The Level Set Method for Acoustic Propagation

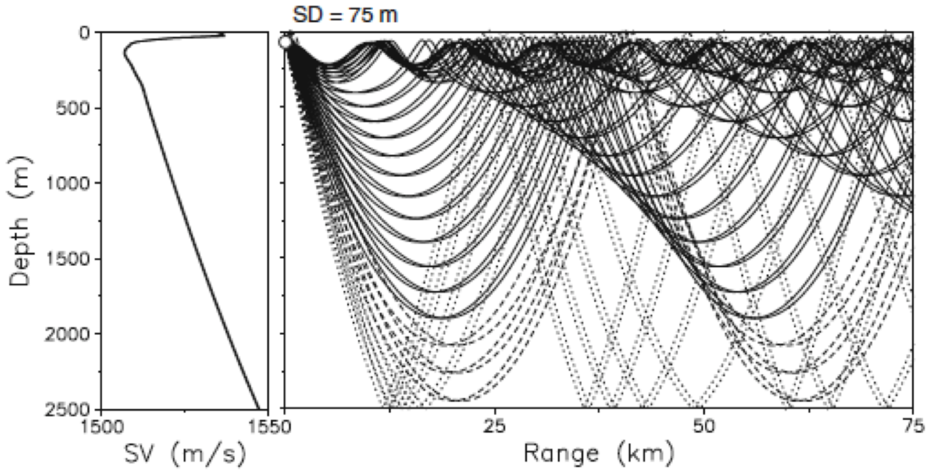
Manmeet Bhabra

mseas
multidisciplinary simulation, estimation, and assimilation systems

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Introduction

- Modeling of high frequency acoustic propagation is prevalent in numerous applications (ex: underwater ocean acoustics).
- Traditionally accomplished using a method known as ray tracing (a Lagrangian approach).
- In this work, we look to investigate an alternate Eulerian approach to tackle this problem through the Level Set Method.



The Acoustic Wave Equation

- Define the following variables:

$$\rho_0 = Equilibrium\ Density \qquad p_0 = Equilibrium\ Pressure \qquad \mathbf{v} = Particle\ Velocity$$

$$\rho = Instantaneous\ Density \qquad p = Instantaneous\ Pressure$$

$$\rho_a = \rho - \rho_0 = Acoustic\ Density \qquad p_a = p - p_0 = Acoustic\ Pressure$$

- Now consider the propagation of acoustic waves:

Continuity Equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0$$

$$\frac{\partial \rho_a}{\partial t} + \nabla \cdot \rho_0 \mathbf{v} = 0$$

Euler's Equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p$$

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{\rho_0}\nabla p_a = 0$$

Equation of State

$$p = p_0 + \rho_a \left(\frac{\partial p}{\partial \rho}\right)_s + \frac{1}{2}\rho_a^2 \left(\frac{\partial^2 p}{\partial \rho^2}\right)_s + \ldots$$

$$p_a = \rho_a \left(\frac{\partial p}{\partial \rho}\right)_s$$

Assumption *: Acoustic Pressure and Density Small*

Acoustic Wave Equation

$$\rho_a \nabla \cdot \left(\frac{1}{\rho_a}\nabla p_a\right) - \frac{1}{c^2}\frac{\partial^2 p_a}{\partial t^2} = 0$$

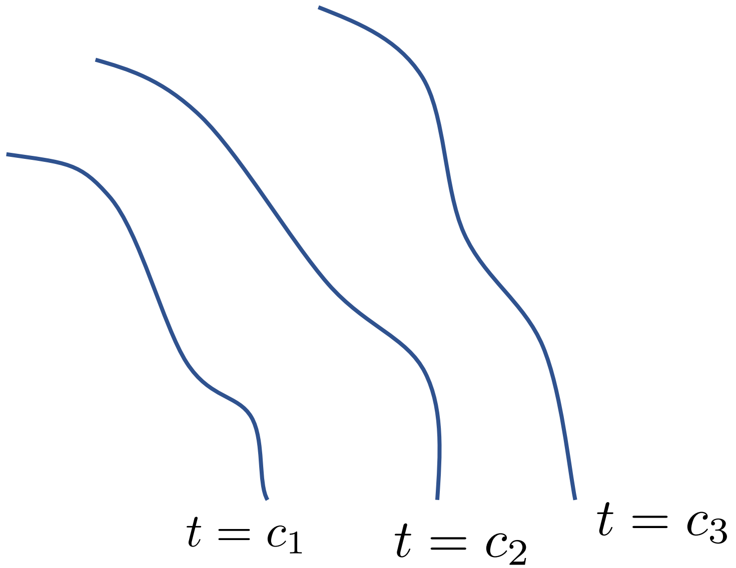
$$\nabla^2 p_a - \frac{1}{c^2}\frac{\partial^2 p_a}{\partial t^2} = 0$$

