MIT LABORATORY FOR AVIATION AND THE ENVIRONMENT

CAMBRIDGE | May 16th, 2018

Discontinuous Galerkin Methods using Strongly-Stability Preserving Runge-Kutta methods

Thibaud FRITZ

Historical background

LA-UR-73-479

TITLE: TRIANGULAR MESH METHODS FOR THE NEUTRON TRANSPORT EQUATION

AUTHOR(S): W. H. Reed and T. R. Hill

TVB Runge-Kutta Local Projection Discontinuous Galerkin Finite Element Method for Conservation Laws II: General Framework

By Bernardo Cockburn and Chi-Wang Shu

THE RUNGE-KUTTA LOCAL PROJECTION DISCONTINUOUS GALERKIN FINITE ELEMENT METHOD FOR CONSERVATION LAWS IV: THE MULTIDIMENSIONAL CASE

BERNARDO COCKBURN, SUCHUNG HOU, AND CHI-WANG SHU

THE LOCAL DISCONTINUOUS GALERKIN METHOD FOR TIME-DEPENDENT CONVECTION-DIFFUSION SYSTEMS*

BERNARDO COCKBURN† AND CHI-WANG SHU‡

The Development of Discontinuous Galerkin Methods

Bernardo Cockburn[1], George E. Karniadakis[2], and Chi-Wang Shu[2]

Motivation and framing

Motivation and Context

- Solving non-linear hyperbolic conservation laws:

$$\boldsymbol{U}_t + \boldsymbol{F}(\boldsymbol{U})_x = \boldsymbol{0}$$

- Explicit (parallelizable), stable high-order time integration scheme

- Total variation diminishing = No-new maxima introduced by the solver.

Different options:

-Spectral methods

-WENO/ENO (Weighted Essentially Non-Oscillatory) scheme

-Discontinuous Galerkin methods

- Parallelizable
- High order accuracy
- BUT can lead to numerical oscillations

Methods

Step 1: **Space discretization**

$$\int_K (u_h)_t v_h dx - \int_K f(u_h) \cdot \nabla v_h dx + \int_{\partial K} \hat{f}(u_h) \cdot n_K v_h ds = 0$$

Step 2: **RK time discretization** $\quad \mathrm{d_t}(\mathrm{u_h}) = L(u_h)$

$$\begin{array}{c|cccc} 0 & & 0 & & \\ c_2 & a_{21} & & & \\ \vdots & \vdots & \ddots & & \\ c_s & a_{s1} & \cdots & a_{s,s-1} & \\ \hline & b_1 & \cdots & \cdots & b_s \end{array}$$

Step 3: **Generalized Slope Limiter**

Use of a non-linear projector $\Lambda\Pi_h$

Methods

Step 1: Space discretization

$$\int_K (u_h)_t v_h dx - \int_K f(u_h) \cdot \nabla v_h dx + \int_{\partial K} \hat{f}(u_h) \cdot n_K v_h ds = 0$$

Step 2: RK time discretization $\quad \mathrm{d_t}(\mathrm{u_h}) = L(u_h)$

$$\begin{array}{c|cccc}
0 & 0 & & & \\
c_2 & a_{21} & & & \\
\vdots & \vdots & \ddots & & \\
c_s & a_{s1} & \cdots & a_{s,s-1} & \\
\hline
 & b_1 & \cdots & \cdots & b_s
\end{array}$$

Step 3: Generalized Slope Limiter

Use of a non-linear projector $\Lambda\Pi_h$

Methods: Space discretization

$$\partial_t u \; + \; \boldsymbol{\nabla} \cdot \boldsymbol{f}(u) \; = \; 0$$

$$\forall T_{k=1..N}, \qquad V_h^k = \left\{ \varphi : \Omega \to \mathbb{R}, \qquad \varphi = \left(\sum_{l=1}^{P_k} a_l \psi_l^k \right) \chi_k \right\}$$

$$u_h = \sum_{k=1}^{N} \chi_k(\boldsymbol{x}) \sum_{j=1}^{P_k} u_k^j(t) \psi_k^j(\boldsymbol{x})$$

$$\forall k' \in [\![1, N]\!], \forall i \in [\![1, P_{k'}]\!], \qquad \int_\Omega \left(\partial_t u_h(t, \boldsymbol{x}) + \boldsymbol{\nabla} \cdot \boldsymbol{f}(u_h(t, \boldsymbol{x})) \right) \psi_{k'}^i(\boldsymbol{x}) = 0$$

$$\sum_{j=1}^{P_k} \left(\int_{T_k} \psi_k^j \psi_k^i \, d\boldsymbol{x} \right) d_t u_k^j(t) - \int_{T_k} \boldsymbol{f}(u_h\,(t, \boldsymbol{x})) \cdot \nabla\, \psi_k^i(\boldsymbol{x}) d\boldsymbol{x} + \int_{\partial T_k} \psi_k^i(s) \boldsymbol{f}\big(u_h(t, \boldsymbol{s})\big) \cdot \boldsymbol{n} ds \; = 0$$

We often consider that: $\boldsymbol{f}(u_h(t, \boldsymbol{x})) = \sum \boldsymbol{f}\left(u_k^j(t) \right) \psi_k^j(\boldsymbol{x})$

Methods

Step 1: Space discretization

$$\int_K (u_h)_t v_h dx - \int_K f(u_h) \cdot \nabla v_h dx + \int_{\partial K} \hat{f}(u_h) \cdot n_K v_h ds = 0$$

Step 2: RK time discretization

$$\mathrm{d_t(u_h)} = L(u_h)$$

$$\begin{array}{c|cccc} 0 & & 0 & & \\ c_2 & a_{21} & & & \\ \vdots & \vdots & \ddots & & \\ c_s & a_{s1} & \cdots & a_{s,s-1} & \\ \hline & b_1 & \cdots & \cdots & b_s \end{array}$$

