Application of Numerical Dissipation: A FV Solver for 2D Burgers Equation on Unstructured Meshes Generated from Matlab PDE Toolbox

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Burgers Equation

- For Newtonian Fluid + incompressible + constant μ:

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g}$$

$$\nabla \cdot \mathbf{v} = 0$$

- Due to incompressibility,

$$\nabla \cdot (\rho \mathbf{v}\mathbf{v}) = \rho \mathbf{v} \cdot \nabla \mathbf{v}$$

- If we further neglect the pressure term and gravity term,

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = \frac{\mu}{\rho} \nabla^2 \mathbf{v} = \mu' \nabla^2 \mathbf{v}$$

Burgers Equation

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = \mu' \nabla^2 \mathbf{v}$$

- It should be noted that we can no longer impose the incompressibility condition, because we have eliminated an unknown variable.
- We will see how the numerical solution violates the incompressibility condition.
- In 2D,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \mu'\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \mu'\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

Numerical Dissipation/Diffusion

- Upwind scheme introduces numerical dissipation because the leading truncation error is a diffusive term.
- We have already known that upwind scheme is good for 1D Burgers Equation in the sense of no oscillation.
- The main idea is to generalize it into multi-dimensional case.

Numerical Dissipation/Diffusion

- Consider 1D scalar conservation law:

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}=0$$

- Corresponding conservative form is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_{i-1/2}}^{x_{i+1/2}} u\mathrm{d}V + f(u)\Big|_{x_{i-1/2}}^{x_{i+1/2}} = 0$$

- The upwind scheme reads

$$\text{If } \frac{\mathrm{d}f}{\mathrm{d}u}>0,\ F_{x_{i+1/2}} = f(u)\Big|_{x_i}$$

$$\text{If } \frac{\mathrm{d}f}{\mathrm{d}u}<0,\ F_{x_{i+1/2}} = f(u)\Big|_{x_{i+1}}$$

Numerical Dissipation/Diffusion

$$\text{If } \frac{\mathrm{d}f}{\mathrm{d}u} > 0,\ F_{x_{i+1/2}} = f(u)\big|_{x_i} \qquad \text{If } \frac{\mathrm{d}f}{\mathrm{d}u} < 0,\ F_{x_{i+1/2}} = f(u)\big|_{x_{i+1}} \quad \Leftrightarrow \quad F_{x_{i+1/2}} = \frac{f(u_i)+f(u_{i+1})}{2} + \left|\frac{\mathrm{d}f}{\mathrm{d}u}\right| \frac{x_{i+1}-x_i}{2} \frac{u_i-u_{i+1}}{x_{i+1}-x_i}$$

- The last term can be rewritten as

$$\left|\frac{\mathrm{d}f}{\mathrm{d}u}\right| \frac{x_{i+1}-x_i}{2} \frac{u_i-u_{i+1}}{x_{i+1}-x_i} = -\nu \frac{u_{i+1}-u_i}{x_{i+1}-x_i}$$

Derivative at the interface

- It can be seen as a diffusive flux with diffusivity ν.

$$\nu = \left|\frac{\mathrm{d}f}{\mathrm{d}u}\right| \frac{x_{i+1}-x_i}{2}$$

Numerical Dissipation/Diffusion

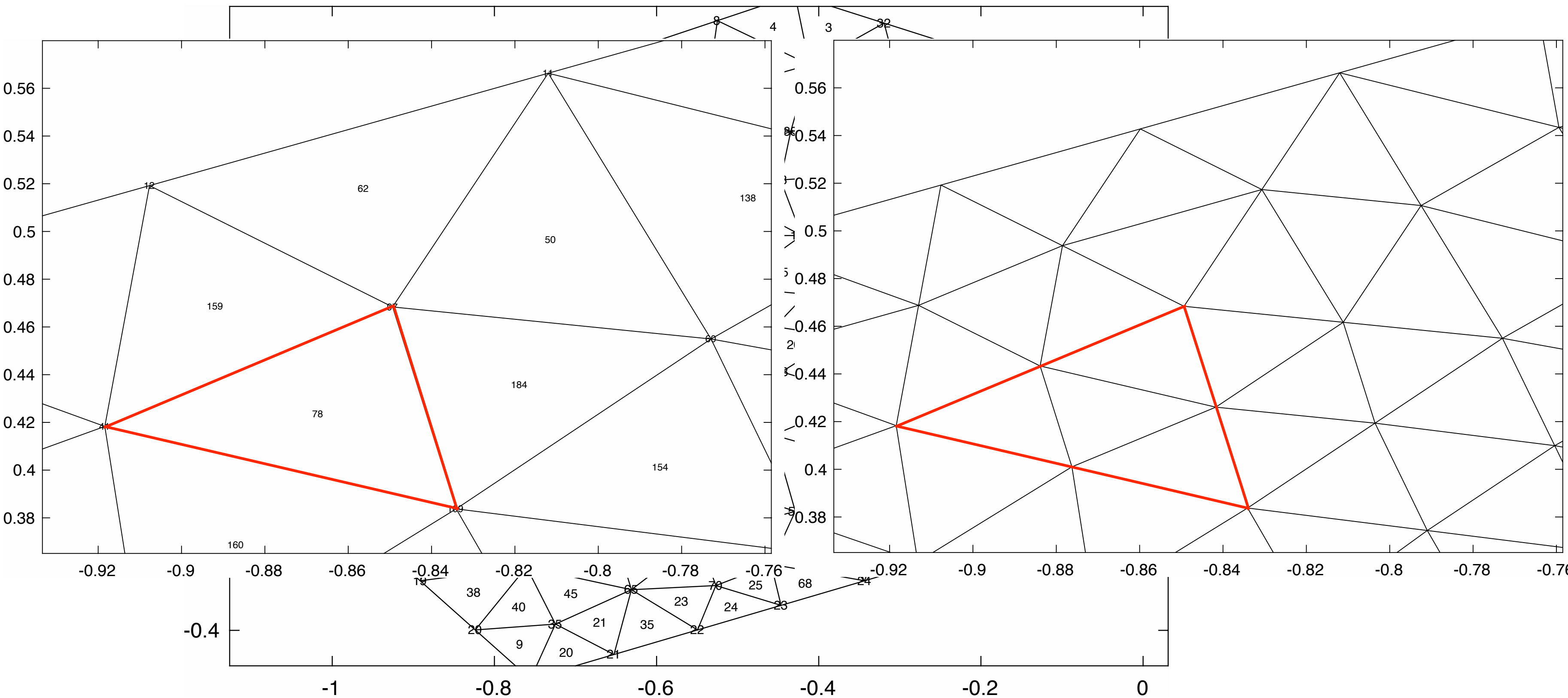
- In 2D or 3D,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{CV} \mathbf{u}\,\mathrm{d}V + \int_{CS} \mathbf{F}(\mathbf{u})\cdot\mathbf{n}\,\mathrm{d}S = 0$$

