Solitary wave solutions of the KdV equation

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2.29 Final Project

I'll be discussing the (nonlinear) KdV equation.

The Korteweg-de Vries equation is nonlinear, which makes numerical solution important.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \delta^2\frac{\partial^3 u}{\partial x^3} = 0$$

Nonlinear term (think of Burgers' Equation).

Linear dispersion term.

The equation (and derivatives) appears in applications including shallow-water waves and plasma physics.

I was mainly interested in studying a case presented by *Zabusky and Kruskal*

In their 1965 paper they use numerical analysis to study "solitary-wave pulses"

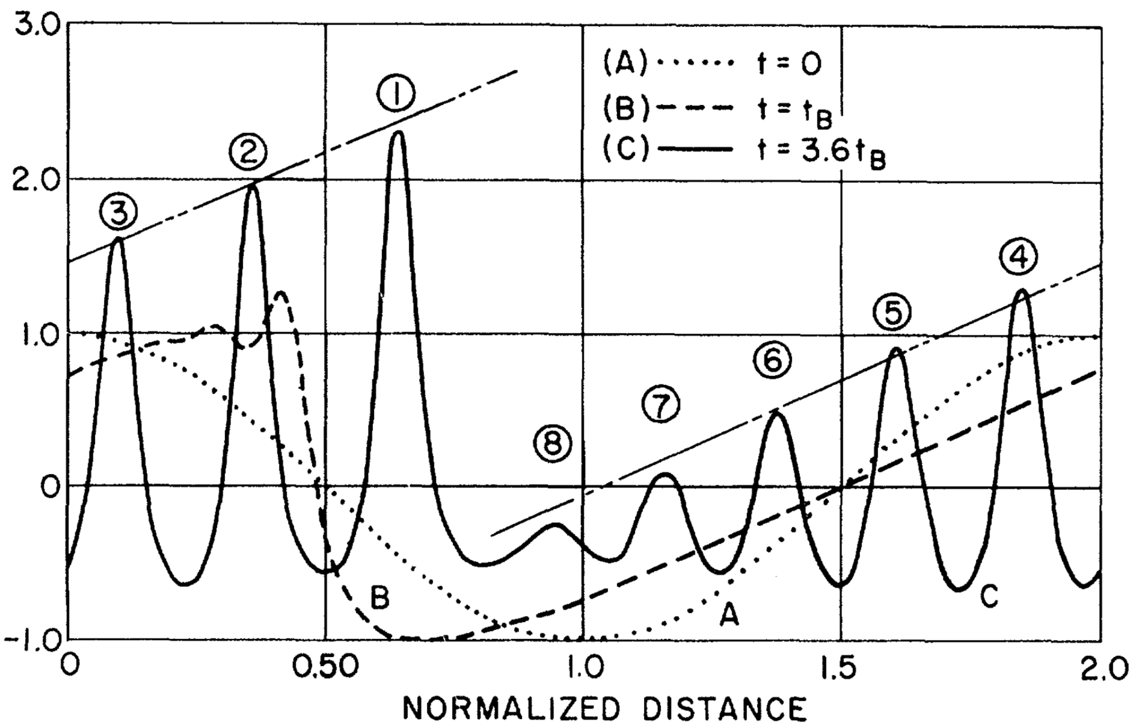


Figure 1 from *Zabusky and Kruskal* [1]

Though the wave starts off as a cosine pattern, it quickly devolves into a train of pulses.

There is a finite-difference method used in the paper:

A local average is used for the nonlinear 'u'

$$u_i^{j+1} = u_i^{j-1} - \tfrac{1}{3}(k/h)(u_{i+1}^j + u_i^j + u_{i-1}^j)(u_{i+1}^j - u_{i-1}^j) - (\delta^2 k/h^3)(u_{i+2}^j - 2u_{i+1}^j + 2u_{i-1}^j - u_{i-2}^j),$$

The method is second-order in time…

…and second-order in space.