Step 3: Generalized Slope Limiter

Use of a non-linear projector $\Lambda\Pi_h$

MIT LABORATORY FOR **AVIATION AND THE ENVIRONMENT**

Methods: Time integration

1. $u_h^{(0)} = u_h^n$

2. $u_h^{(i)} = \sum_{l=0}^{i-1} \alpha_{il} w_h^{il}, \qquad w_h^{il} = u_h^{(l)} + \frac{\beta_{il}}{\alpha_{il}} \Delta \mathrm{t_n} \mathrm{L_h} \left(\mathrm{u_h^{(l)}}\right)$

3. $u_h^{n+1} = u_h^K$

Methods

Step 1: Space discretization

$$\int_K (u_h)_t v_h dx - \int_K f(u_h) \cdot \nabla v_h dx + \int_{\partial K} \hat{f}(u_h) \cdot n_K v_h ds = 0$$

Step 2: RK time discretization $\quad \mathrm{d_t}(\mathrm{u_h}) = L(u_h)$

$$\begin{array}{c|cccc} 0 & & 0 & & \\ c_2 & a_{21} & & & \\ \vdots & \vdots & \ddots & & \\ c_s & a_{s1} & \cdots & a_{s,s-1} & \\ \hline & b_1 & \cdots & \cdots & b_s \end{array}$$

Step 3: Generalized Slope Limiter

Use of a non-linear projector $\Lambda\Pi_h$

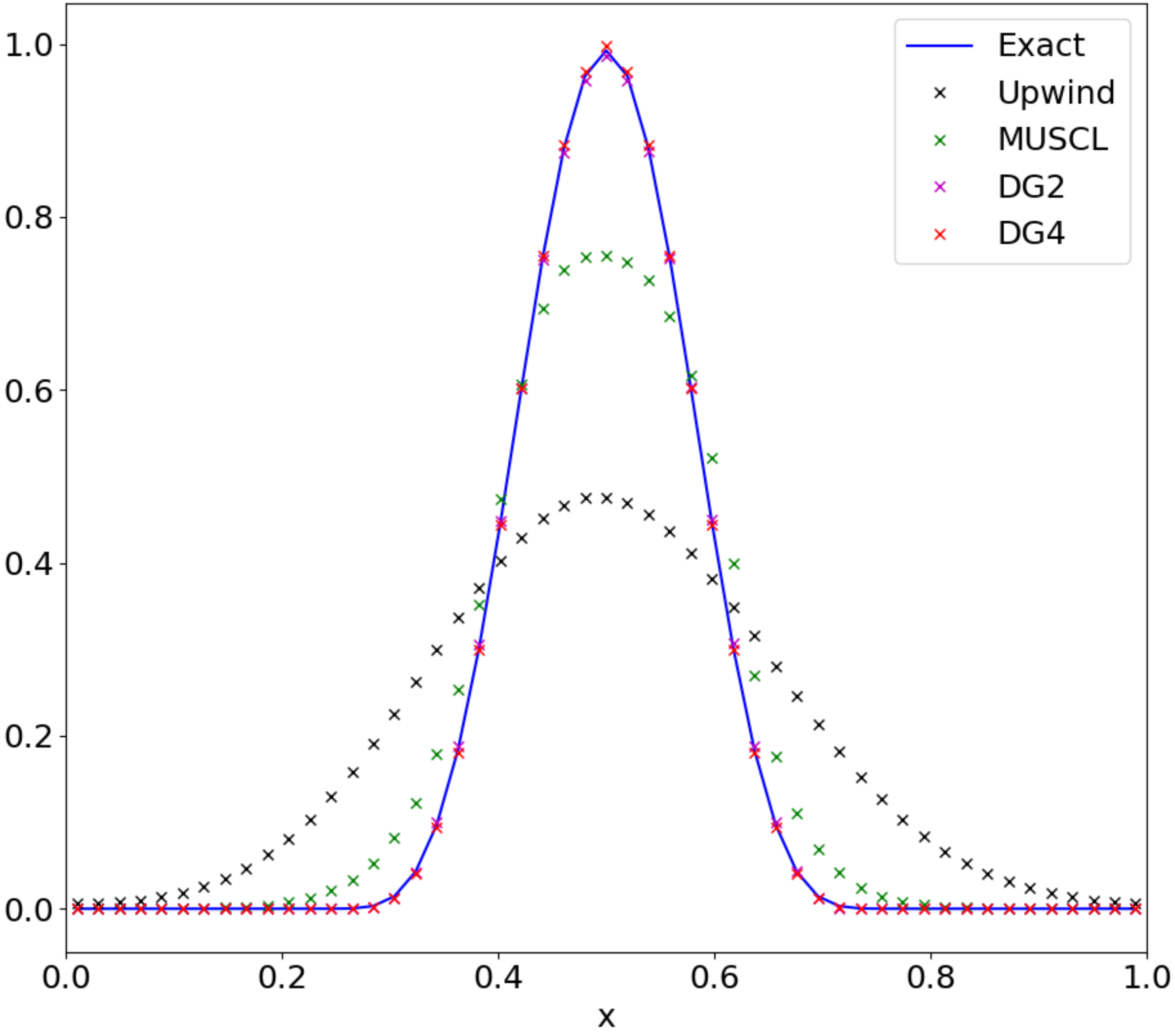
Methods: Slope limiter (Modified time integration)