The High Frequency Solution

- We consider now solving for the evolution of acoustic waves at high frequencies.
  - Assume a solution of the following form:

$$p(\mathbf{x}, t) = e^{i\omega S(\mathbf{x},t)} \sum_{k=0}^{\infty} A_k(\mathbf{x}, t)(i\omega)^{-k}$$

  - $S(\boldsymbol{x}, t)$ is the phase function. At any given point of time, the level surfaces of $S(\boldsymbol{x}, t)$ correspond to points at which the phase is the same (surfaces of constant phase).
    - We refer to these surfaces of constant phase as the wavefront as well.



  - Substitution into the wave equation, (and retaining the highest order terms in ω) we obtain:

$$S(\mathbf{x}, t) + c|\nabla S(\mathbf{x}, t)| = 0$$

  - This has the form a Hamilton-Jacobi Equation.

Hamilton-Jacobi Equation General Form

$$\frac{\partial u(\mathbf{x}, t)}{\partial t} + H(\mathbf{x}, \nabla u) = 0$$

Approaches to Evolve the Wavefront

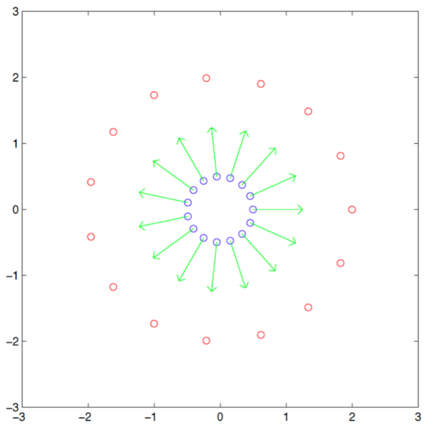
Lagrangian Approach

- Considers a discrete set of particles on the initial wavefront:
  - Trajectory of each point governed by characteristics of the Hamiltonian system:

$$\frac{d\mathbf{x}}{dt} = c(\mathbf{x})^2\mathbf{p}$$

$$\frac{d\mathbf{p}}{dt} = -\frac{1}{c(\mathbf{x})}\nabla_x c(\mathbf{x})$$

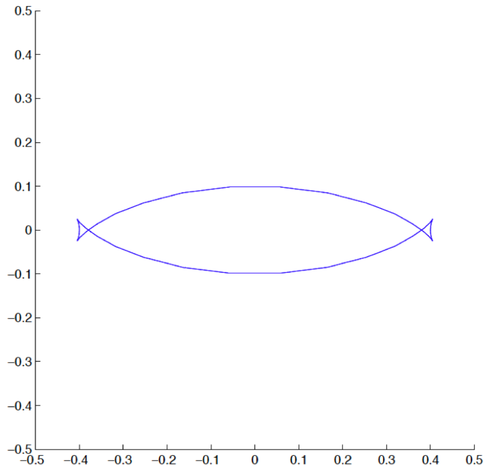
- **Difficulty:** Points originally close together may diverge at later times (poor wavefront resolution)

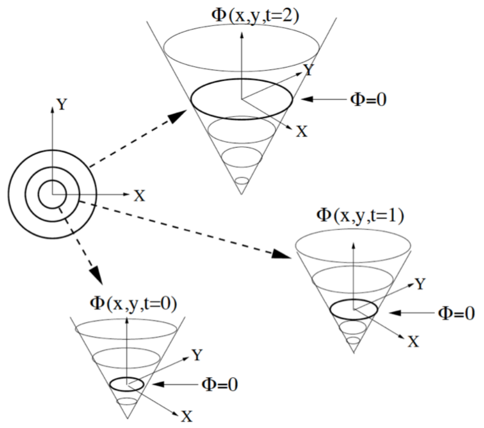


Eulerian Approach

- Consider the front to be the level surface of an implicit function (ϕ) in a higher dimension:
  - Evolve the implicit function by solving a PDE governing the implicit functions evolution (the Hamilton-Jacobi equations).