However, there are two main disadvantages: (1) it requires two initial conditions and (2) it has an onerous stability condition [2]:

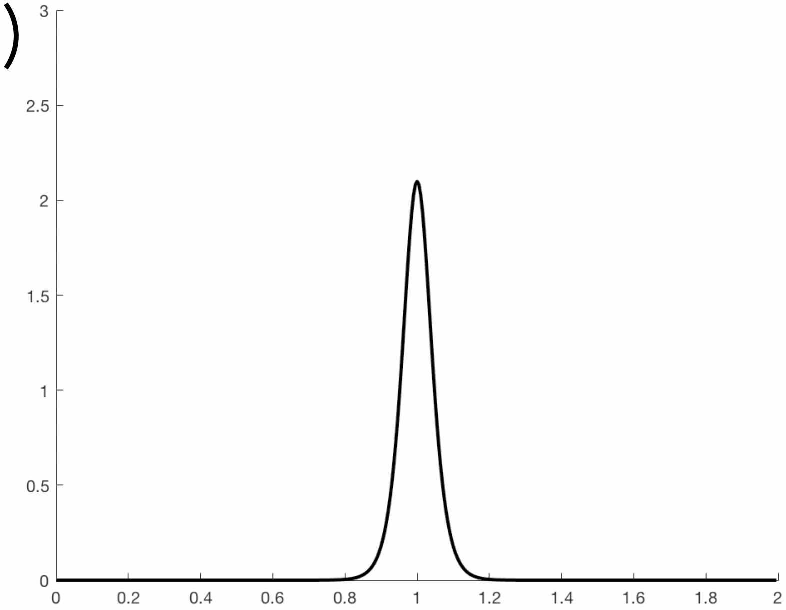
$$\frac{\Delta t}{\Delta x}\left|-2u_0 + \frac{1}{(\Delta x)^2}\right| \leq \frac{2}{3\sqrt{3}}$$

But how can we check that everything is working? We need some test cases!

Fortunately, there is an **analytic solution** to the KdV equation! (many in-fact)

Speed, amplitude and width are linked.

$$u(x,t) = 3c_s \mathrm{sech}^2\left(\frac{\sqrt{c_s}}{2\delta}(x - c_s t - x_0)\right)$$



The dispersion exactly balances the shock formation caused by the nonlinear term.

I implemented a number of methods to simulate the scenario proposed in the paper:

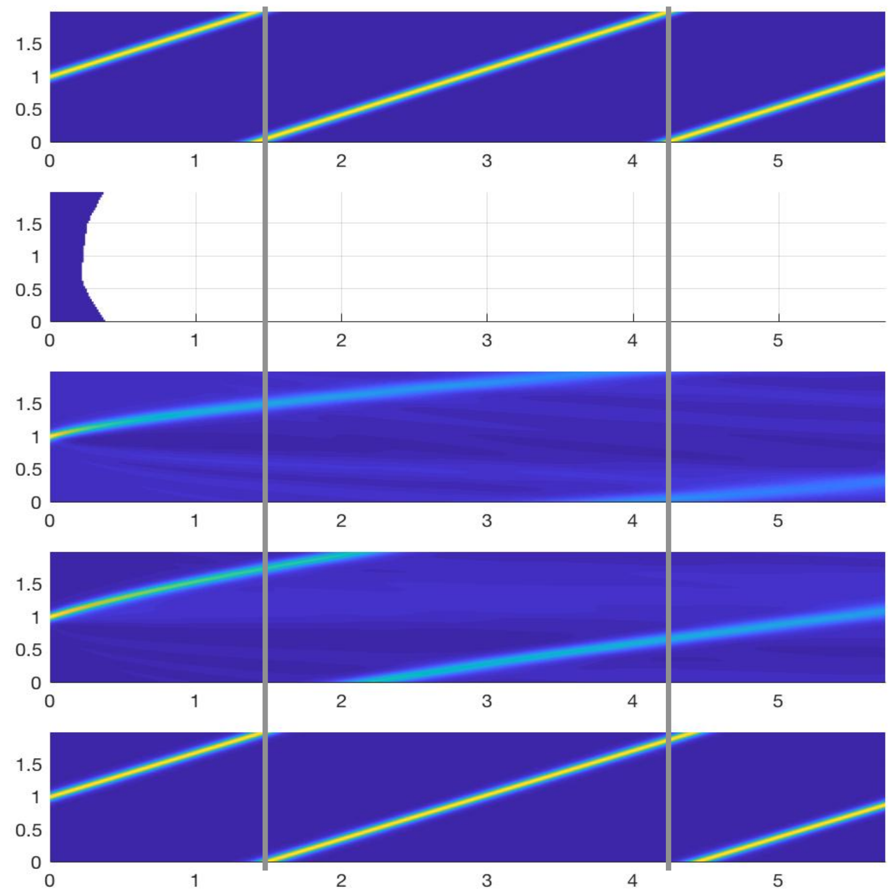
Analytic:

ZK approach:

Implicit Backward:

Implicit Backward (iterative):

Crank-Nicholson (iterative):



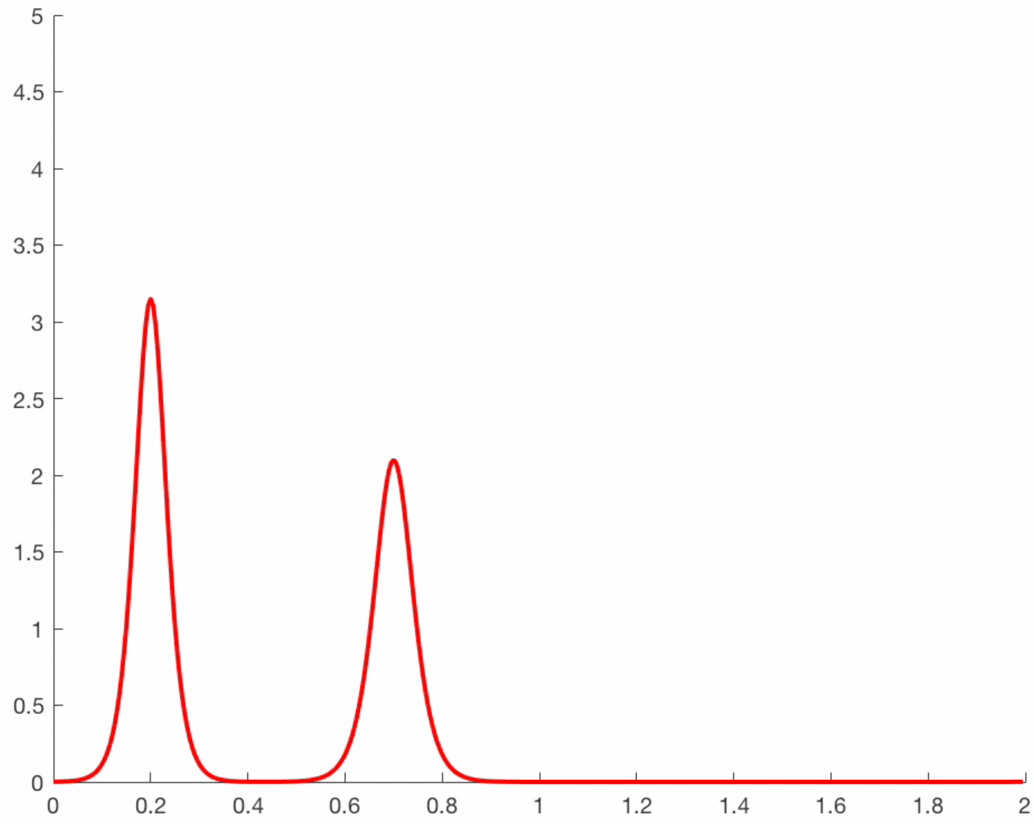
For this set of parameters, the ZK method is unstable.

The "backwards" approaches are only first-order accurate in time.

The Crank-Nicholson approach is 2nd-order accurate in time.