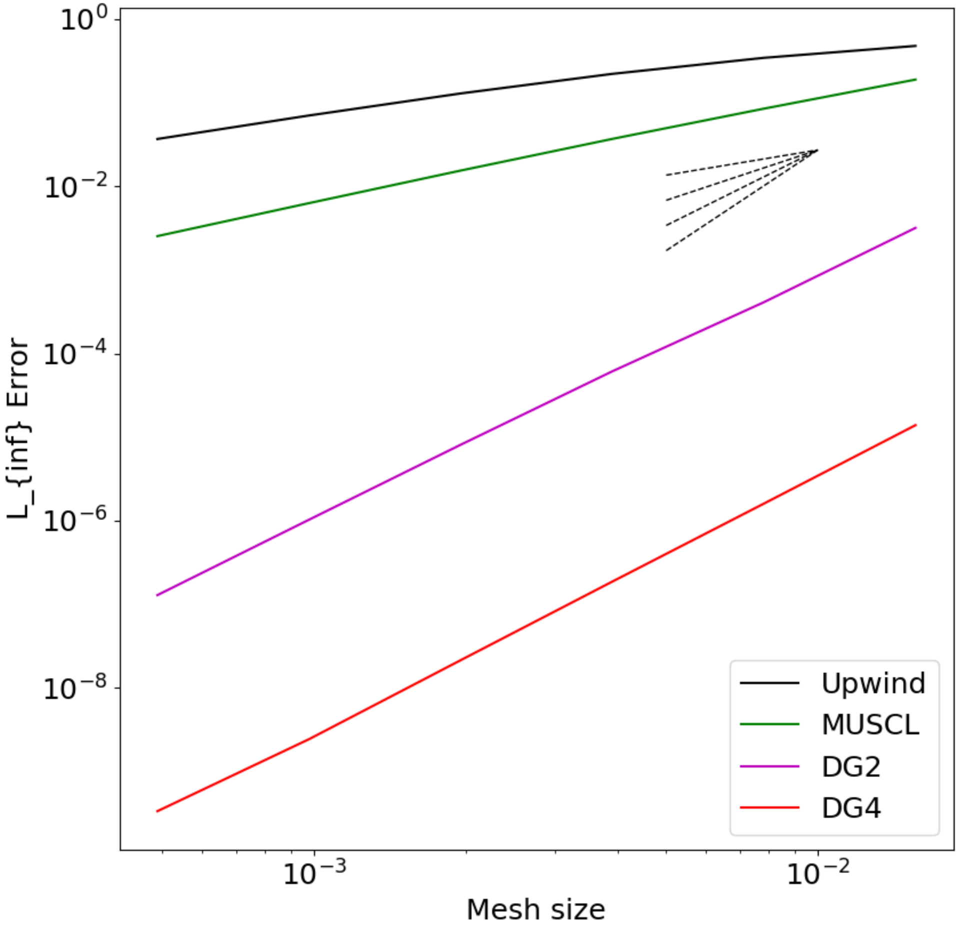
$$1.\ \ u_h^{(0)} = u_h^n$$

$$2.\ \ u_h^{(i)} = \Lambda\Pi_h\left(\sum_{l=0}^{i-1} \alpha_{il} w_h^{il}\right), \qquad w_h^{il} = u_h^{(l)} + \frac{\beta_{il}}{\alpha_{il}} \Delta \mathrm{t_n L_h} \left(\mathrm{u_h^{(l)}}\right)$$

