Solving Stochastic Advection Diffusion Equation Using HDG Method

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Numerical Fluid Mechanics

Outline

- Stochastic Advection Diffusion Equation
- Hybridized Discontinuous Galerkin Method
  - Background
  - Discretization of The Advection Diffusion Equation
  - Global Equation
  - Examples
- Dynamically Orthogonal Field Equations
  - Definition
  - DO Field Equations for The Advection Diffusion Equation

Stochastic Advection Diffusion Equation

- Advection Diffusion Equation

$$\frac{\partial f}{\partial t} + \nabla\left(uf\right) = \nabla.(D\nabla f)$$

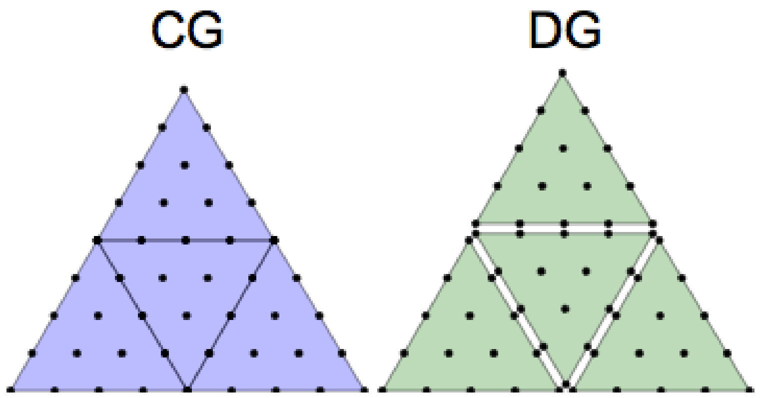
- The flow field is assumed stochastic, i.e. $u = u(\boldsymbol{x}, t; \omega)$ and $f = f(\boldsymbol{x}, t; \omega)$

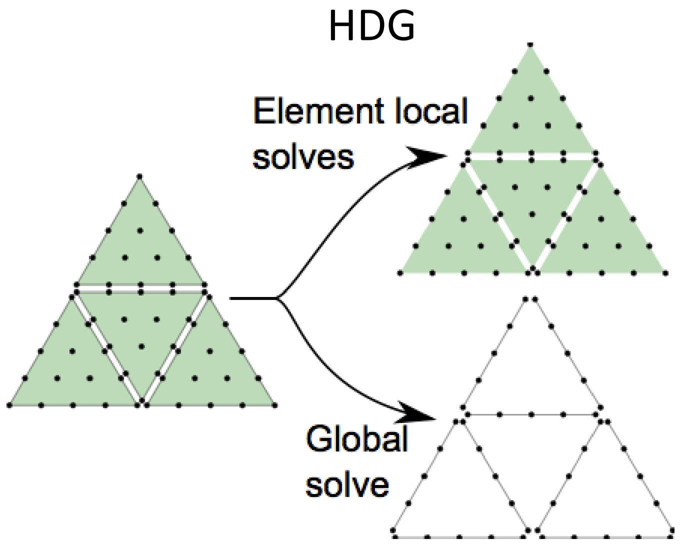
- The 1D Stochastic Advection Diffusion Equation is

$$\frac{\partial f(\mathbf{x}, t; \omega)}{\partial t} + \frac{\partial\left(u(\mathbf{x}, t; \omega) f(\mathbf{x}, t; \omega)\right)}{\partial x} = \frac{\partial}{\partial x}\left(\frac{D(\mathbf{x}, t) \partial f(\mathbf{x}, t; \omega)}{\partial x}\right)$$

Hybridized Discontinuous Galerkin Method

- Finite Element Methods are are accurate but slow
- HDG is competitive to CG while retaining the properties of DG
- Each element can be solved locally given the boundary conditions
- Solving for the boundary condition is possible by equating the fluxes on each edge to the total flux into the system





Spatial Discretization – 1D

- Definitions
  - $(a, b)_{\Delta x_i} = \int_{x_i}^{x_{i+1}} ab\, dx$
  - $\lambda = \hat{f}$ is the value of f at the boundary
  - $f = \sum_{i=1}^{N_e} f_i \theta_i$
- Let θ be a test function, then multiply the equation by the test function and integrate

$$\int_{x_i}^{x_{i+1}} \frac{\partial f}{\partial t} \theta dx + \int_{x_i}^{x_{i+1}} \frac{\partial (uf)}{\partial x} \theta dx - \int_{x_i}^{x_{i+1}} \frac{\partial}{\partial x} \left(D \frac{\partial f}{\partial x} \right) \theta dx = 0$$

