Solving Partial Differential Equations with Neural Networks

Mohammed Aljashmi

Outline

- Machine Learning Overview
- Physics Informed Neural Networks
- Results
    - Wave Equation
    - Burgers Equation
    - NN architecture vs error
    - Conclusions

Neural Networks are universal function approximators

Output ← $y = f(x)$

Mapping

Data

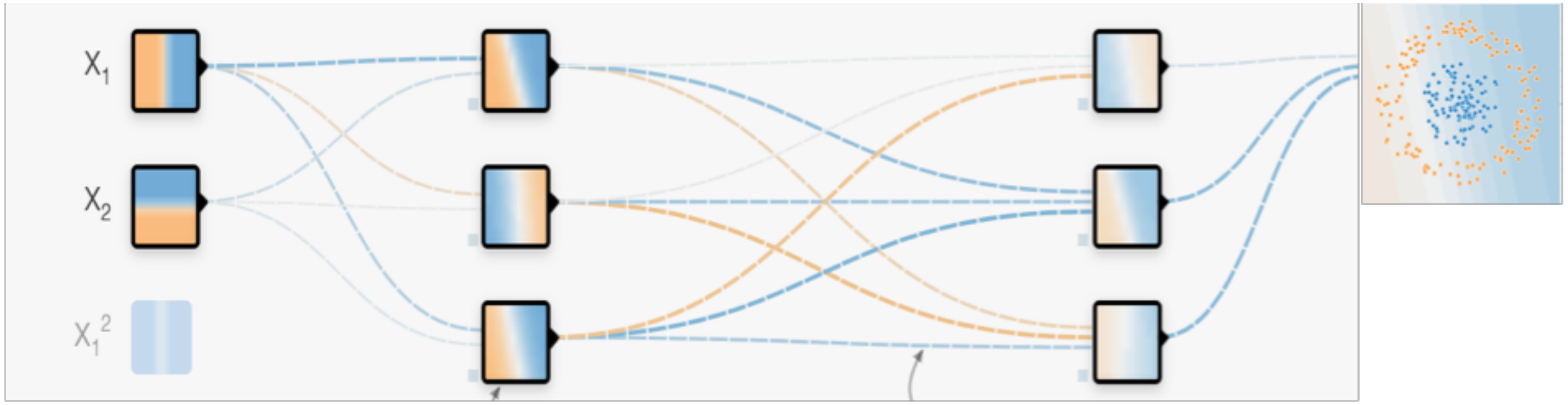
Initialized randomly

Input

Weights

Weights

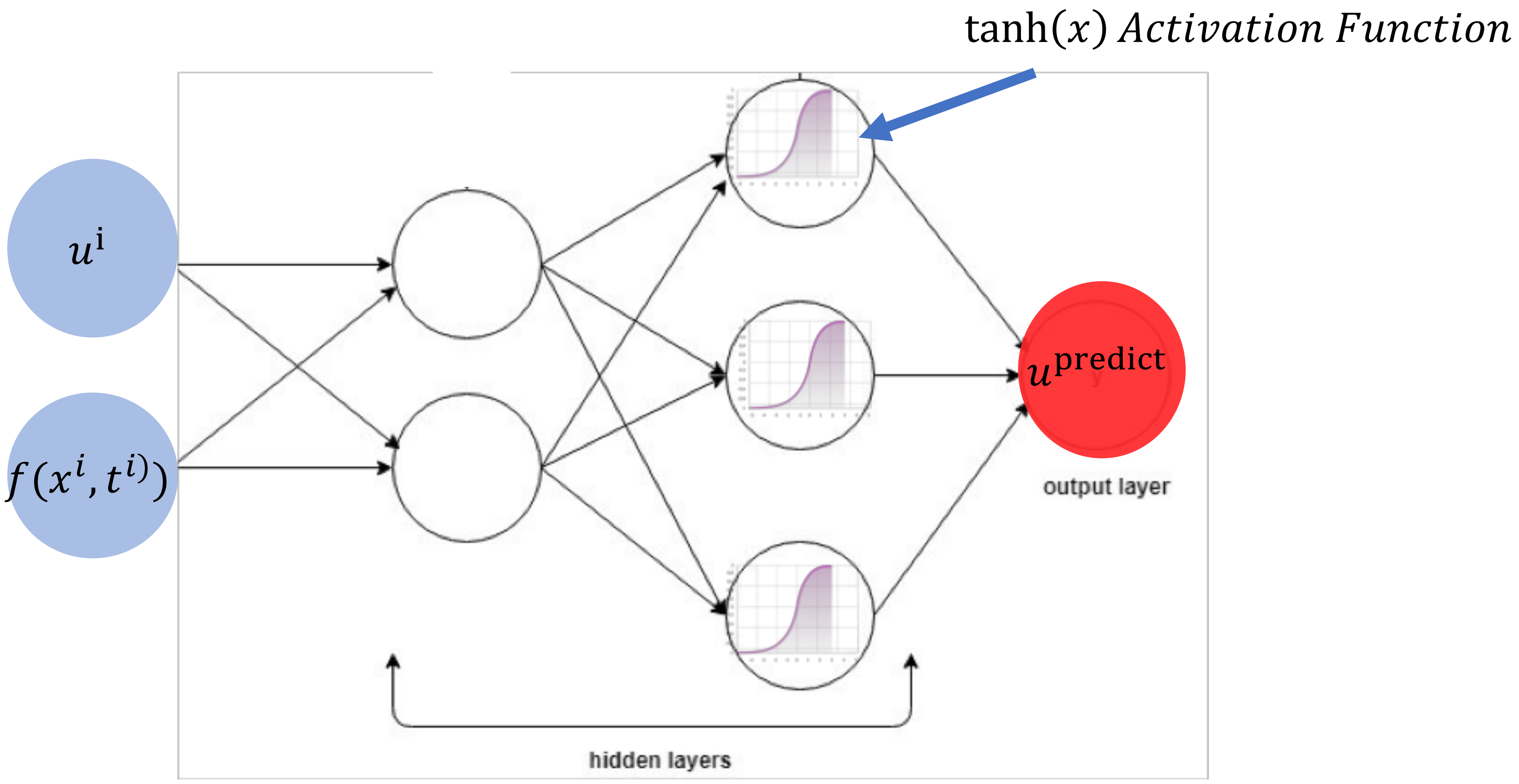
Output



Hidden layers

Non linear Activation function

Physics informed Neural Network



Training the neural network