$$3.\ \ u_h^{n+1} = u_h^K$$

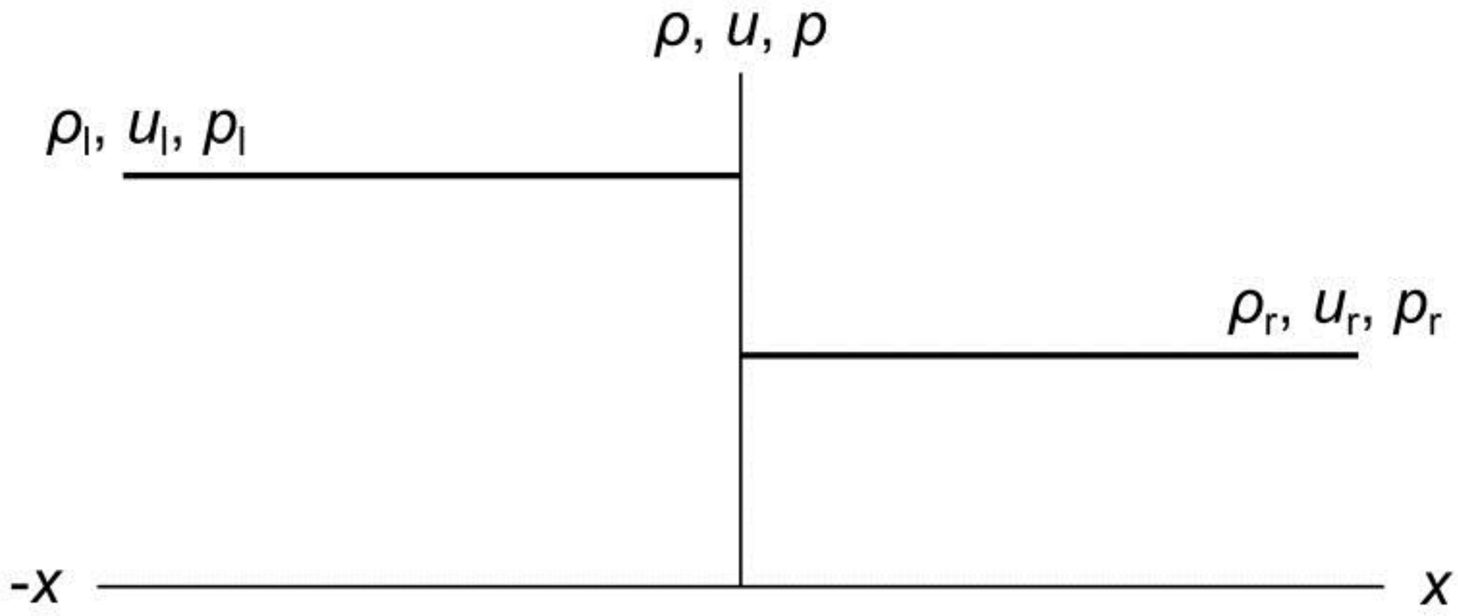
Results: 1D Linear advection





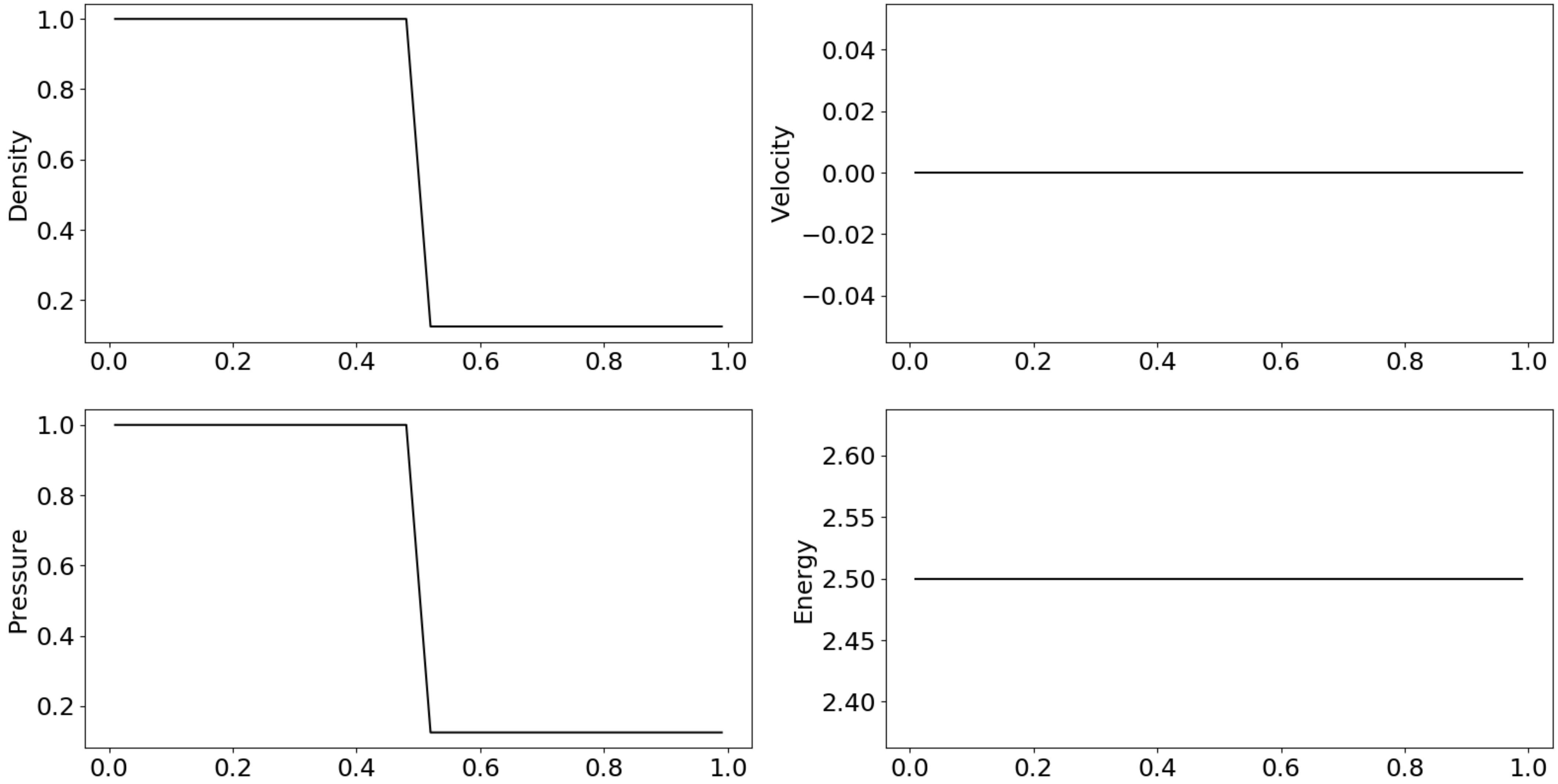
Results: Riemann Problems

Initial flow discontinuity.