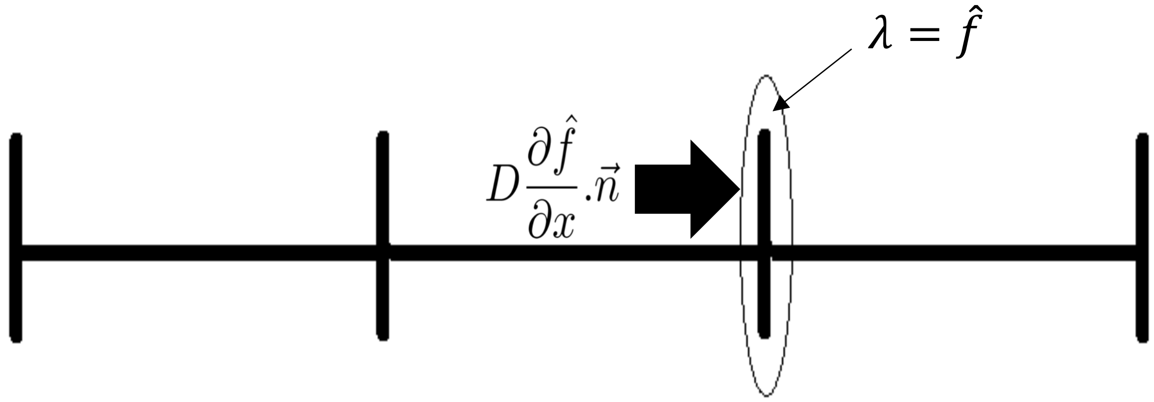
Spatial Discretization -1D

- Integrate the diffusion term by parts twice

$$\int_{x_i}^{x_{i+1}} \frac{\partial}{\partial x}\left(D\frac{\partial f}{\partial x}\right)\theta dx = D\frac{\partial \hat{f}}{\partial x}.\vec{n}\theta\Big|_{x_i}^{x_{i+1}} - D\frac{\partial f}{\partial x}.\vec{n}\theta\Big|_{x_i}^{x_{i+1}} + \int_{x_i}^{x_{i+1}} \frac{\partial}{\partial x}\left(D\frac{\partial f}{\partial x}\right)\theta dx$$

- The Diffusion is modeled as follows

$$D\frac{\partial \hat{f}}{\partial x} = D\frac{\partial f}{\partial x}.\vec{n} - \tau(f-\lambda)\vec{n}$$



Spatial - time Discretization – 1D

- The strong form of the Advection Diffusion Equation is

$$\frac{(f,\theta)_{\Delta x_i}^{k+1}}{\Delta t} + \tau\theta f^{k+1}\Big|_{x_i} + \tau\theta f^{k+1}\Big|_{x_{i+1}} - \left(\frac{\partial}{\partial x}\left(D\frac{\partial f}{\partial x}\right),\theta\right)_{\Delta x_i}^{k+1}$$

$$= \tau\lambda^{k+1}\theta\Big|_{x_i} + \tau\lambda^{k+1}\theta\Big|_{x_{i+1}} - \left(\frac{\partial\left(uf\right)}{\partial x},\theta\right)_{\Delta x_i}^{k} + \frac{(f,\theta)_{\Delta x_i}^{k}}{\Delta t}$$

- Final Finite Element Equation

$$f_i^{k+1}\left[\frac{(\theta_i,\theta_j)_{\Delta x_i}}{\Delta t} + \tau\theta_i\theta_j\Big|_{x_i} + \tau\theta_i\theta_j\Big|_{x_{i+1}} - \left(\frac{\partial}{\partial x}\left(D\frac{\partial\theta_i}{\partial x}\right),\theta_j\right)_{\Delta x_i}\right]$$

$$= \tau\lambda^{k+1}\theta_j\Big|_{x_i} + \tau\lambda^{k+1}\theta_j\Big|_{x_{i+1}} - \left(\frac{\partial\left(uf^k\right)}{\partial x},\theta_j\right)_{\Delta x_i} + \frac{(f^k,\theta_j)_{\Delta x_i}}{\Delta t}$$