- Training Set:
  - IC & BC $\{t_u^i, x_u^i, u^i\}_{i=1}^{N_u}$

Non-linear operator

Define $f := u_t + \mathcal{N}[u]$

- Objective Function:
  - Loss: $\mathcal{L}(f(x^i; W), y^i)$, in our case: Mean Squared error

$$MSE = MSE_u + MSE_f$$

$$MSE_u = \frac{1}{N_u}\sum_{i=1}^{N_u}\left|u(t_u^i, x_u^i) - u^i\right|^2$$ → $IC\ \&\ BC$

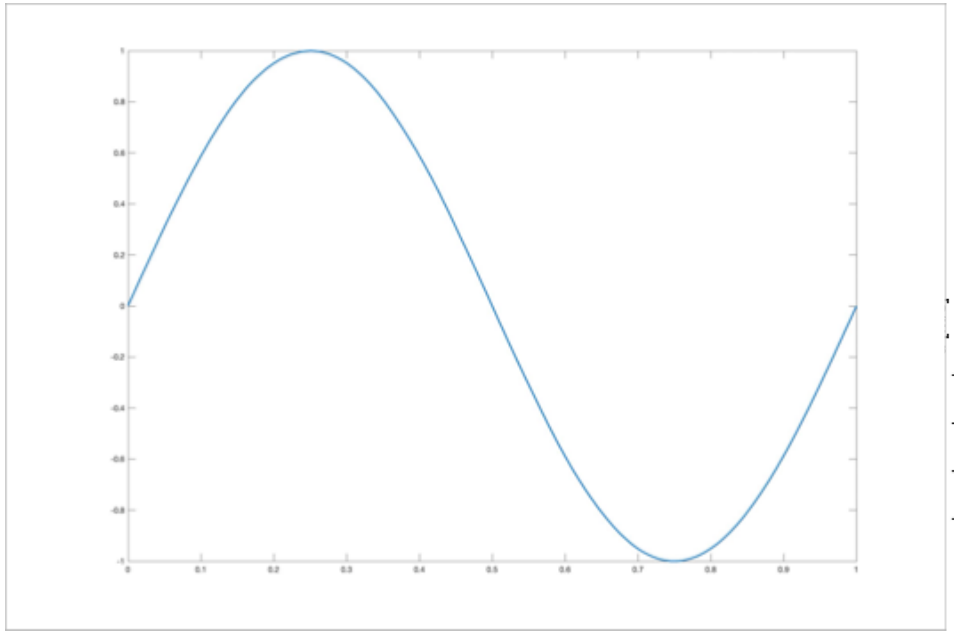
$$MSE_f = \frac{1}{N_f}\sum_{i=1}^{N_f}\left|f(t_u^i, x_u^i)\right|^2$$ → $Regularization\ using\ PDE$

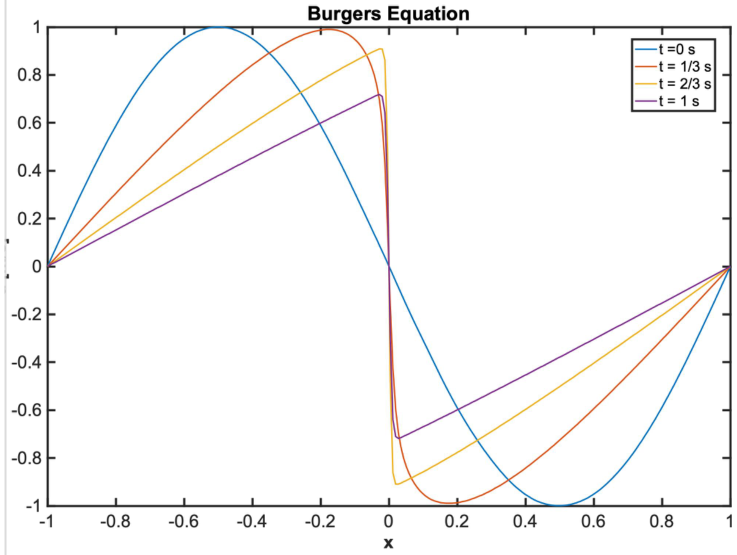
Overfitting: Solutions are not generalizable and are affected by noise