$$\nu = \left|\frac{\mathrm{d}f}{\mathrm{d}u}\right|\frac{x_{i+1}-x_i}{2} \qquad \Rightarrow \boxed{\nu = \left|\frac{\mathrm{d}(\mathbf{F}\cdot\mathbf{n})}{\mathrm{d}\mathbf{u}}\right|\frac{\Delta}{2}}$$

- We need to calculate the **maximum absolute value of the eigenvalues** of $\boxed{\frac{\mathrm{d}(\mathbf{F}\cdot\mathbf{n})}{\mathrm{d}\mathbf{u}}}$, and Δ is the distance between two cell centers.

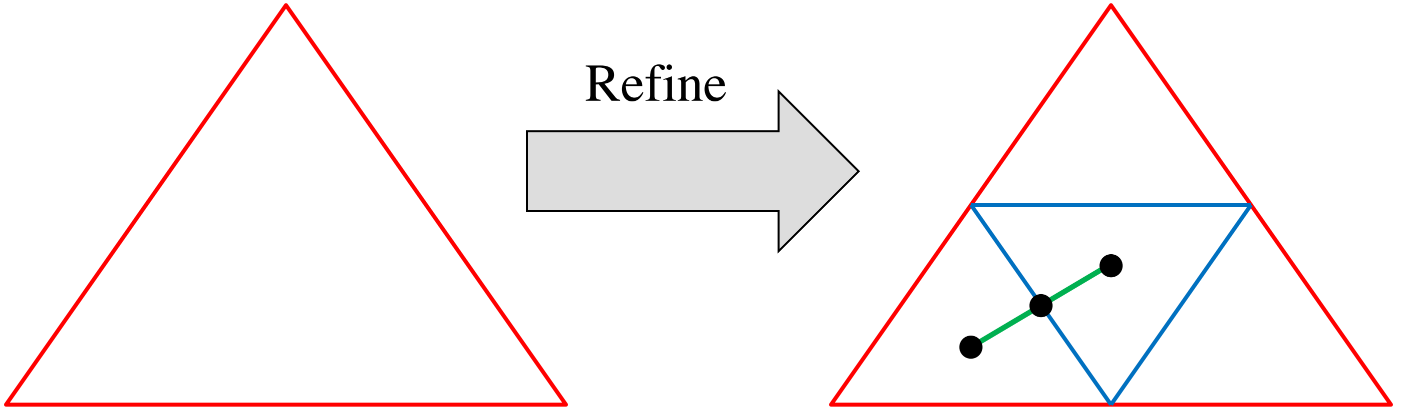
Triangular Grid



![MIT logo] Massachusetts Institute of Technology

Triangular Grid

- When we refine the grid, each triangle is divided into four identical triangles.



FV Discretization

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\mu'\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=\mu'\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

$$\mathbf{F}_{ax}=\begin{bmatrix} u^2/2 \\ uv \end{bmatrix},\ \mathbf{F}_{dx}=-\mu'\frac{\partial}{\partial x}\begin{bmatrix} u \\ v \end{bmatrix}$$

$$\mathbf{F}_{ay}=\begin{bmatrix} uv \\ v^2/2 \end{bmatrix},\ \mathbf{F}_{dy}=-\mu'\frac{\partial}{\partial y}\begin{bmatrix} u \\ v \end{bmatrix}$$

$$\mathbf{s}=\begin{bmatrix} u\dfrac{\partial v}{\partial y} \\ v\dfrac{\partial u}{\partial x} \end{bmatrix}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{CV}\begin{bmatrix} u \\ v \end{bmatrix}\mathrm{d}V+\int_{CS}\left[(\mathbf{F}_{ax}+\mathbf{F}_{dx})n_x+(\mathbf{F}_{ay}+\mathbf{F}_{dy})n_y\right]\mathrm{d}S=\int_{CV}\mathbf{s}\,\mathrm{d}V$$

FV Discretization

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{CV}\begin{bmatrix} u \\ v \end{bmatrix}\mathrm{d}V + \int_{CS}\left[(\mathbf{F}_{ax}+\mathbf{F}_{dx})n_x + (\mathbf{F}_{ay}+\mathbf{F}_{dy})n_y\right]\mathrm{d}S = \int_{CV}\mathbf{s}\,\mathrm{d}V$$