Local Finite Element Equation Summery

$$A_{ij}^{local} = \frac{(\theta_i, \theta_j)_{\Delta x_i}}{\Delta t} + \tau \theta_i \theta_j \Big|_{x_i} + \tau \theta_i \theta_j \Big|_{x_{i+1}} - \left(\frac{\partial}{\partial x} \left(D \frac{\partial \theta_i}{\partial x} \right), \theta_j \right)_{\Delta x_i}$$

$$b_j^{local} = \tau \lambda^{k+1} \theta_j \Big|_{x_i} + \tau \lambda^{k+1} \theta_j \Big|_{x_{i+1}} - \left(\frac{\partial \left(u f^k \right)}{\partial x}, \theta_j \right)_{\Delta x_i} + \frac{(f^k, \theta_j)_{\Delta x_i}}{\Delta t}$$

$$A^{local} f = b^{local}$$

Global Equation

- Definition
  - $[[a.\vec{n}]] = a^+.\vec{n}^+ + a^-.\vec{n}^-$
  - $< a, b >_e = \int_e \; ab \; de$
  - $< a, b >_\epsilon = \sum_{e \in \epsilon} < a, b >_e$

- By Equating the fluxes of the boundary to the total flux we arrive at the global Flux Equation

$$\left\langle \left[\left[\left(D\frac{\partial f}{\partial x}.\vec{n} - \tau(f-\lambda)\vec{n}\right).\vec{n}\right]\right], \theta_\epsilon \right\rangle_\epsilon =< g_N, \theta_\epsilon >_\epsilon$$

Global Finite Element Equation

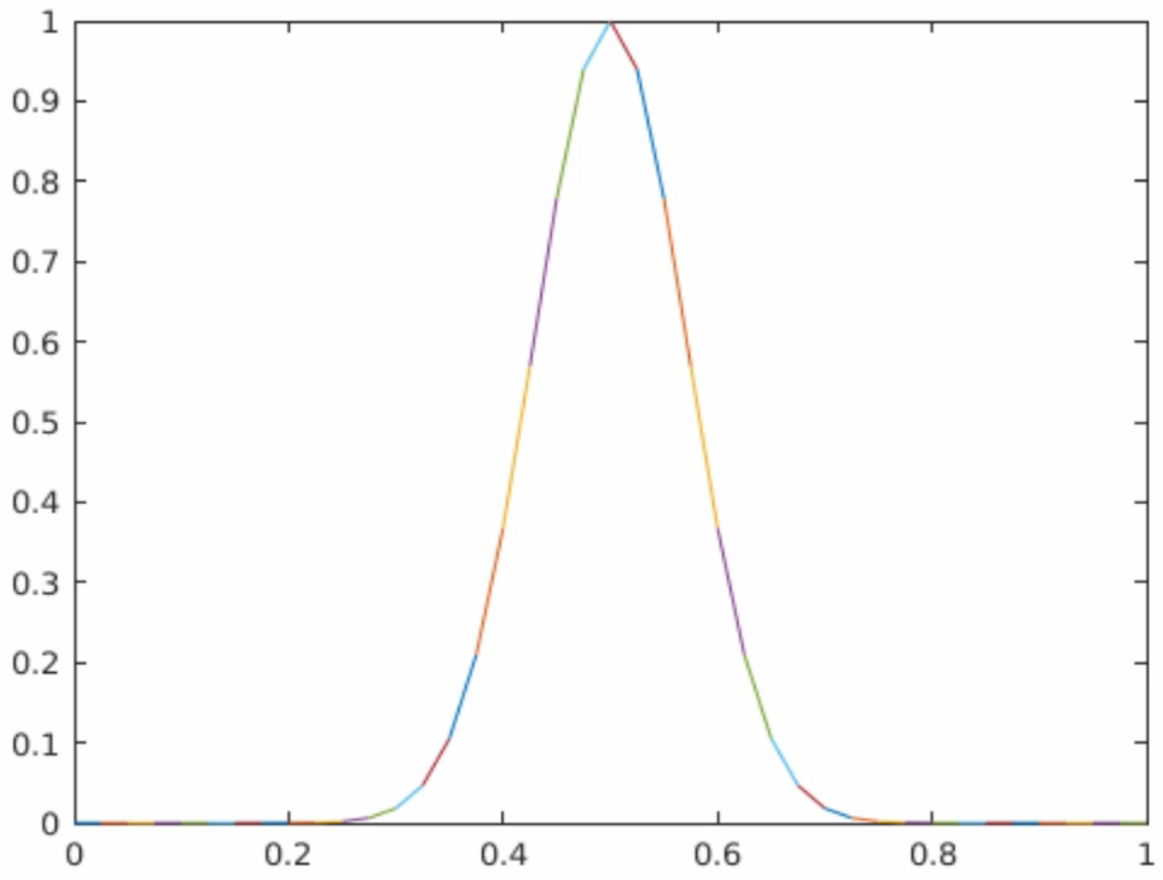
- Assume $f \approx f^F + \sum f^{\lambda_i}$
- f^F is the effect of the forcing terms only
- f^{λ_i} is the effect of the boundary only