- **Difficulty:** The Level Set method has issues when wavefronts become multivalued.

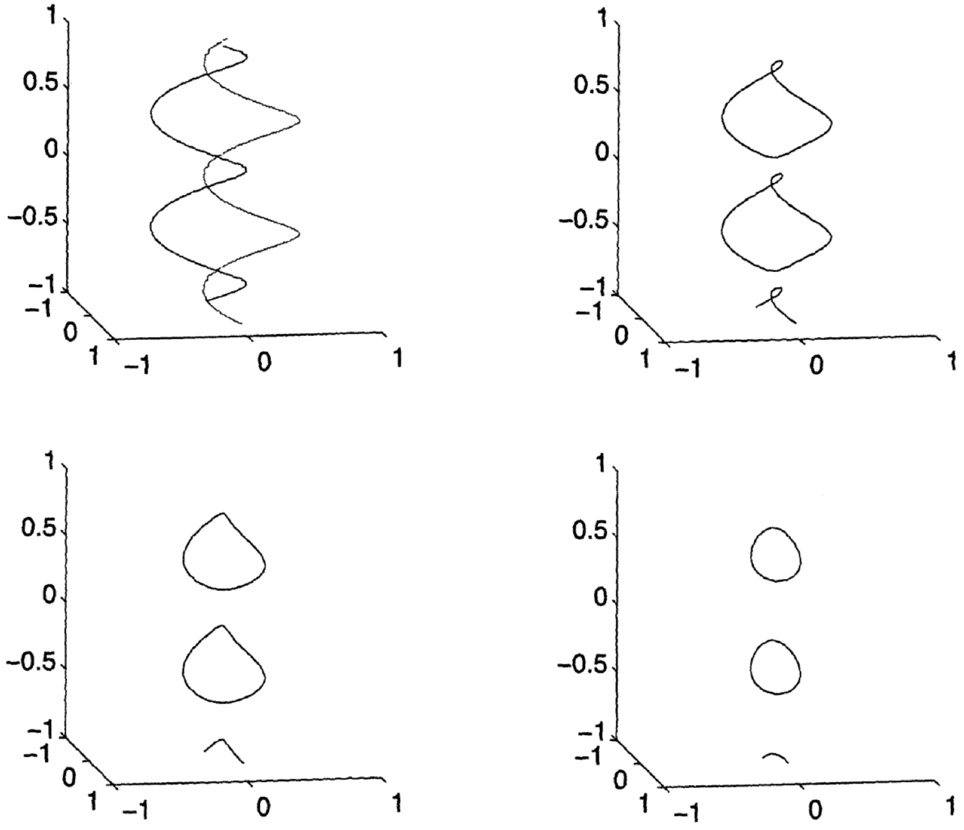




Level Set Methods with Codimension-Two Objects

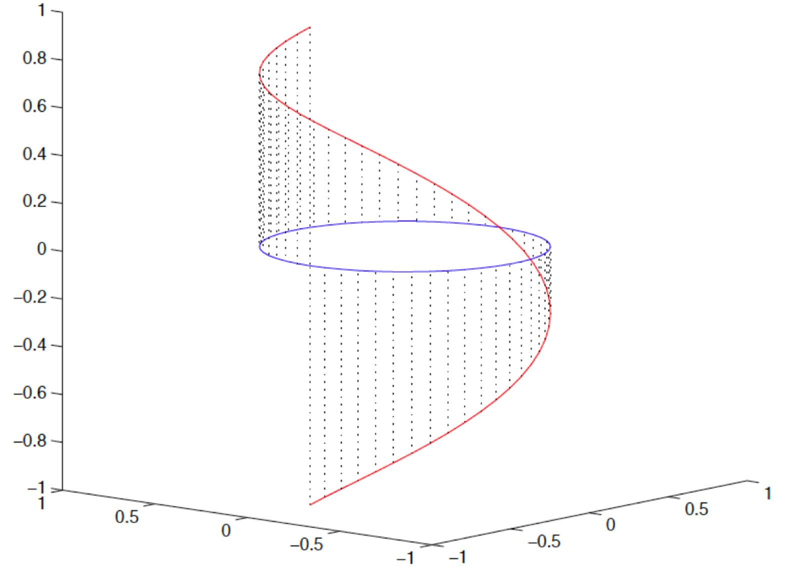
- Traditional Level Set Methods use codimension-one objects:
  - **Example**: If we are interested in solving for the evolution of an interface in $\mathbb{R}^2$, evolve the surface as the zero isocontour of an implicit function in $\mathbb{R}^3$.