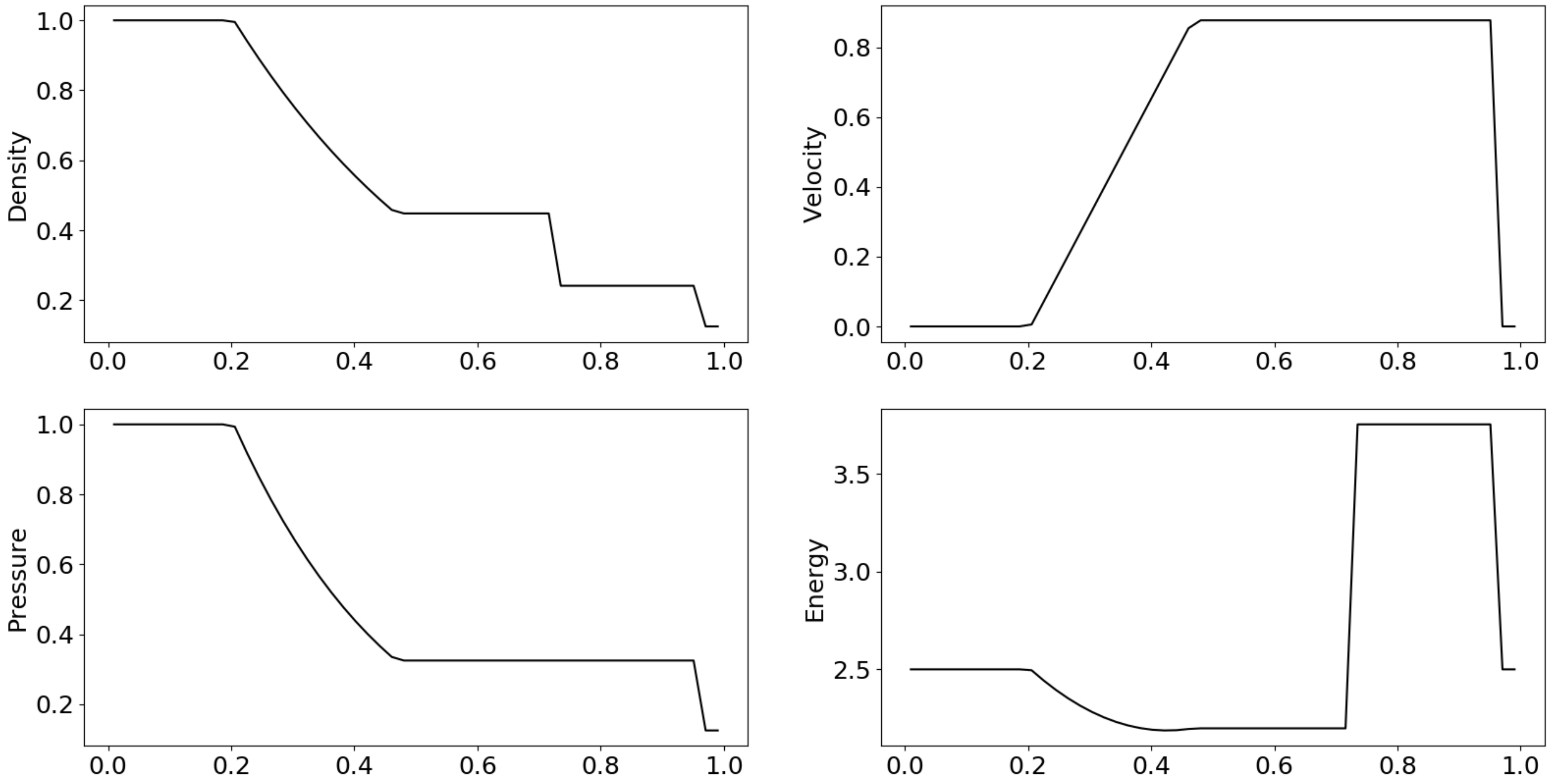


Each cell edge can be seen as a Riemann problem.

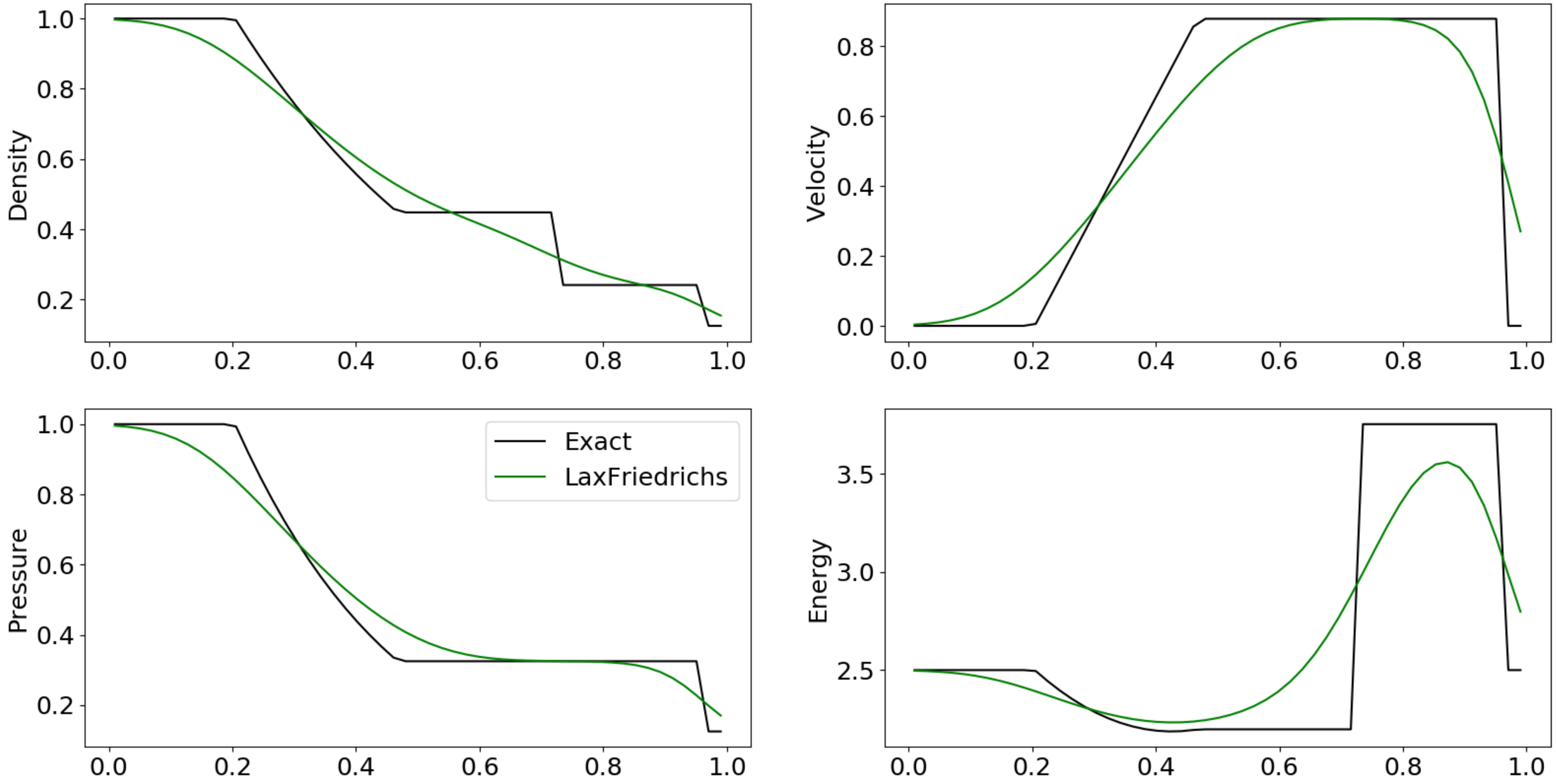
Results: 1D Euler equations - Shock-capturing



