Time-optimal path planning in stochastic flows

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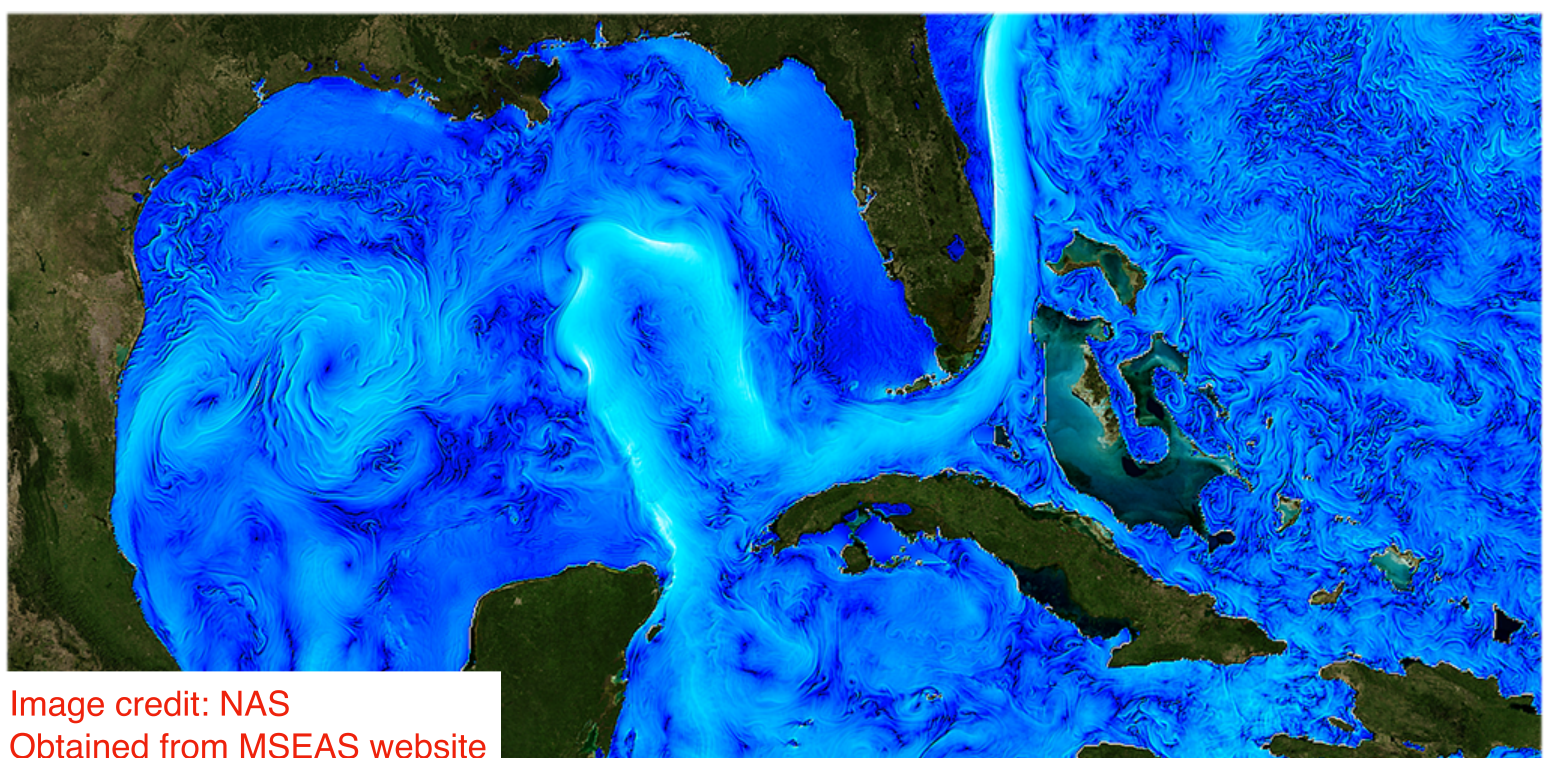
2.29 Numerical Fluid Mechanics
Final Project