- Codimension-two objects in Level Set Methods:
  - **Example:** Points in $\mathbb{R}^2$, Curves in $\mathbb{R}^3$.
  - Represent the object in the level set framework as the **intersection** of the isocontours of **two** functions ϕ_1 and ϕ_2.
    - Evolve each level set function as in the standard approach.
    - Treatment may be needed to evolve the level sets while keeping them as orthogonal as possible.



Level Set Formulation in a Reduced Phase Space

- We look to evolve the acoustic wavefront as a curve/strip in a higher dimensional reduced phase space with coordinates (x, y, θ).
  - θ gives the normal direction of a given point on the wavefront (measured from the $+x$ axis).
  - The wavefront is then found by projecting this curve onto the (x, y) plane.



- The velocity field for the strip's propagation can be derived to be:

$$\mathbf{V}(\mathbf{x}, \theta) = \begin{bmatrix} c(\mathbf{x}) \cdot cos(\theta) \\ c(\mathbf{x}) \cdot sin(\theta) \\ \frac{\partial c(\mathbf{x})}{\partial x} \cdot sin(\theta) - \frac{\partial c(\mathbf{x})}{\partial y} \cdot cos(\theta) \end{bmatrix}$$

- The strip is represented as the intersection of two level set functions whose evolution is governed by:

$$\frac{\partial \phi_1}{\partial t} + \mathbf{V}(x, y, \theta) \cdot \nabla \phi_1 = 0$$
$$\frac{\partial \phi_2}{\partial t} + \mathbf{V}(x, y, \theta) \cdot \nabla \phi_2 = 0$$
$$\longleftarrow \quad \left[\begin{array}{l} \phi_1 = \phi_1(x, y, \theta, t) \\ \phi_2 = \phi_2(x, y, \theta, t) \end{array} \right.$$

- The wavefront (surfaces of constant phase) is given as:

$$W(x, y, t) = \{(x, y) \mid \phi_1(x, y, \theta, t) = \phi_2(x, y, \theta, t) = 0\}$$

Implementation

$$\frac{\partial \phi_1}{\partial t} + \mathbf{V}(x, y, \theta) \cdot \nabla \phi_1 = 0$$
$$\frac{\partial \phi_2}{\partial t} + \mathbf{V}(x, y, \theta) \cdot \nabla \phi_2 = 0$$

Discretize explicitly in time. In this work, have used TVD Runge-Kutta Schemes.

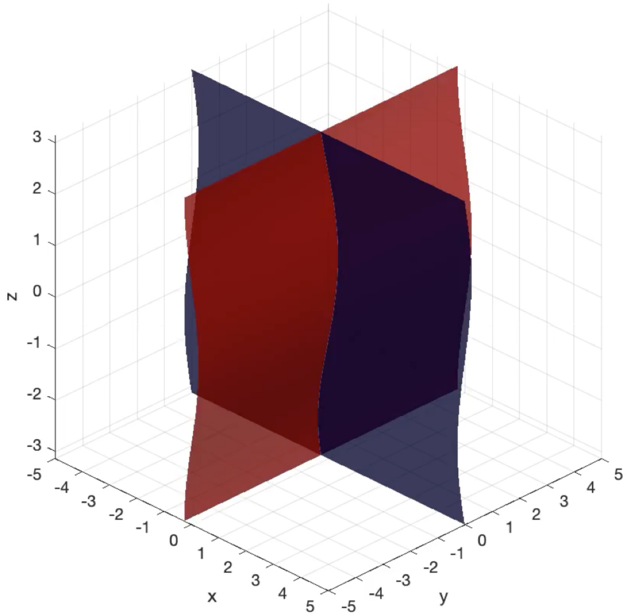
Discretize in space using upwinding for all spatial derivatives.