→ Gradient Descent to Minimize Loss function

Burger's Equation

$$\frac{du}{dt} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0$$

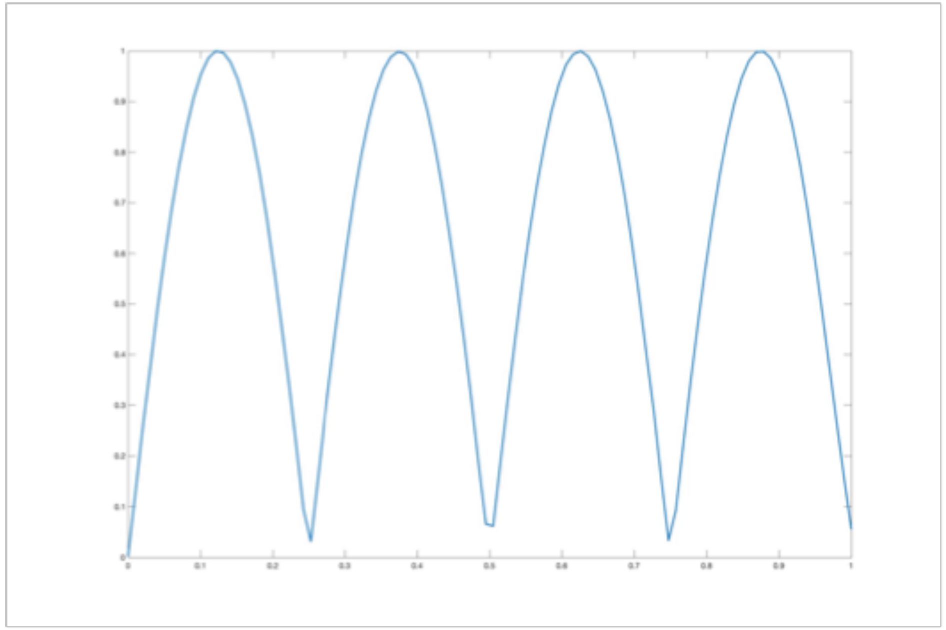


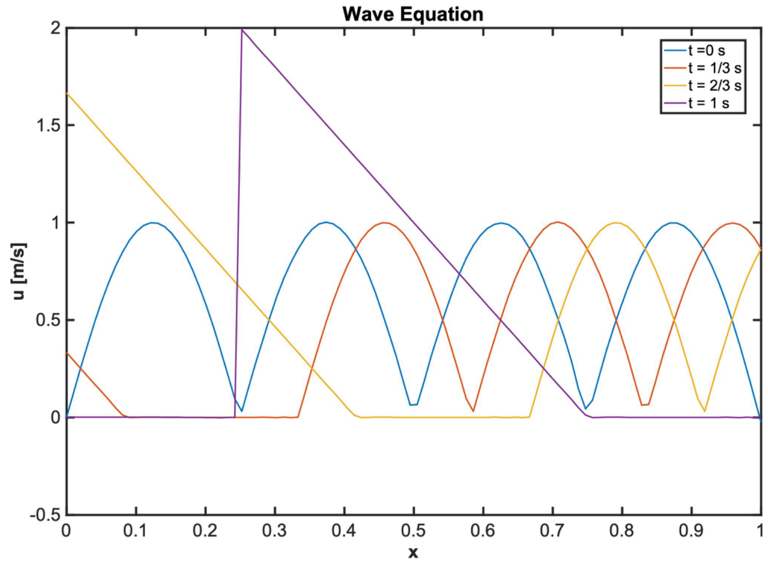


- Can lead to shock formation that's hard to resolve by classical numerical methods

Wave Equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$





No CFL condition → No Numerical Dispersion

Neural Network Architecture

Layers / Neurons	7	8	9	10
10	1.31E-01	4.77E-02	1.29E-01	1.75E-01
20	2.65E-02	4.89E-03	2.63E-02	4.97E-02
40	4.23E-03	2.39E-03	3.06E-03	2.01E-03

Training data vs Error

N_f / $\boldsymbol{N_u}$	**1000**	**2500**
200	1.18E-01	4.40E-01
300	3.00E-01	1.11E-01
500	9.9E-03	

Conclusions

- Ability to solve the PDE exactly without discretization error
- Ability to Regularize the neural network by incorporating the physics from the PDE
- We don’t encounter numerical dispersion or dissipation due to violating CFL
- The neural network architecture is parallelizable
- No convergence guarantees

References

[1] Raissi, M., Perdikaris, P., and Karniadakis, G. E., 2017, "Physics Informed Deep Learning (Part II): Data-Driven Discovery of Nonlinear Partial Differential Equations," arXiv:1711.10566 [cs, math, stat].