Motivation: First Arrival


Image credit: Melissa Bailey
Obtained from Healthcare Finance News Website

Image credit: Franklin Heijnen
Obtained from Wikipedia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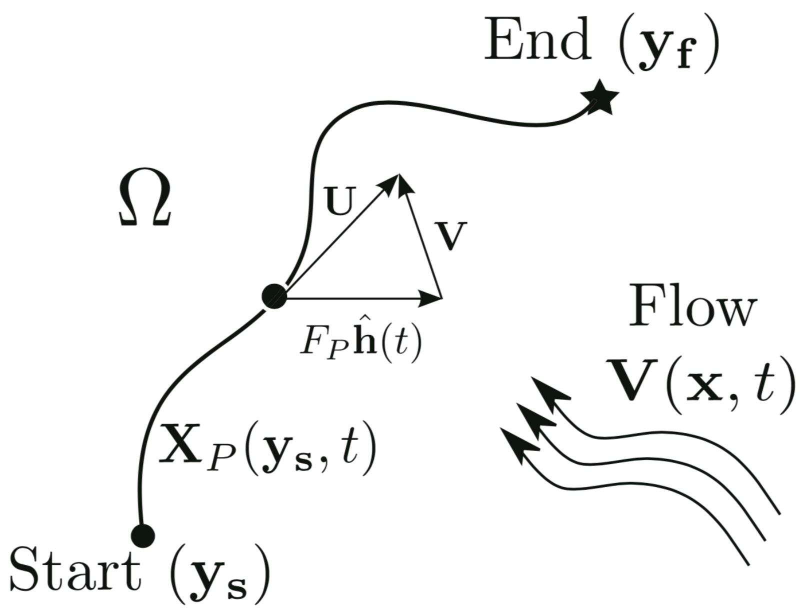
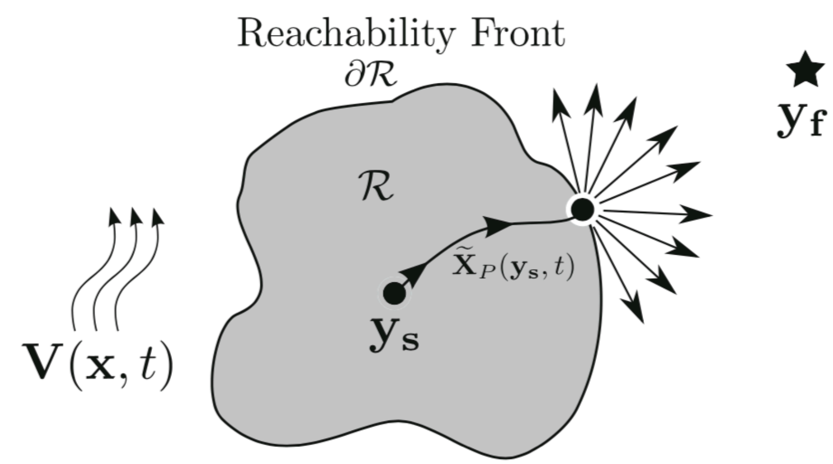
Level-Set Equation

There is no road in the ocean: infinite choices



Lolla & Lermusiaux et al, 2014.

Fig. 1 Motion of P in an unsteady flow field, $\mathbf{V}(\mathbf{x}, t)$. Its trajectory $\mathbf{X}_P(\mathbf{y_s}, t)$ connects the start $(\mathbf{y_s})$ and end $(\mathbf{y_f})$ points and satisfies (1), (2). The total velocity, $\mathbf{U}$, is the vector sum of the steering velocity $F_P(t)\,\hat{\mathbf{h}}(t)$ and flow field $\mathbf{V}(\mathbf{x}, t)$



Lolla & Lermusiaux et al, 2014.