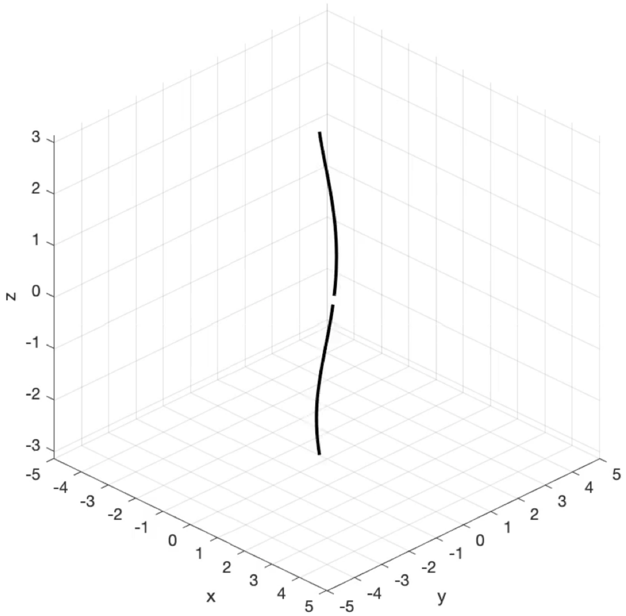
- Boundary Conditions:
  - Absorbing BCs implemented by imposing a zero-Neumann condition on the boundary.
  - Reflecting BCs implemented by imposing a Dirichlet BC given by:

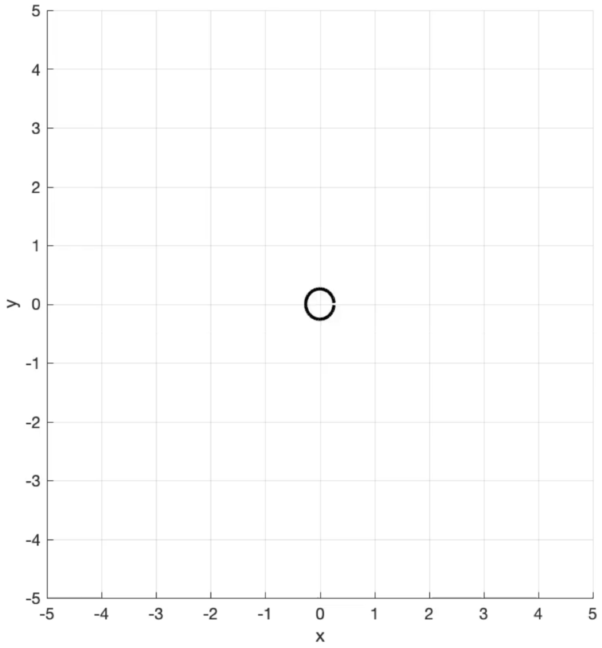
$$\phi_1(x, y, \theta_{refl}, t) = \phi_1(x, y, \theta_{inc}, t)$$
$$\phi_2(x, y, \theta_{refl}, t) = \phi_2(x, y, \theta_{inc}, t)$$
$$\theta_{refl} = 2\theta_B - \theta_{inc} - \pi$$

Case 1: Constant Wave Speed (Open Domain)

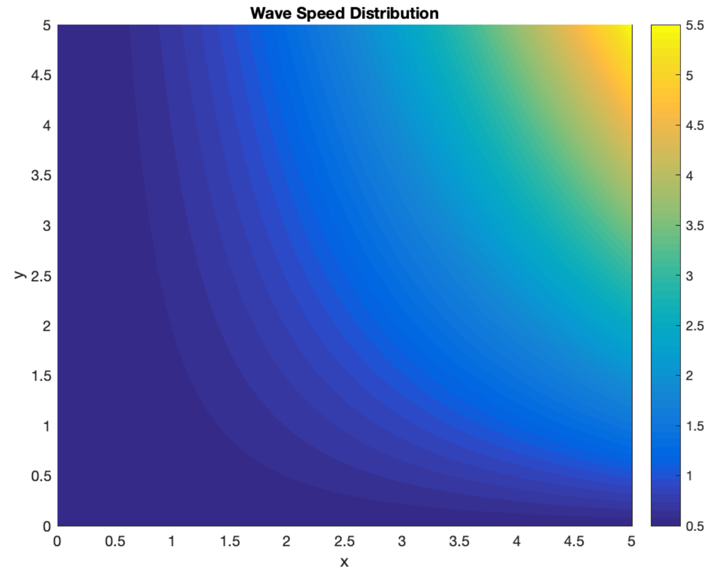
Wave Speed: Constant value of 1
Spatial Discretization: $N_x = N_y = N_z = 40$
Temporal Discretization: $t \in [0.0, 6.0], \quad \Delta t = 1 * 10^{-2}$
Time Stepping Scheme: Forward Euler
Spatial Derivatives Scheme: First Order Upwind Approximation







Case 2: Variable Wave Speed (Open Domain)