$$A_{ij}^{global} = \left\langle \left[\left[\left(D\frac{\partial f^{\lambda_i}}{\partial x}.\vec{n} - \tau(f^{\lambda_i} - \delta_{ij}\theta_{\epsilon,i})\vec{n}\right).\vec{n}\right]\right], \theta_{\epsilon,j} \right\rangle_{\epsilon}$$

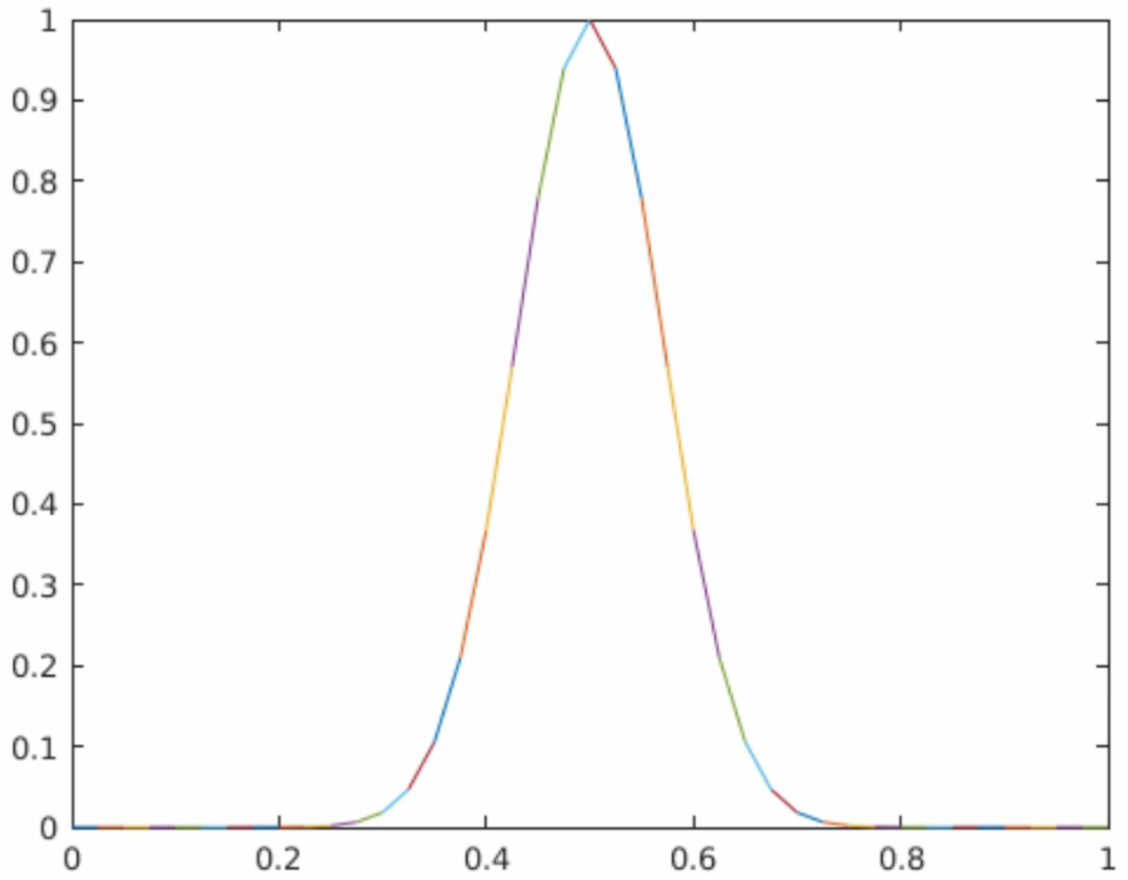
$$b_j^{global} = < g_N, \theta_{\epsilon,j} >_{\epsilon} - \left\langle \left[\left[\left(D\frac{\partial f^{F}}{\partial x}.\vec{n} - \tau(f^{F})\vec{n}\right).\vec{n}\right]\right], \theta_{\epsilon,j} \right\rangle_{\epsilon}$$