Fig. 2 Reachability front $\partial\mathcal{R}(\mathbf{y_s}, t)$ and infinite possible steering directions: $\partial\mathcal{R}$ denotes the boundary of the reachable set $\mathcal{R}(\mathbf{y_s}, t)$ *(set of points that can be visited at time t)*

Level-set equations:

$$\frac{\partial \phi}{\partial t} + F\left|\nabla\phi\right| + \mathbf{v}\cdot\nabla\phi = 0$$

$$\phi(\mathbf{x},0) = \left|\mathbf{x} - \mathbf{y}_s\right|$$

Scalar field $\phi(\mathbf{x}, t)$

Contour $\phi(\mathbf{x}, t) = 0$

Backtrack:

$$\frac{\mathrm{d}\mathbf{X}_P}{\mathrm{d}t} = F\frac{\nabla\phi(\mathbf{X}_P, t)}{\left|\nabla\phi(\mathbf{X}_P, t)\right|} + \mathbf{v}(\mathbf{X}_P, t)$$

Finite Difference

Spatial discretization: CDS

$$\left(\nabla\phi\right)^n_{i,j} = \left(\frac{\phi^n_{i+1,j} - \phi^n_{i-1,j}}{2\Delta x}, \frac{\phi^n_{i,j+1} - \phi^n_{i,j-1}}{2\Delta y}\right)$$

$$\left(\mathbf{v}\cdot\nabla\phi\right)^n_{i,j} = \left(u^n_{i,j}\frac{\phi^n_{i+1,j} - \phi^n_{i-1,j}}{2\Delta x}, v^n_{i,j}\frac{\phi^n_{i,j+1} - \phi^n_{i,j-1}}{2\Delta y}\right)$$

Temporal discretization: fractional step

Lolla & Lermusiaux et al, 2014.

$$\frac{\bar{\phi} - \phi(\mathbf{x},t)}{\Delta t/2} = -F\left|\nabla\phi(\mathbf{x},t)\right|$$

$$\frac{\tilde{\phi} - \bar{\phi}}{\Delta t} = -\mathbf{v}(\mathbf{x}, t+\Delta t/2)\cdot\nabla\bar{\phi}$$

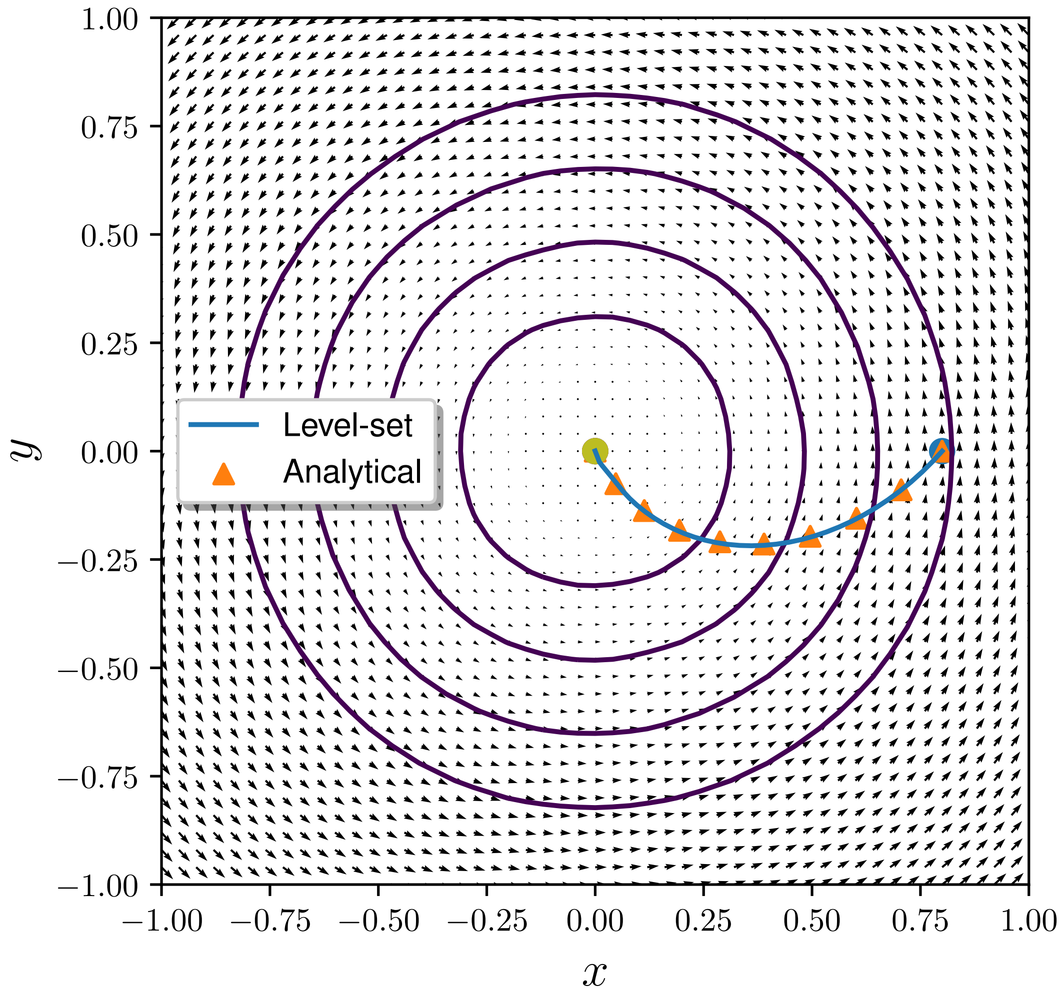
$$\frac{\phi(\mathbf{x}, t+\Delta t) - \tilde{\phi}}{\Delta t/2} = -F\left|\nabla\tilde{\phi}\right|$$