- For the i-th control volume (triangle), we have

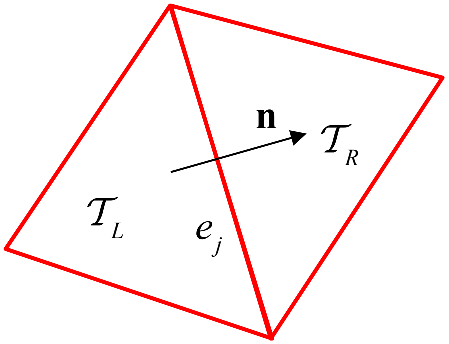
$$\Delta V_i \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u_i \\ v_i \end{bmatrix} + \sum_{j=1}^{3}\left[(\mathbf{F}_{ax}+\mathbf{F}_{dx})n_x + (\mathbf{F}_{ay}+\mathbf{F}_{dy})n_y\right]S_{e_j} = \Delta V_i \begin{bmatrix} u_i \left.\dfrac{\partial v}{\partial y}\right|_i \\ v_i \left.\dfrac{\partial u}{\partial x}\right|_i \end{bmatrix}$$

FV Discretization

- Advective Flux (using Numerical Dissipation)

$$\Delta V_i \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u_i \\ v_i \end{bmatrix} + \sum_{j=1}^{3}\left[(\mathbf{F}_{ax}+\mathbf{F}_{dx})n_x + (\mathbf{F}_{ay}+\mathbf{F}_{dy})n_y\right]S_{e_j} = \Delta V_i \begin{bmatrix} u_i \left.\frac{\partial v}{\partial y}\right|_i \\ v_i \left.\frac{\partial u}{\partial x}\right|_i \end{bmatrix}$$

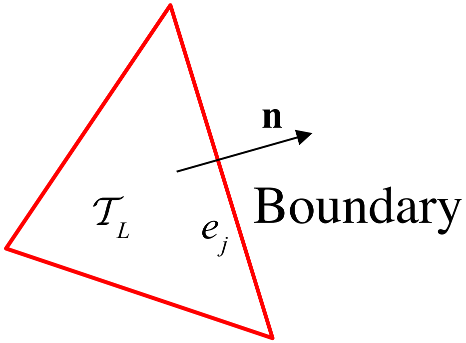
$$\mathbf{F}_a = \mathbf{F}_{ax} n_x + \mathbf{F}_{ay} n_y$$



$$\mathbf{F}_a|_{e_j} = \frac{\mathbf{F}_a|_{\mathcal{T}_L} + \mathbf{F}_a|_{\mathcal{T}_R}}{2} + \nu \frac{\mathbf{u}_{\mathcal{T}_L} - \mathbf{u}_{\mathcal{T}_R}}{\Delta}$$

$$\nu = \left\{ |un_x + vn_y| + \sqrt{|uvn_x n_y|} \right\} \frac{\Delta}{2}$$

$$\max\left|\lambda\left(\frac{\mathrm{d}\mathbf{F}_a}{\mathrm{d}\mathbf{u}}\right)\right|$$



FV Discretization

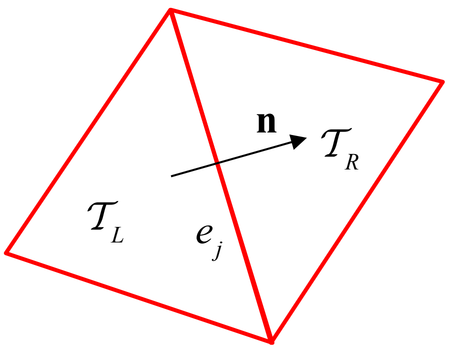
- Diffusive Flux

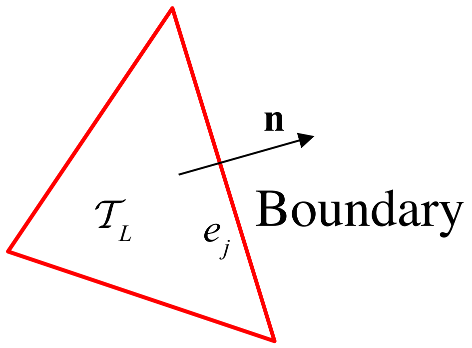
$$\Delta V_i \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u_i \\ v_i \end{bmatrix} + \sum_{j=1}^{3}\left[(\mathbf{F}_{ax}+\mathbf{F}_{dx})n_x + (\mathbf{F}_{ay}+\mathbf{F}_{dy})n_y\right]S_{e_j} = \Delta V_i \begin{bmatrix} u_i \left.\frac{\partial v}{\partial y}\right|_i \\ v_i \left.\frac{\partial u}{\partial x}\right|_i \end{bmatrix}$$

$$\mathbf{F}_d = \mathbf{F}_{dx} n_x + \mathbf{F}_{dy} n_y$$

$$\mathbf{F}_{dx} = -\mu' \frac{\partial}{\partial x}\begin{bmatrix} u \\ v \end{bmatrix}$$

$$\mathbf{F}_{dy} = -\mu' \frac{\partial}{\partial y}\begin{bmatrix} u \\ v \end{bmatrix}$$



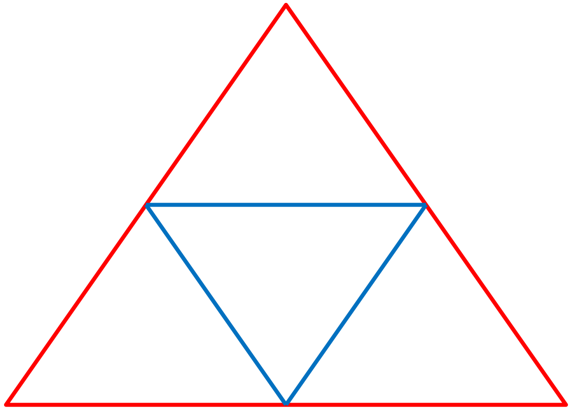


FV Discretization

- Diffusive Flux

$$\mathbf{F}_{dx} = -\mu' \frac{\partial}{\partial x}\begin{bmatrix} u \\ v \end{bmatrix}$$