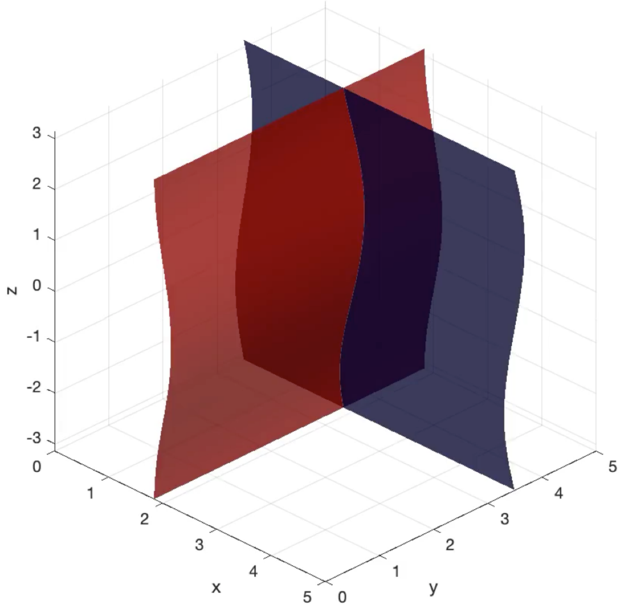
Wave Speed: Variable Distribution

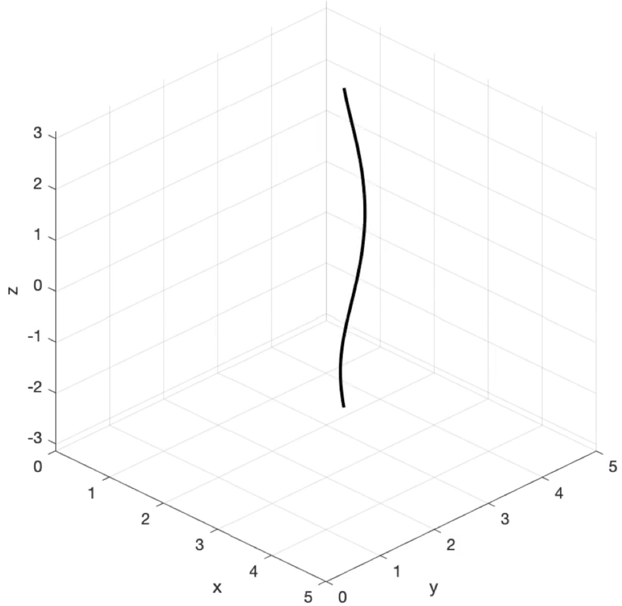
Spatial Discretization: $N_x = N_y = N_z = 50$

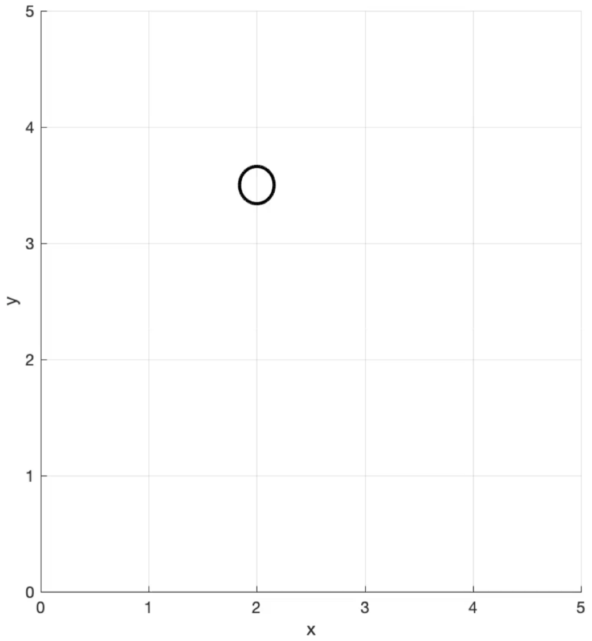
Temporal Discretization: $t \in [0.0, 3.2]$, $\Delta t = 1 * 10^{-2}$