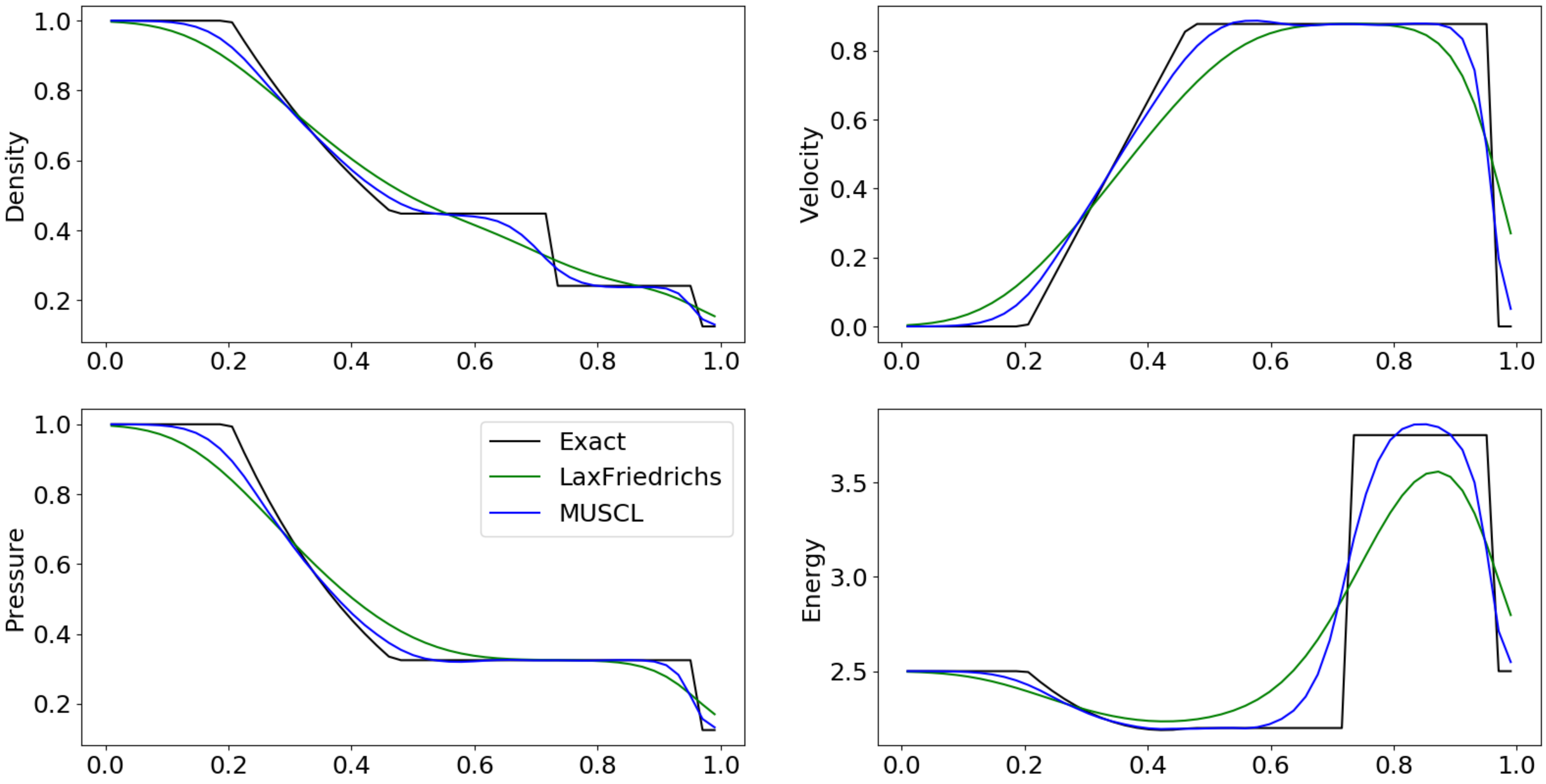
Results: 1D Euler equations - Shock-capturing



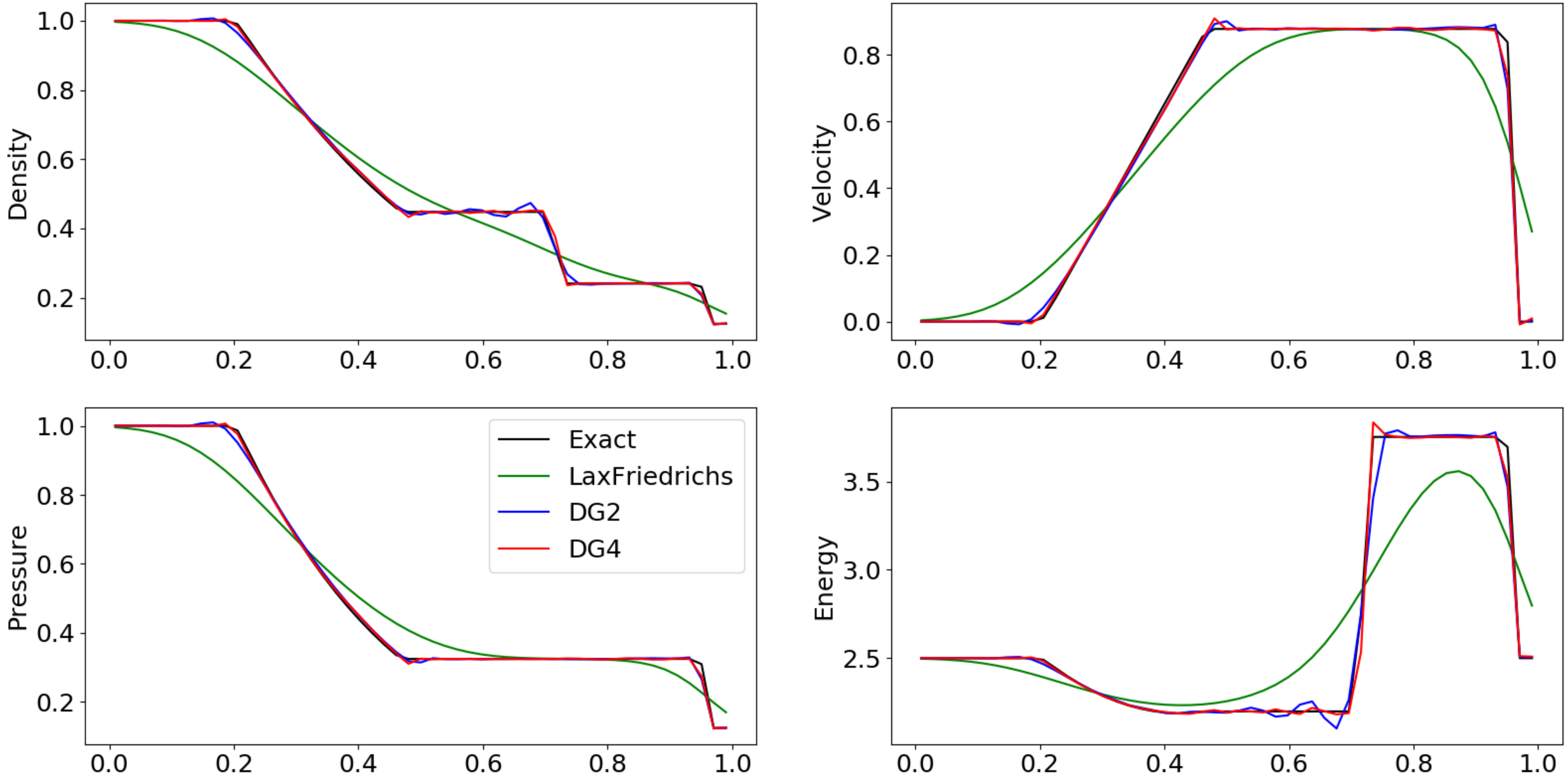
Results: 1D Euler equations - Shock-capturing