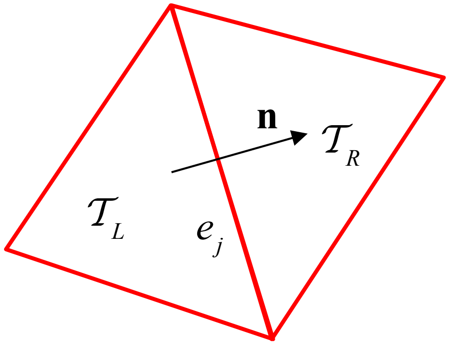
$$\mathbf{F}_{dy} = -\mu' \frac{\partial}{\partial y}\begin{bmatrix} u \\ v \end{bmatrix}$$



- Divergence Theorem

$$\Delta V_i \left.\frac{\partial u}{\partial x}\right|_i = \sum_{j=1}^{3}\left(\frac{u_L + u_R}{2}\right) n_x S_{e_j}$$

$$\left.\frac{\partial u}{\partial x}\right|_{e_j} = \frac{1}{2}\left(\left.\frac{\partial u}{\partial x}\right|_{\mathcal{T}_L} + \left.\frac{\partial u}{\partial x}\right|_{\mathcal{T}_R}\right)$$



FV Discretization

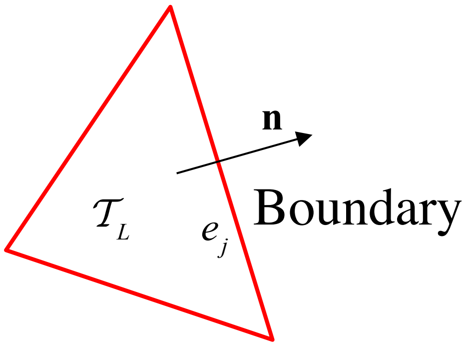
- Source Term

$$\Delta V_i \begin{bmatrix} u_i \left.\frac{\partial v}{\partial y}\right|_i \\ v_i \left.\frac{\partial u}{\partial x}\right|_i \end{bmatrix}$$

- Boundary Condition: Open boundary

$$\mathbf{F}_a |_{e_j} = \mathbf{F}_a |_{\mathcal{T}_L}$$

$$\left.\frac{\partial u}{\partial x}\right|_{e_j} = \left.\frac{\partial u}{\partial x}\right|_{\mathcal{T}_L}$$

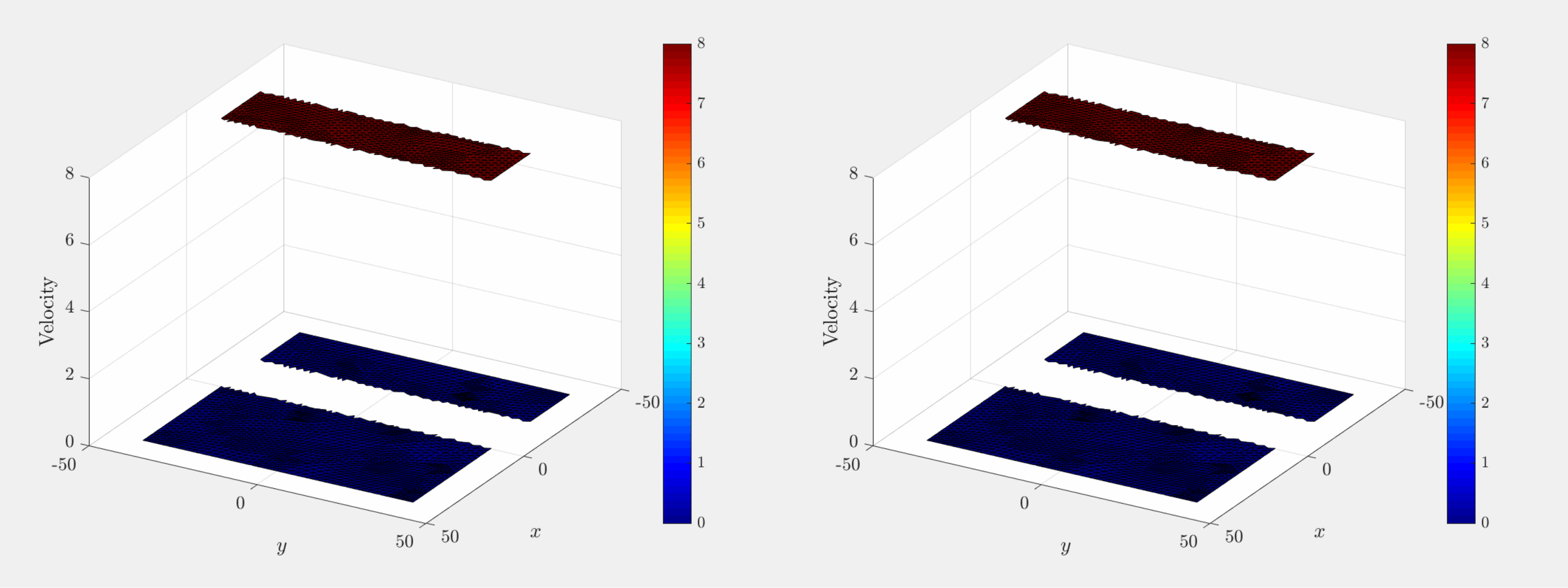


- Time Integration: ODE45

Numerical Solutions: *U* plots

$$\nu = 0$$

$\mu' = 0$ $\mu' = 5$



Numerical Solutions: Div of [U,V]