Time Stepping Scheme: Second order TVD RK

Spatial Derivatives Scheme: Second Order Upwind Approximation

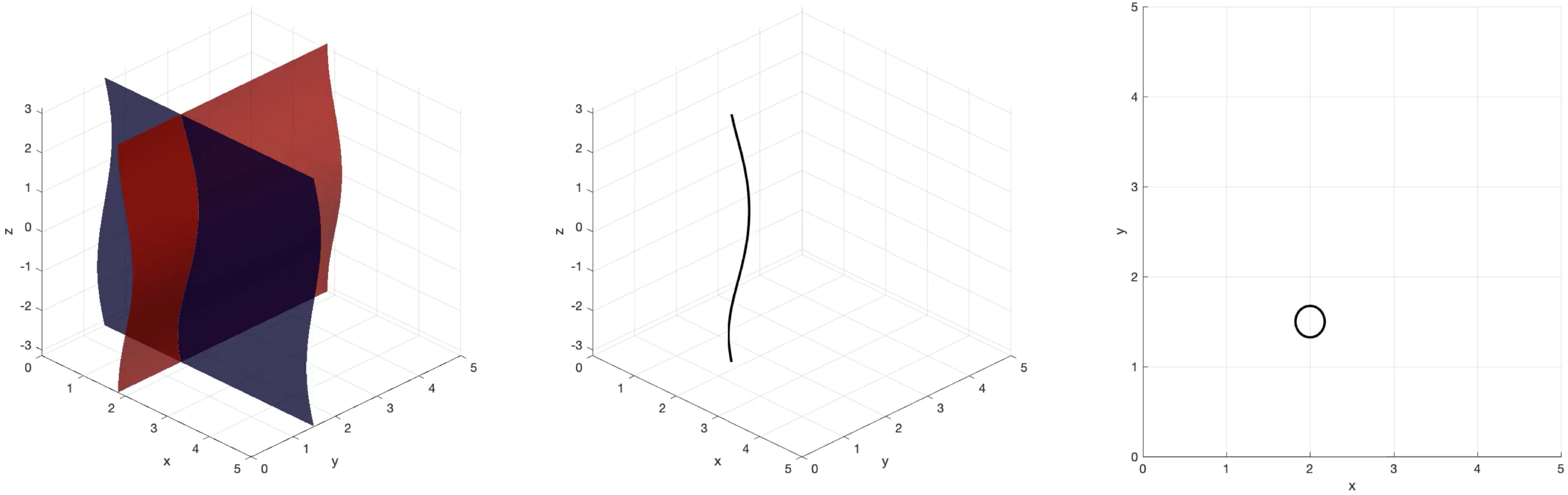




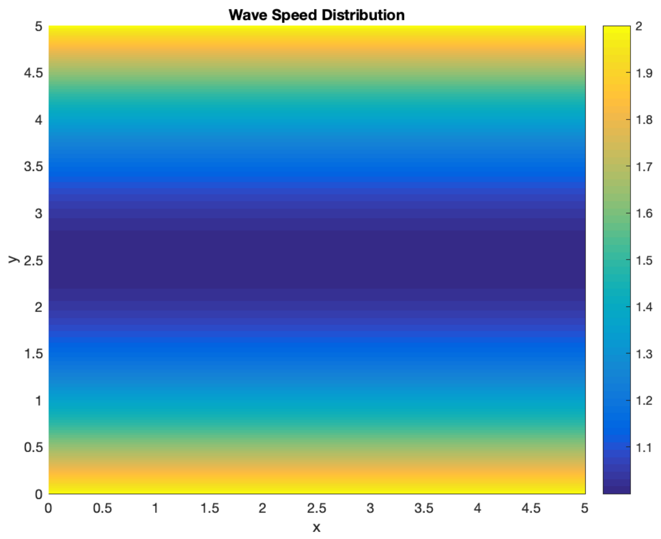


Case 3: Constant Wave Speed (Reflecting Boundary)

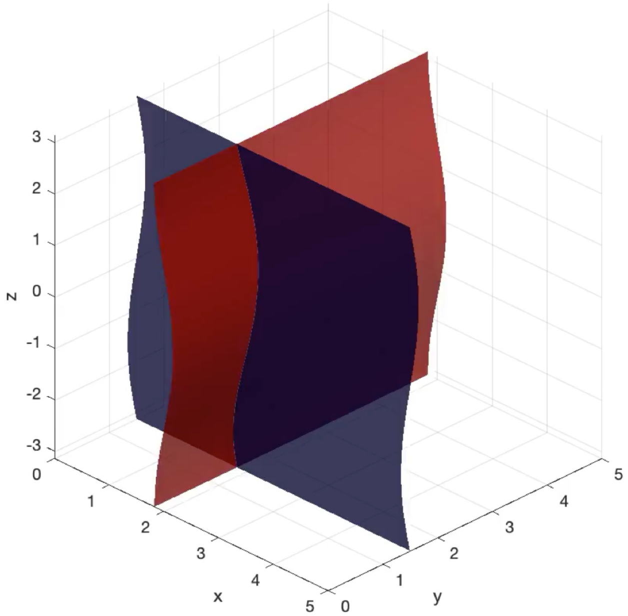
Wave Speed: Constant value of 2.5
Spatial Discretization: $N_x = N_y = N_z = 50$
Temporal Discretization: $t \in [0.0, 5.2], \quad \Delta t = 1 * 10^{-2}$
Time Stepping Scheme: Third order TVD RK
Spatial Derivatives Scheme: Fifth Order Upwind Approximation
(using WENO scheme)