Backtracking

Lolla & Lermusiaux et al, 2014.

$$\frac{\mathbf{X}_P(\mathbf{y}_s, t-\Delta t) - \mathbf{X}_P(\mathbf{y}_s, t)}{\Delta t} = -\mathbf{v}(\mathbf{X}_p, t) - F\frac{\nabla\phi(\mathbf{X}_P, t)}{\left|\nabla\phi(\mathbf{X}_P, t)\right|}$$

Rankine vortex benchmark



Analytical Solution

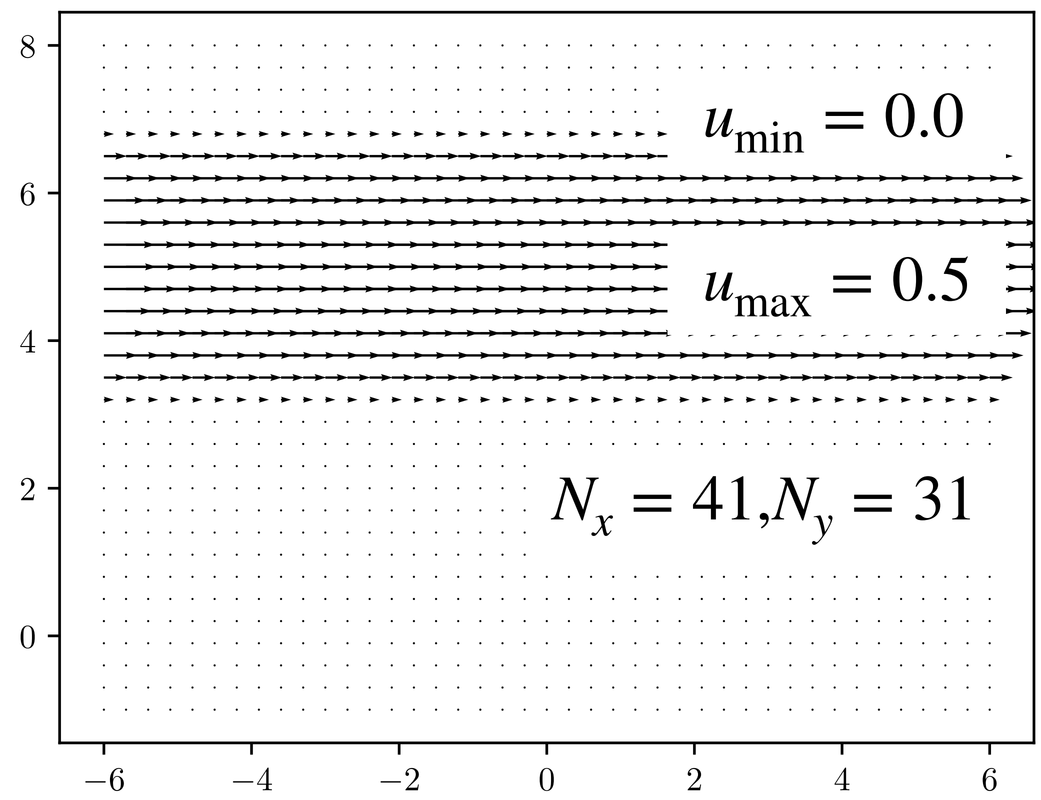
Lolla & Lermusiaux et al, 2014.