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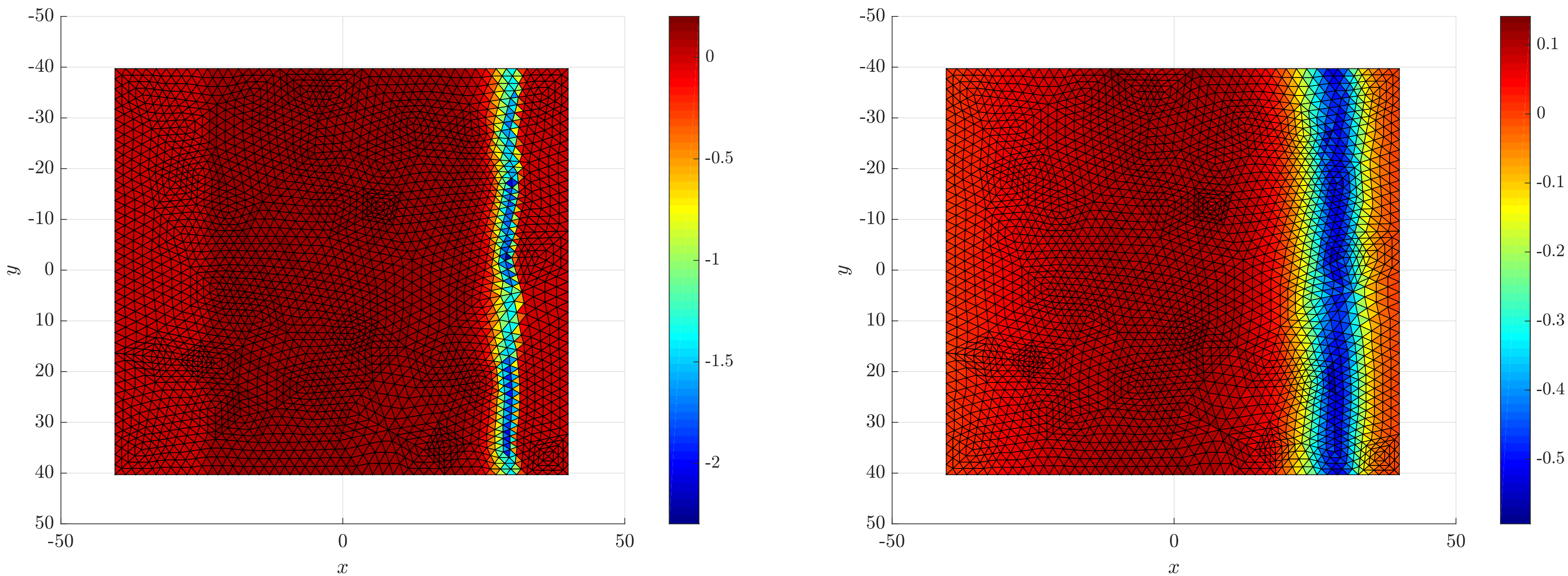

Numerical Solutions: *U* plots

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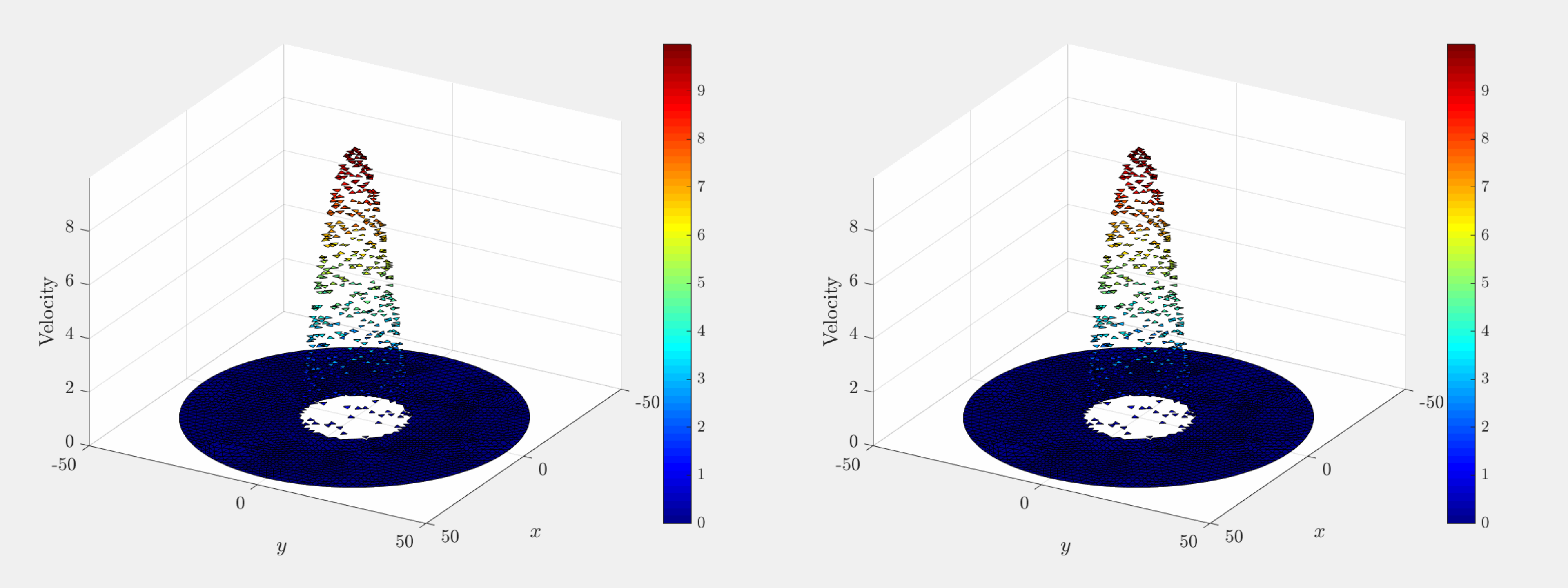