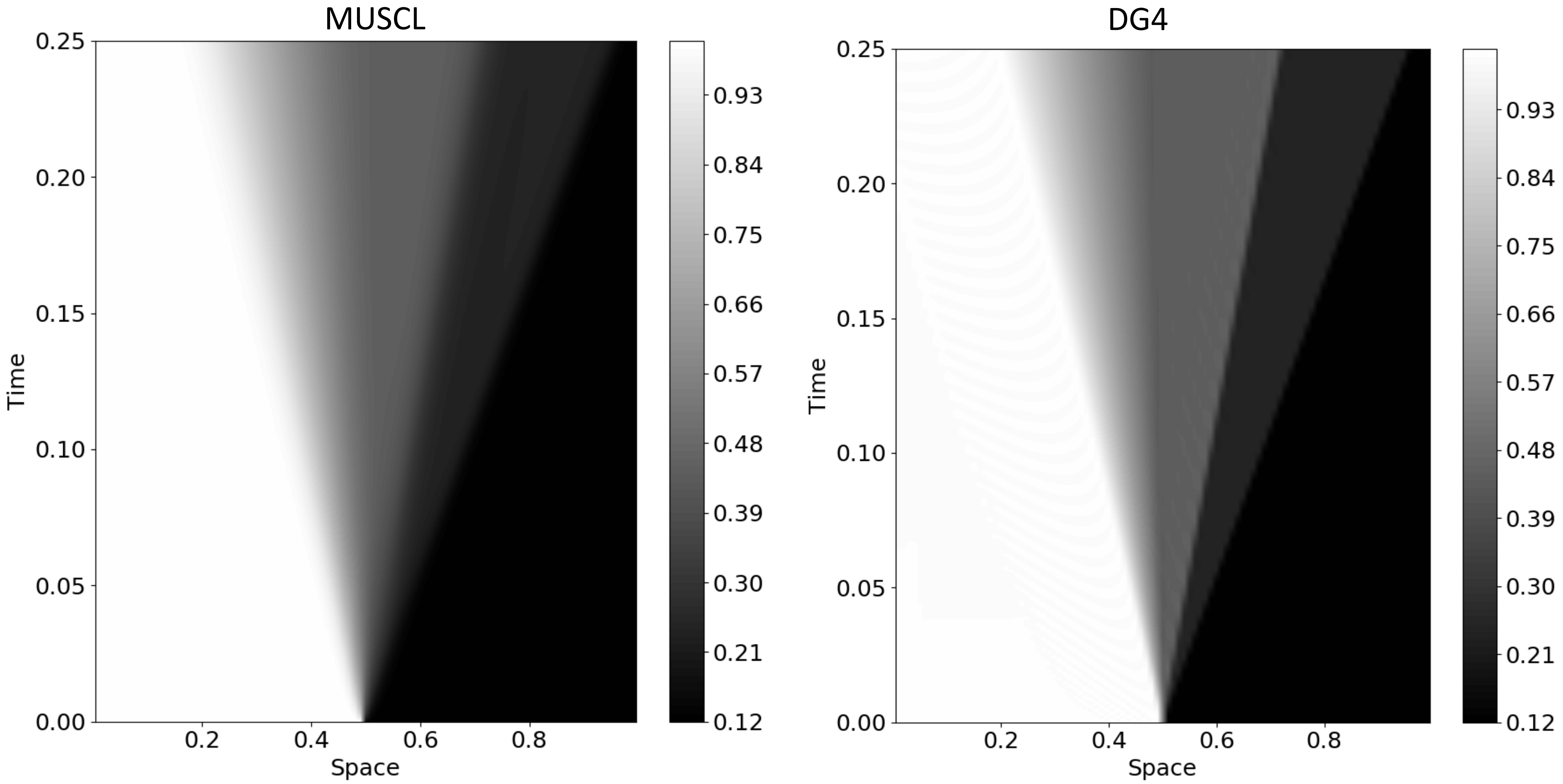
Results: 1D Euler equations - Shock-capturing