$$r^*(t) = Ft \qquad \theta^*(t) = \frac{\Gamma(Ft - R)}{2\pi F\sigma^2}$$

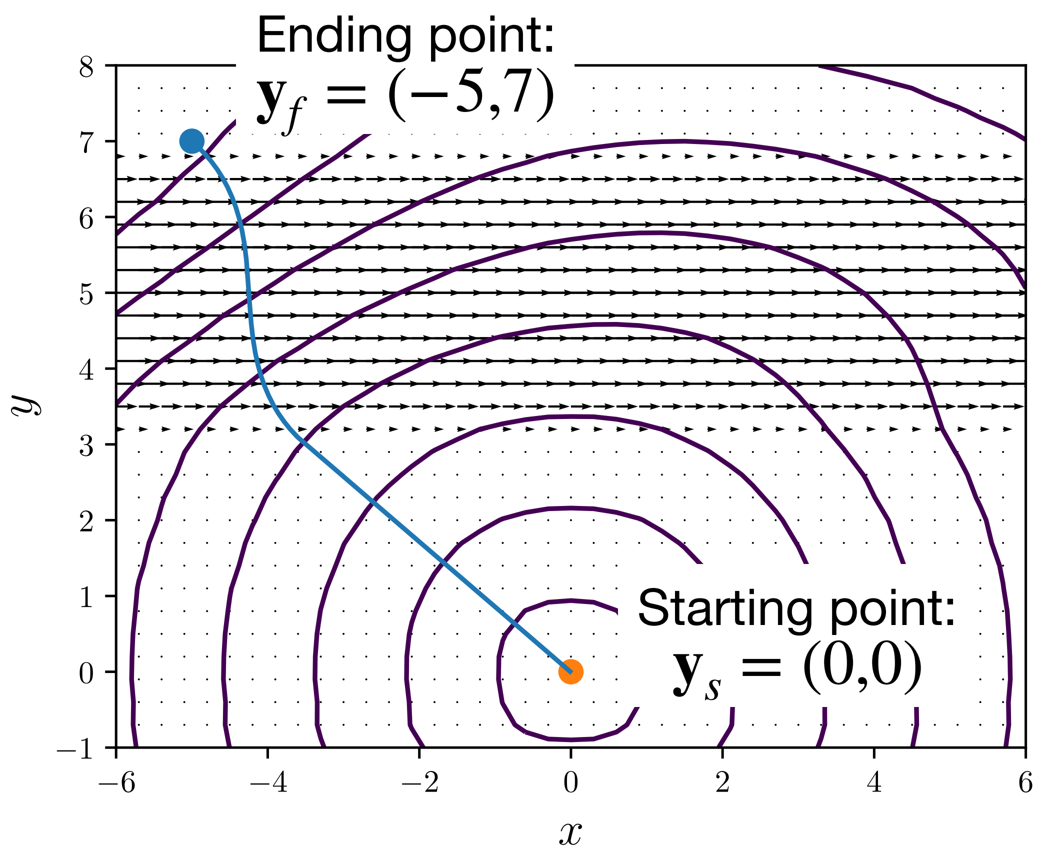
Application: Crossing River

Parabolic velocity profile: "flow between plates"

$$\bar{u}(y) = \begin{cases} -\frac{1}{8}y^2 + \frac{5}{4}y - \frac{21}{8} & \textbf{for } 3 \leq y \leq 7 \\ 0 & \textbf{otherwise} \end{cases}$$