$$A^{global}\lambda = b^{global}$$

Example 1



Example 2



Dynamically Orthogonal Field Equations

- The response of the dynamical system is assumed to have the form

$$f(\mathbf{x}, t; \omega) = \bar{f}(\mathbf{x}, t) + \sum_{i=1}^{r_f} \zeta_i(t; \omega) f_i(t, x) = \bar{f} + \zeta_i f_i$$

- And the stochastic term u can be written as

$$u(\mathbf{x}, t; \omega) = \bar{u}(\mathbf{x}, t) + \sum_{k=1}^{r_u} \beta_k(t; \omega) u_k(t, x) = \bar{u} + \beta_k u_k$$

- ζ_i and β_k are zero mean stochastic processes

DO Condition

- The DO condition is defined as

$$\left\langle \frac{\partial f_i(\mathbf{x},t)}{\partial t}, f_j(\mathbf{x},t) \right\rangle = 0$$

- The above condition implies

$$\frac{\partial}{\partial t}\left\langle f_i(\mathbf{x},t), f_j(\mathbf{x},t) \right\rangle = 0$$

- $\{f_i(\boldsymbol{x},t)\}_{i=1}^{s}$ are deterministic fields which are initially orthonormal

Do Field Equations For The Advection Diffusion Equation

$$\frac{\partial \bar{f}}{\partial t} = \frac{\partial \bar{f}}{\partial x}\left(\frac{\partial D}{\partial x} - \bar{u}\right) + D\frac{\partial^2 f}{\partial x^2} - \frac{\partial f_i}{\partial x} u_k Cov(\beta_k, \zeta_i)$$

$$\frac{\partial \zeta_j}{\partial t} = -\zeta_i < \bar{u}\frac{\partial f_i}{\partial x}, f_j > -\beta_k < \frac{\partial \bar{f}}{\partial x} u_k, f_j > -\beta_k \zeta_i < u_k \frac{\partial f_i}{\partial x}, f_j >$$
$$Cov(\beta_k, \zeta_i) < \frac{\partial f_i}{\partial x} u_k, f_j > +\zeta_i < \frac{\partial D}{\partial x}\frac{\partial f_i}{\partial x}, f_j > +\zeta_i < D\frac{\partial^2 f_i}{\partial x^2}, f_j >$$

Do Field Equations For The Advection Diffusion Equation

$$
\begin{aligned}
\frac{\partial f_j}{\partial t} =& \frac{\partial f_j}{\partial x}\left(\frac{\partial D}{\partial x} - \bar{u}\right) + D\frac{\partial^2 f_j}{\partial x^2} \\
&+ < \frac{\partial f_l}{\partial x}\left(-\frac{\partial D}{\partial x}\bar{u}\right) - D\frac{\partial^2 f_l}{\partial x^2}, f_j > f_j Cov(\zeta_l \zeta_i) Cov^{-1}(\zeta_j, \zeta_i) \\
&\left(< \frac{\partial \bar{f}}{\partial x} u_k, f_j > f_j - \frac{\partial \bar{f}}{\partial x} u_k\right) Cov(\beta_k, \zeta_i) Cov^{-1}(\zeta_j, zeta_i) \\
&+ < u_k \frac{\partial f_l}{\partial x}, f_j > f_j M_3(\beta_k, \zeta_l, \zeta_i) Cov^{-1}(\zeta_j, \zeta_i) \\
&- u_k \frac{\partial f_j}{\partial x} M_3(\beta_k, \zeta_l, \zeta_i) Cov^{-1}(\zeta_j, \zeta_i)
\end{aligned}
$$

Future Work

- Solve higher dimensional problems (2D, 3D) using HDG methods
- Solve the DO Field Equations
- Compare the results with Monte Carlo Simulations