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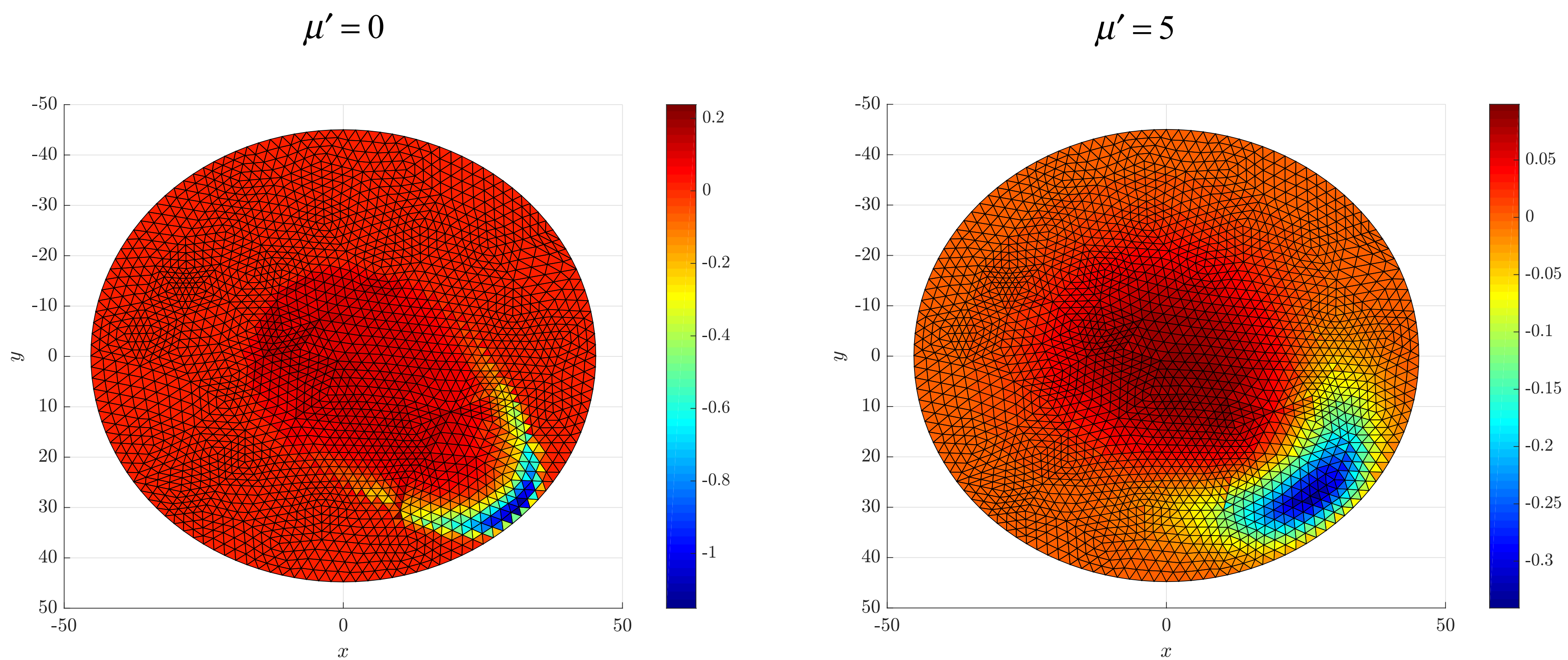

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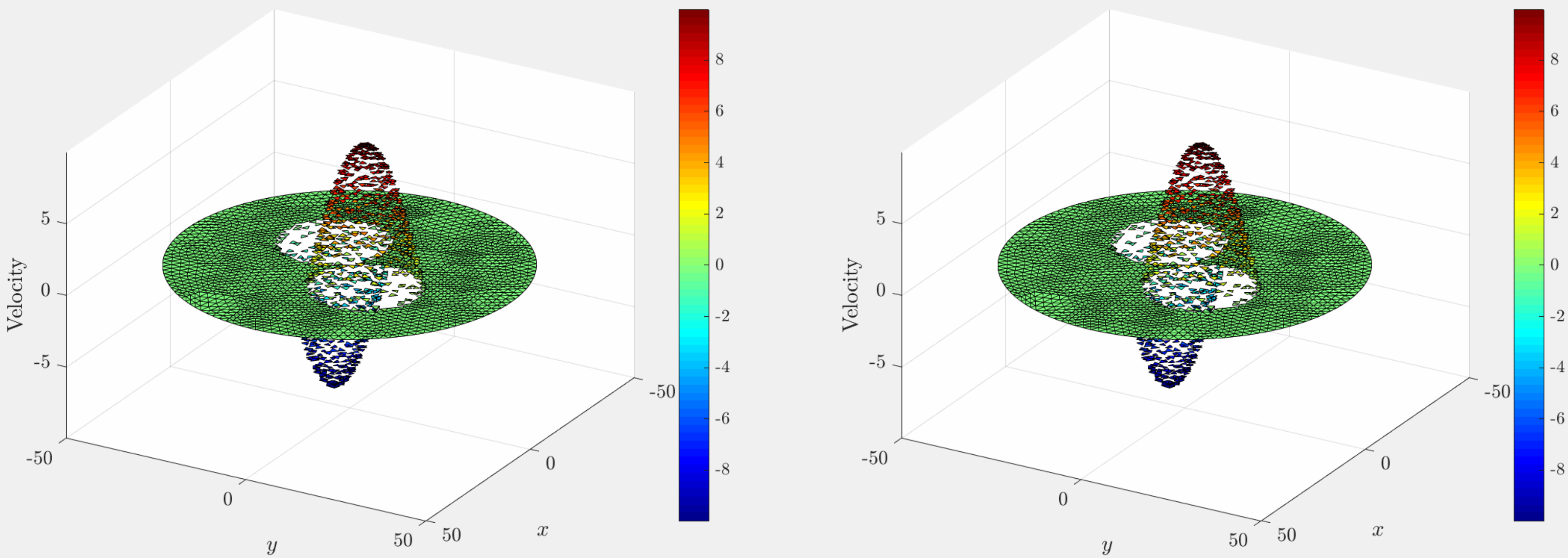