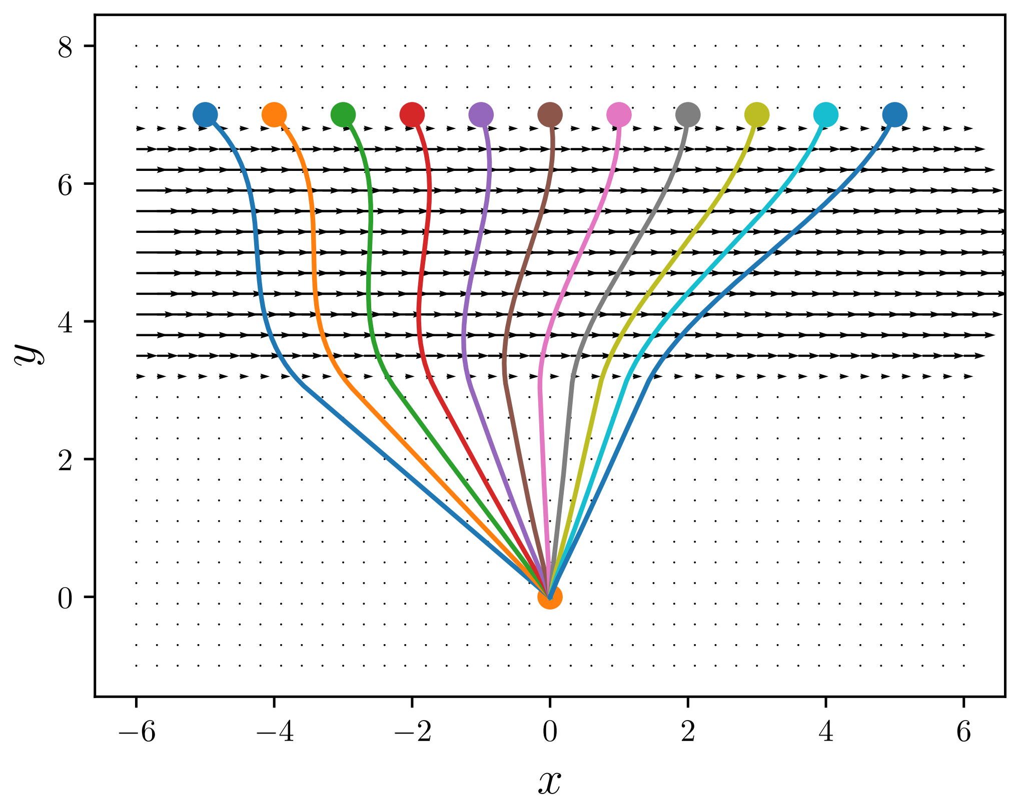
Flow field



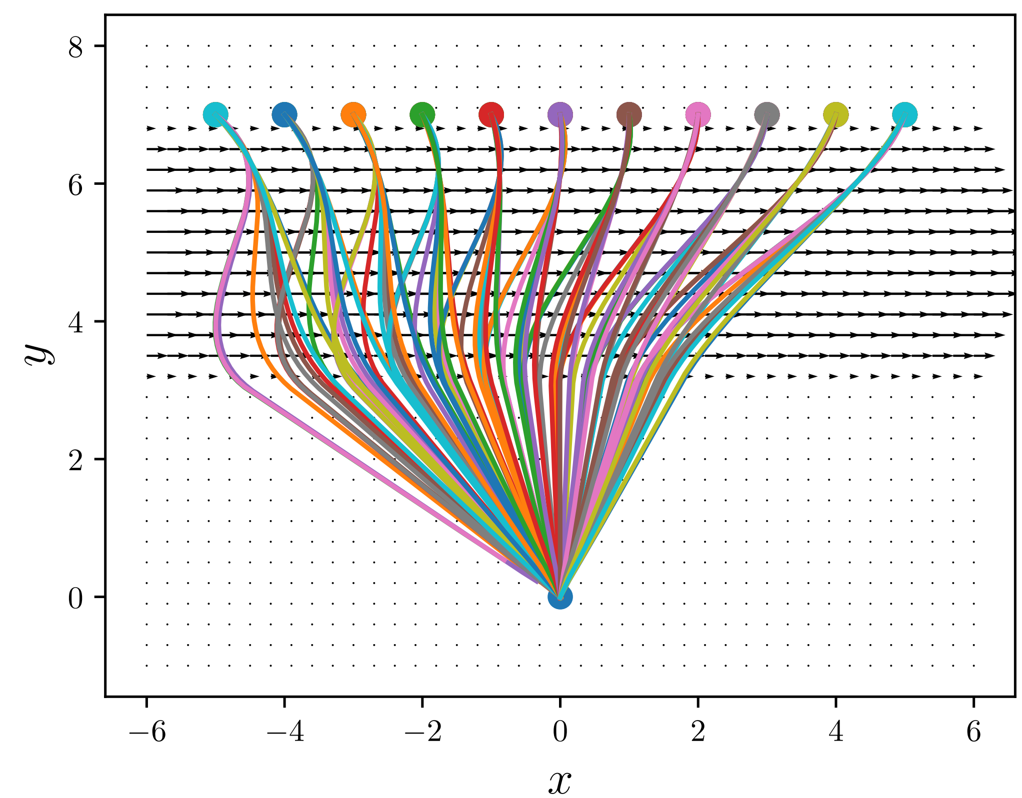
Optimal paths

Application: Crossing River

Deterministic $u(y) = \bar{u}(y)$



Stochastic $u(y) = \alpha \bar{u}(y)$, $\alpha \sim \mathcal{U}(0.4, 1.6)$



The optimal path changes according to flow field.

How can we estimate arrival time to each point given the stochastic flow?

We need large number of realizations.

Narrow-Band