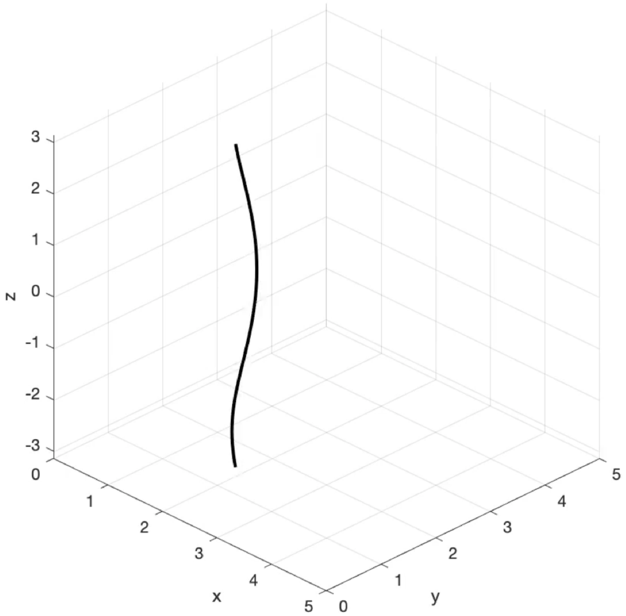


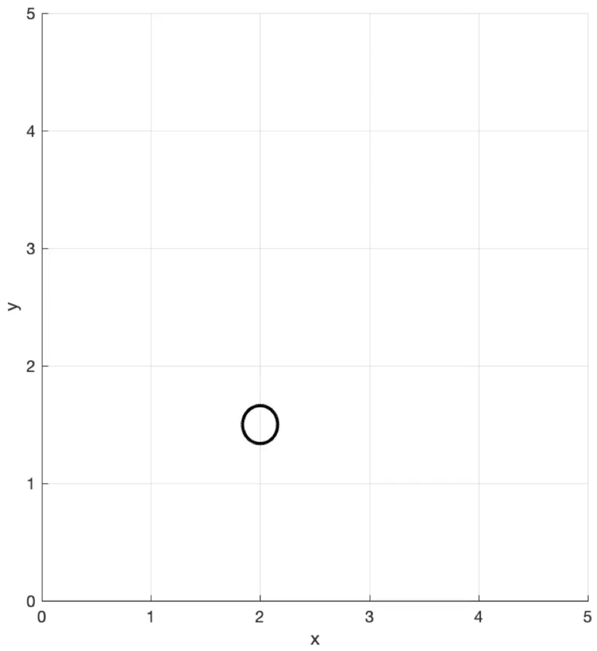
Case 4: Variable Wave Speed (Reflecting Boundary)



Wave Speed: Variable Distribution
Spatial Discretization: $N_x = N_y = N_z = 50$
Temporal Discretization: $t \in [0.0, 5.2]$, $\Delta t = 1 * 10^{-2}$
Time Stepping Scheme: Third order TVD RK
Spatial Derivatives Scheme: Fifth Order Upwind Approximation
(using WENO scheme)







Conclusion

- Level set methods provide an effective approach to propagate wavefronts in an Eulerian framework relative to standard ray tracing methods.
- However, increased computational costs occurred as we are solving in a higher dimensional reduced phase space.
  - For 2D propagation, need to evolve level set implicit functions in $\mathbb{R}^3$
  - For 3D propagation, need to evolve level set implicit functions in $\mathbb{R}^5$

- Future Work:
  - Improve computational cost by incorporating the narrow band level set method.
  - Investigate the incorporation of schemes to compute the amplitude.

Thank You!