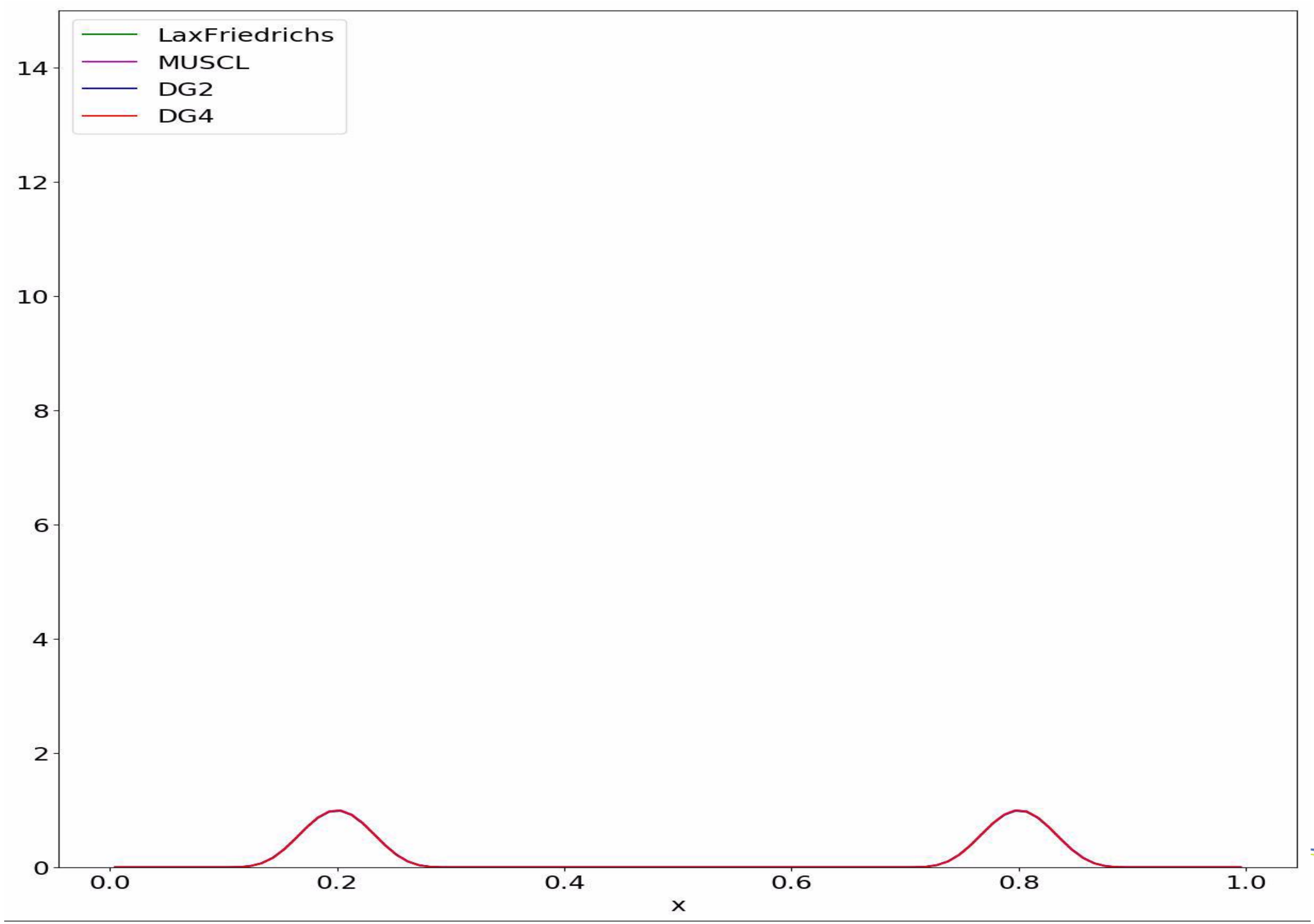
Results: 1D Euler equations - Shock-capturing



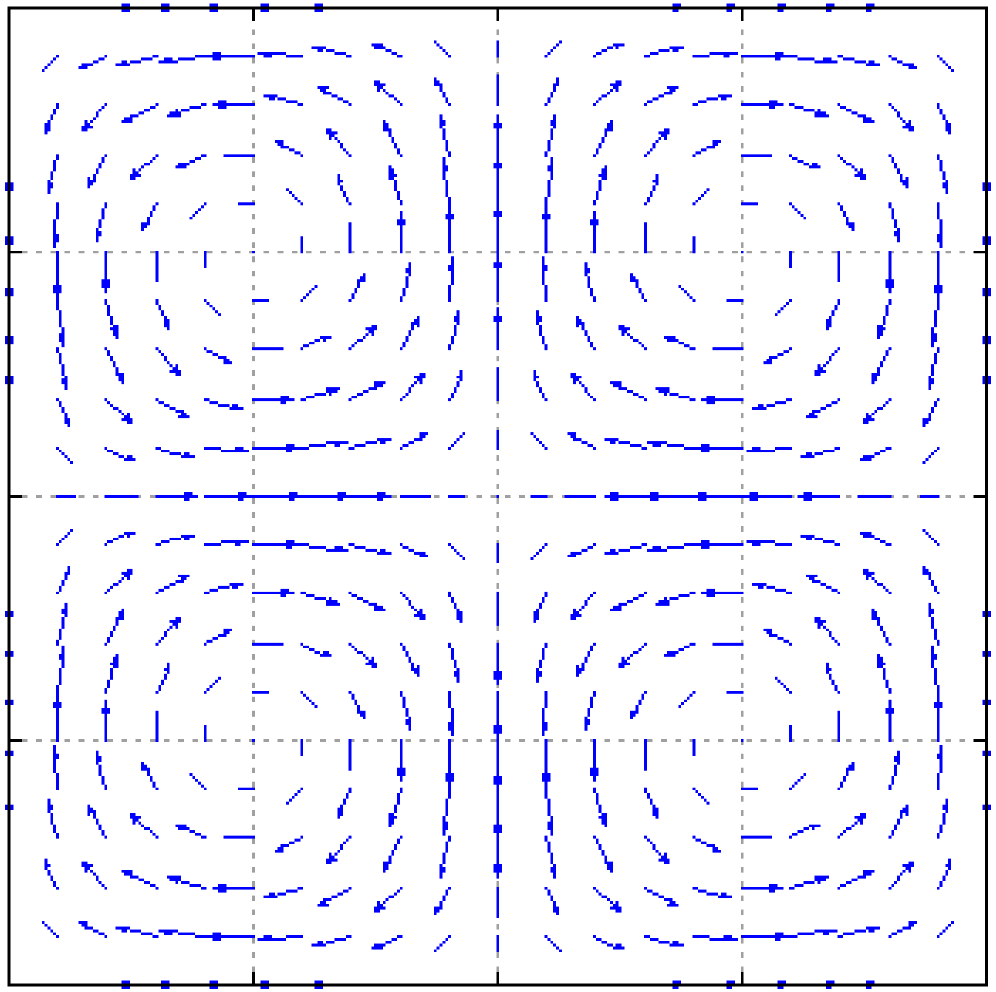
Results: 1D Euler equations - Shock-capturing



Results: 1D Euler equations - Shock-capturing



Results: 2D simulations – Taylor-Green vortices



Results: 2D simulations – Taylor-Green vortices