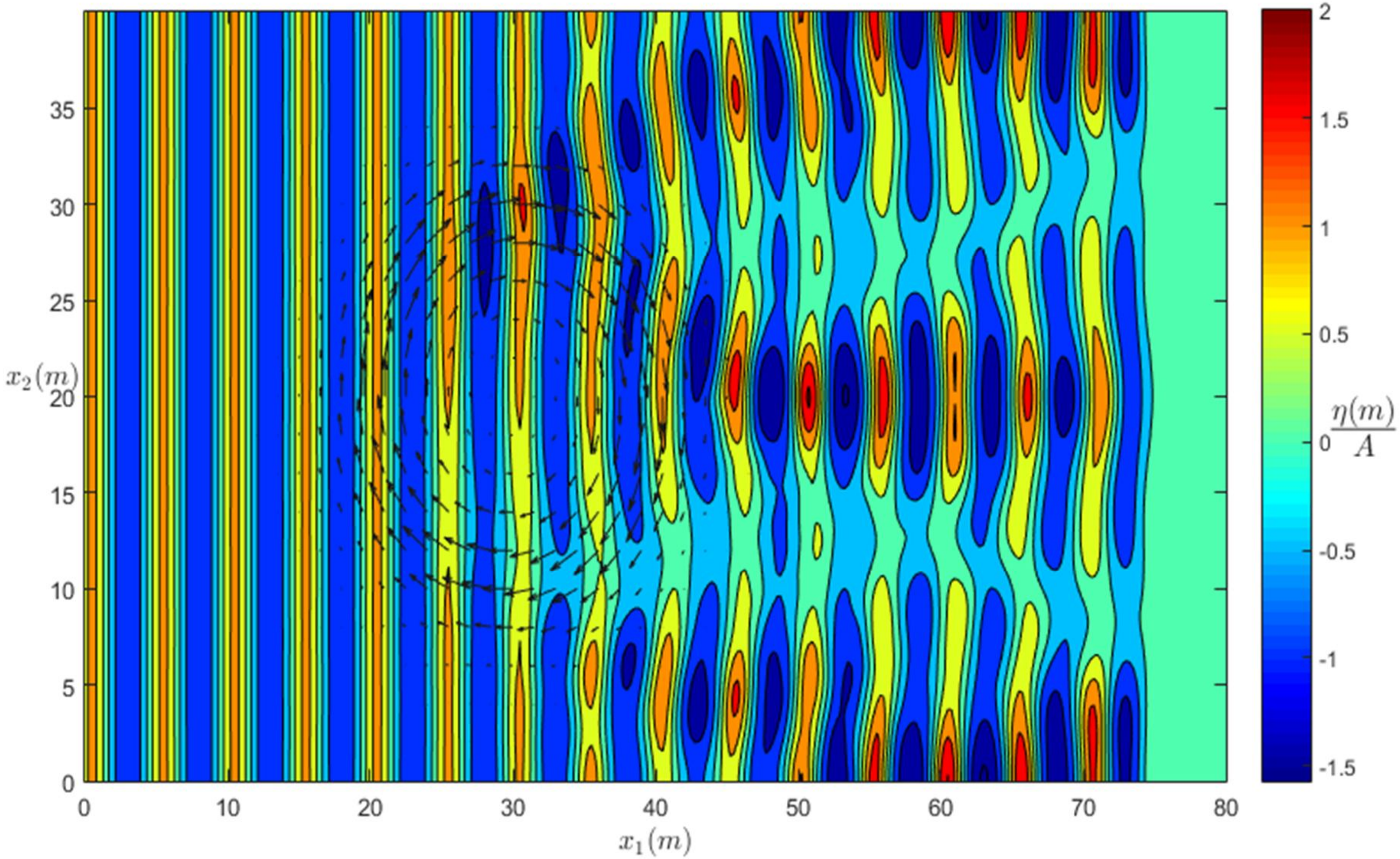


Regular waves interacting with a vortex ring



Thank you

for your

attention !!