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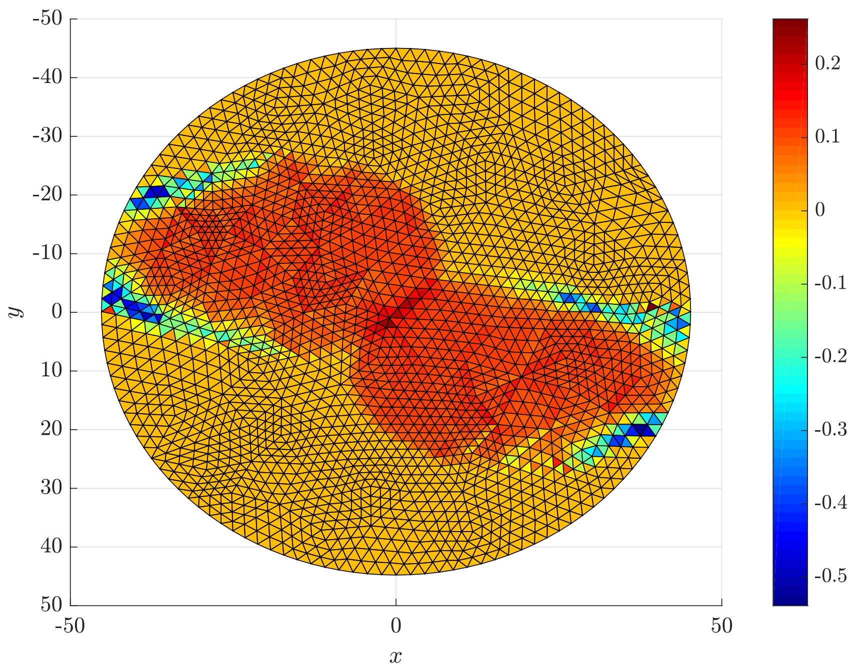

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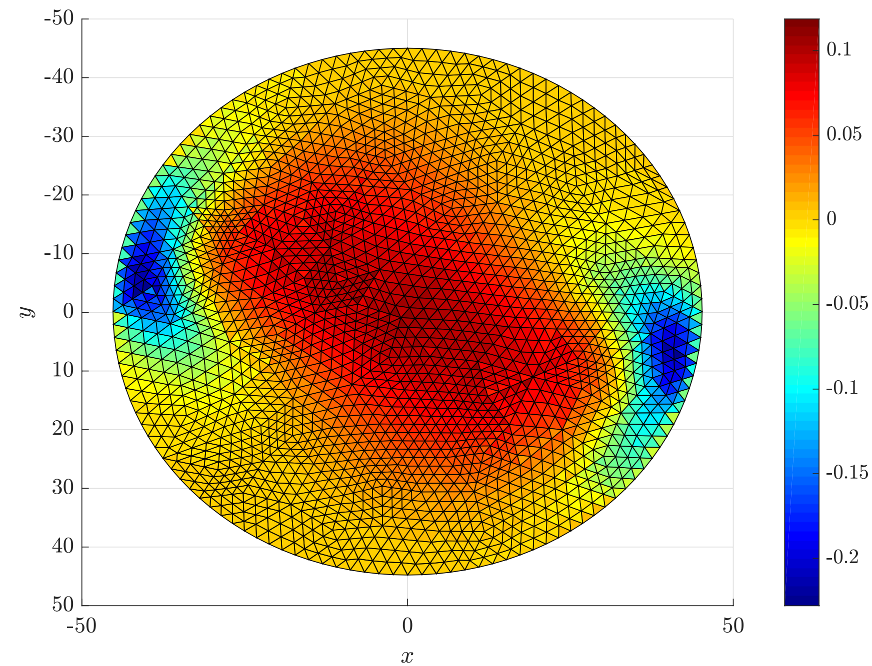

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Numerical Solutions: *U* plots