When we have two solitons, amazingly, they'll interact and then go their separate ways

I next simulated one soliton catching the other:



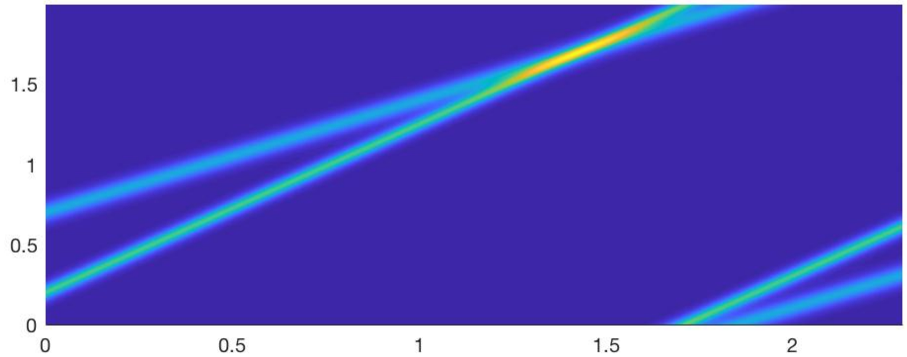
After passing one another, the solitons are unchanged, but accumulate a phase shift.

(There is a closed-form solution for this as well [3])

When we have two solitons, amazingly, they'll interact and then go their separate ways

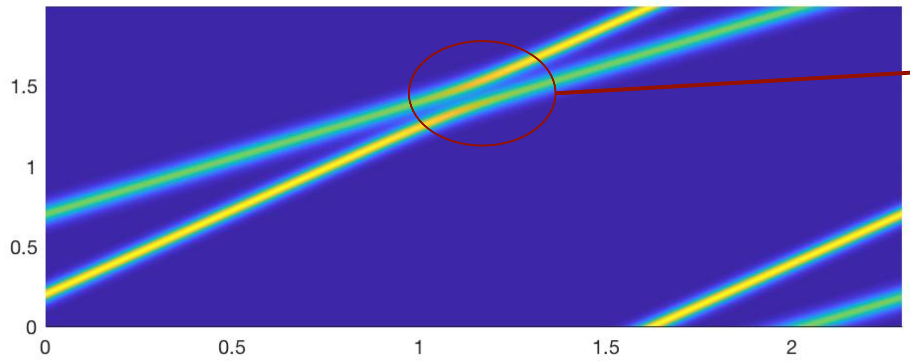
Here is a different view of the same scenario:

No interaction:



After passing one another, the solitons are unchanged, but accumulate a phase shift.

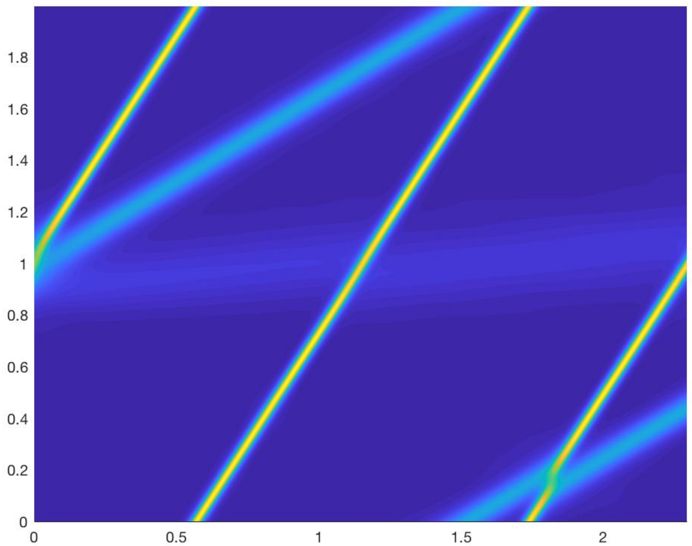
Numerical Solution:

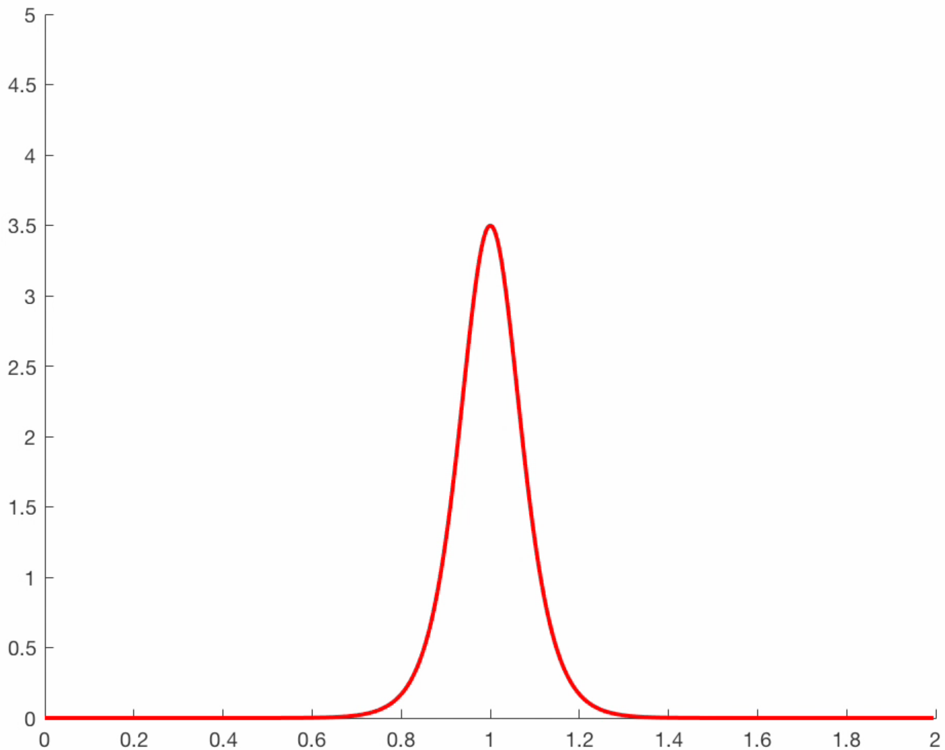


Here, we can see this shift directly.

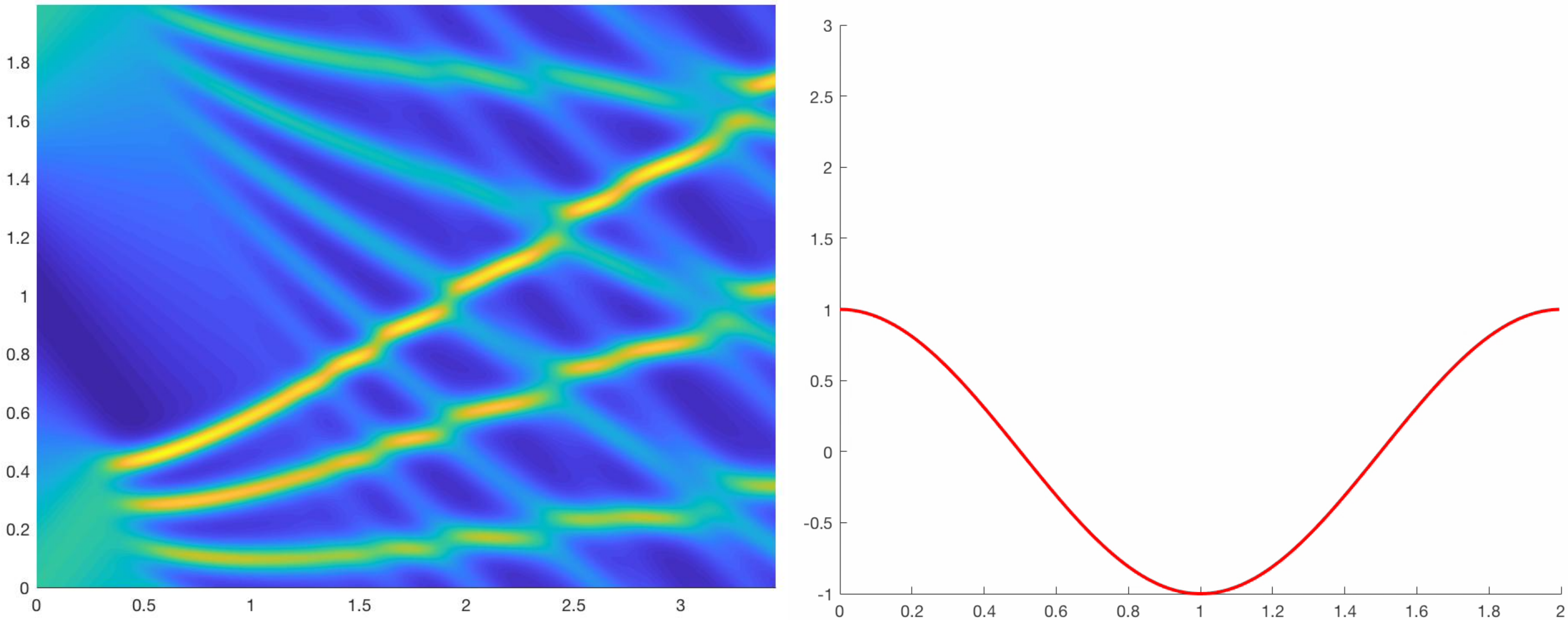
Strong non-soliton pulses will break into a train of soliton solutions.

Even if the wave doesn't begin as a soliton, as time proceeds, the waves will become individual soliton solutions.





Now, we can return to our original problem with a better understanding of the physics.



We can observe the behaviors we'd seen before: (1) Soliton propagation, (2) soliton-soliton interaction and (3) pulse-train formation.

Insights and failures: not everything went as smoothly as planned…

- I also implemented some (explicit) Finite Volume techniques (including Upwind, Centered & QUICK), but none were stable after a time.
- The iterative Crank-Nicholson scheme became very expensive; so I used the ZK scheme for all of the plots, despite stability concerns.
- The ZK scheme *conserves momentum* [1]: this means that solutions can be very accurate, *but* getting the initial conditions correct is essential.
- The averaged "u" in the nonlinear term is important for getting quadratic accuracy in space.

(more) Insights and failures: not everything went as smoothly as planned…

- I had aimed to implement additional schemes, but most improvements beyond the ZK scheme were… complicated [2]:

$$
\begin{aligned}
&\frac{S_n^m}{1-S_n^m} \Big\{ A_-^{(0)} - \sum_{l=-\infty}^{n} \Big[E_{l+1} + S_l^{m+1} W_l (A_-^{(2)} + C_{l-2}) \\
&\quad - \Big\{ D_-^{(4)} \gamma_{l-1} + D_-^{(2)} + \sum_{k=-\infty}^{l-1} (H_k + G_k) \Big\} S_l^{m+1}\gamma_l + (\gamma_l - 1) \Big] W_l^{-1} \Big\} W_n \\
&\quad - \frac{S_n^{m+1}}{1-S_n^{m+1}} \Big\{ D_-^{(0)} + \sum_{l=-\infty}^{n-1} \Big[\frac{-S_l^m}{S_{l+1}^{m+1}} \{ \gamma_{l+1}^{-1} N_{l+1} - N_l + M_l \\
&\quad + S_{l+1}^{m+1} Z_l - S_{l+1}^m \gamma_{l+1}^{-1} N_{l+1} \} + \gamma_l T_{l-2} + (\gamma_l - 1) \Big] W_l^{-1} \Big\} W_{n-1} \\
&\quad + \frac{1}{1-S_n^m} E_{n+1} - \frac{1}{1-S_n^{m+1}} T_{n-2} = \frac{S_n^{m+1} - S_n^m}{(1-S_n^{m+1})(1-S_n^m)},
\end{aligned}
\tag{2.16}
$$

where

$$
\begin{aligned}
E_n &= A_-^{(2)} S_n^m W_{n-1} - S_n^{m+1} D_-^{(2)} + H_n + G_n - S_n^{m+1} \sum_{k=-\infty}^{n} (H_k + G_k) \\
&\quad + S_n^m W_{n-1} C_{n-1} - S_n^m D_-^{(4)}, \qquad C_n = A_-^{(4)} + \sum_{j=-\infty}^{n} P_j W_j^{-1},
\end{aligned}
$$