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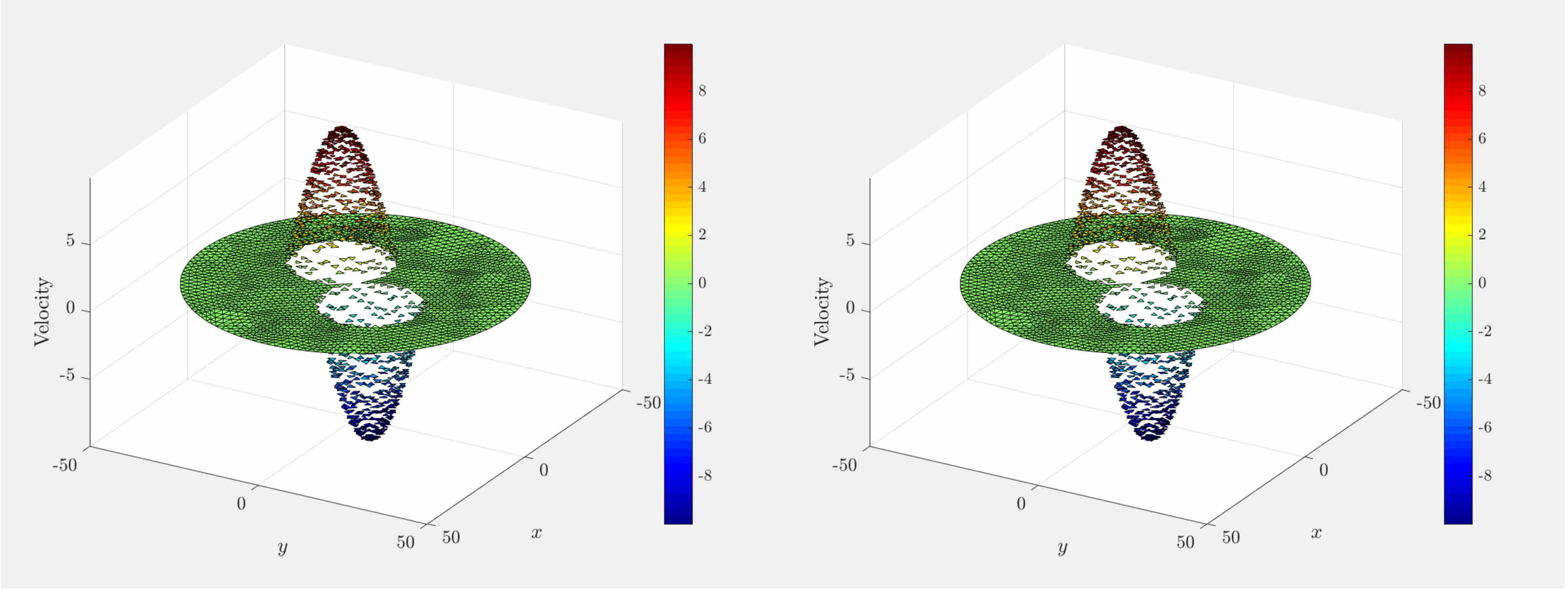

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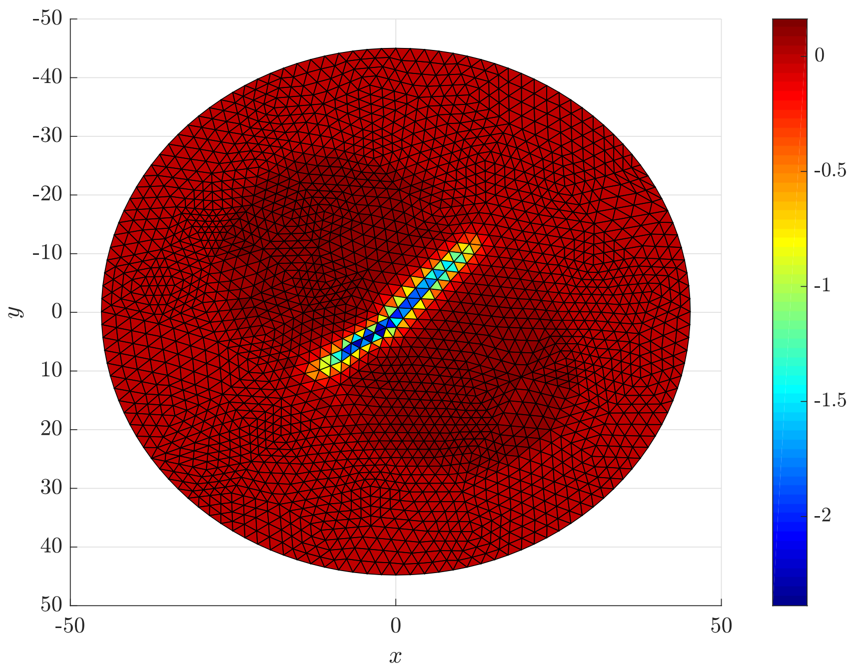

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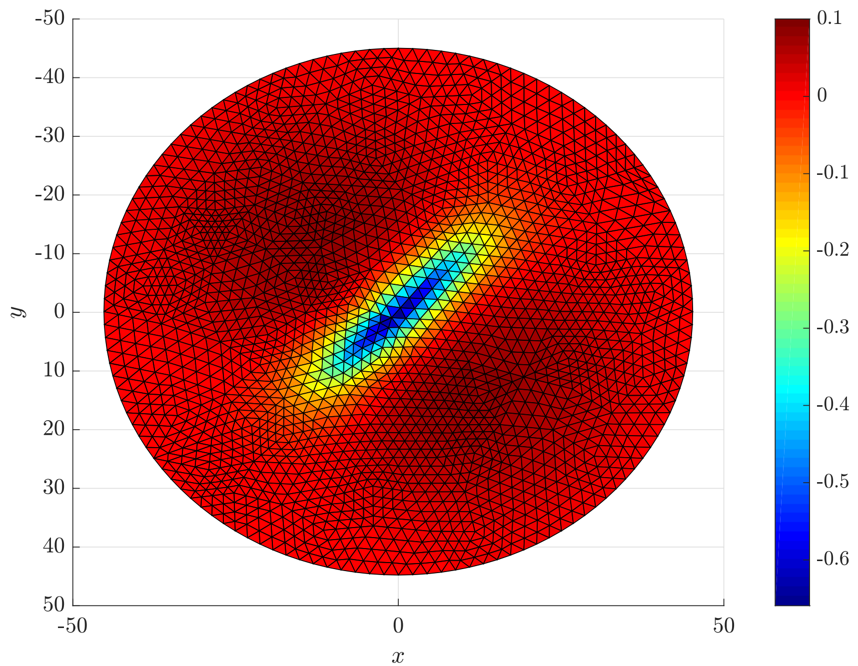

Thanks for your attention.

Q & A