$$T_n = \gamma_{n+1} M_n + S_{n+1}^{m+1} \gamma_{n+1} Z_n - S_{n+1}^m N_{n+1}, \qquad M_n = S_n^{m+1} W_n A_-^{(4)} - S_n^m D_-^{(4)},$$

$$Z_n = (A_-^{(2)} + \sum_{j=-\infty}^{n} Q_j W_j^{-1}) W_n, \qquad N_n = D_-^{(2)} + \sum_{j=-\infty}^{n} F_j,$$

$$W_n = \prod_{i=-\infty}^{n} \gamma_i, \quad \gamma_i = \left(\frac{1-S_i^m}{1-S_i^{m+1}} \right),$$

$$H_k = A_-^{(4)} (S_{k+1}^m \gamma_k - S_k^m) W_{k-1}, \qquad G_k = (S_k^m - S_{k+1}^{m+1}) D_-^{(4)},$$

$$F_j = A_-^{(4)} (S_j^{m+1} W_j - S_{j-1}^{m+1} W_{j-1}) + D_-^{(4)} (S_{j-1}^m - S_j^{m+1}),$$

$$P_j = A_-^{(4)} (S_j^{m+1} - S_{j+1}^m) W_j + D_-^{(4)} (S_{j+1}^{m+1} - S_j^{m+1} \gamma_j),$$

$$Q_j = (S_{j-1}^{m+1} - S_j^m) W_j A_-^{(4)} - (S_{j-1}^m \gamma_j - S_j^m) D_-^{(4)},$$

$$A_-^{(2)} = -\frac{2}{3} A_-^{(0)} + \frac{1}{2}\alpha, \qquad D_-^{(2)} = -\frac{2}{3} A_-^{(0)} - \frac{1}{2}\alpha,$$

$$A_-^{(4)} = \frac{1}{6} A_-^{(0)} - \frac{1}{4}\alpha, \qquad D_-^{(4)} = \frac{1}{6} A_-^{(0)} + \frac{1}{4}\alpha, \quad \alpha = \frac{\Delta t}{(\Delta x)^3},$$

$A_-^{(0)}$ = arbitrary constant and $S_n^m = 1 - e^{-(\Delta x)^2} u_n^m$.

Ok, so why do I care about solitons?

Intuition is built from an understanding how different terms in nonlinear differential equations interact:

$$\begin{aligned}\frac{\partial E}{\partial z} &= \frac{\mathrm{i}}{2k}\nabla_{\perp}^{2}E + \mathrm{i}\frac{k}{2}n_2\,\epsilon_{\mathrm{o}}c|E|^2E \\ &- \frac{1}{2c}\int_{-\infty}^{\tau}\omega_{\mathrm{p}}^{2}E\;\mathrm{d}\tau' - \frac{I_{\mathrm{p}}}{2c\epsilon\,\mathrm{Re}(E)^2}\frac{\partial\rho}{\partial\tau}E.\end{aligned}$$

This is a *very* nonlinear equation I modeled for a paper of mine when I studied strong-field optics.

I like to use numerical methods to develop experimental intuition.

Numerical solutions are important for generating understandable results after experiments are done, but intuition built from simple test cases can be useful for guiding experiments as well.

Thanks!

References

[1] N. J. Zabusky and M. D. Kruskal, Interaction of “Solitons” in a collisionless plasma and the recurrence of initial states, Phys. Rev. Lett. **15** (1965).

[2] T. R. Taha and M. J. Ablowitz, Analytical and numerical aspects of certain nonlinear evolution equations. III. Numerical, Korteweg-de Vries Equation, J. Comp. Phys. **55** (1984).

[3] T. R. Marchant and N. F. Smyth, Soliton interaction for the extended Korteweg-de Vries equation, IMA J. of Applied Mathematics **56** (1996).