Time-consuming if calculate the whole field for lots of realizations:

1. **Dynamically Orthogonal equations**
   Subramani, Wei, & Lermusiaux, 2018.
2. **Narrow-band: only zero contour is used**

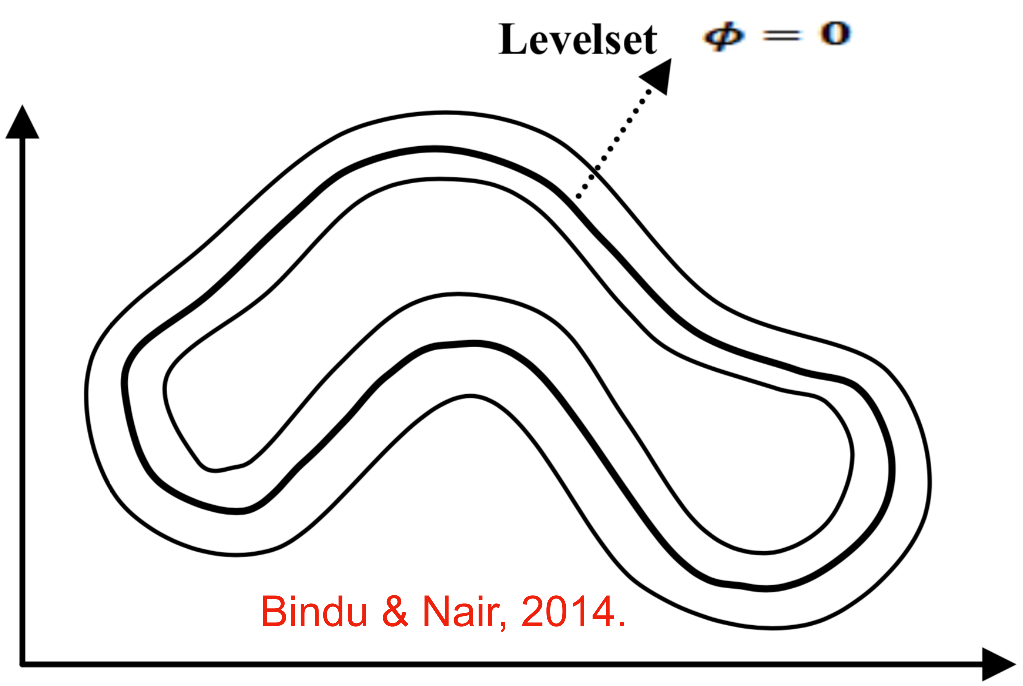


Fig. 1. Narrow band of width δ around level set Φ=0.

Only points near zero level are calculated

We can do this because historical information is carried along with the zero contour and its neighbors

Method	Entire Domain	Narrow-Band
Time (s)	1.845	0.879

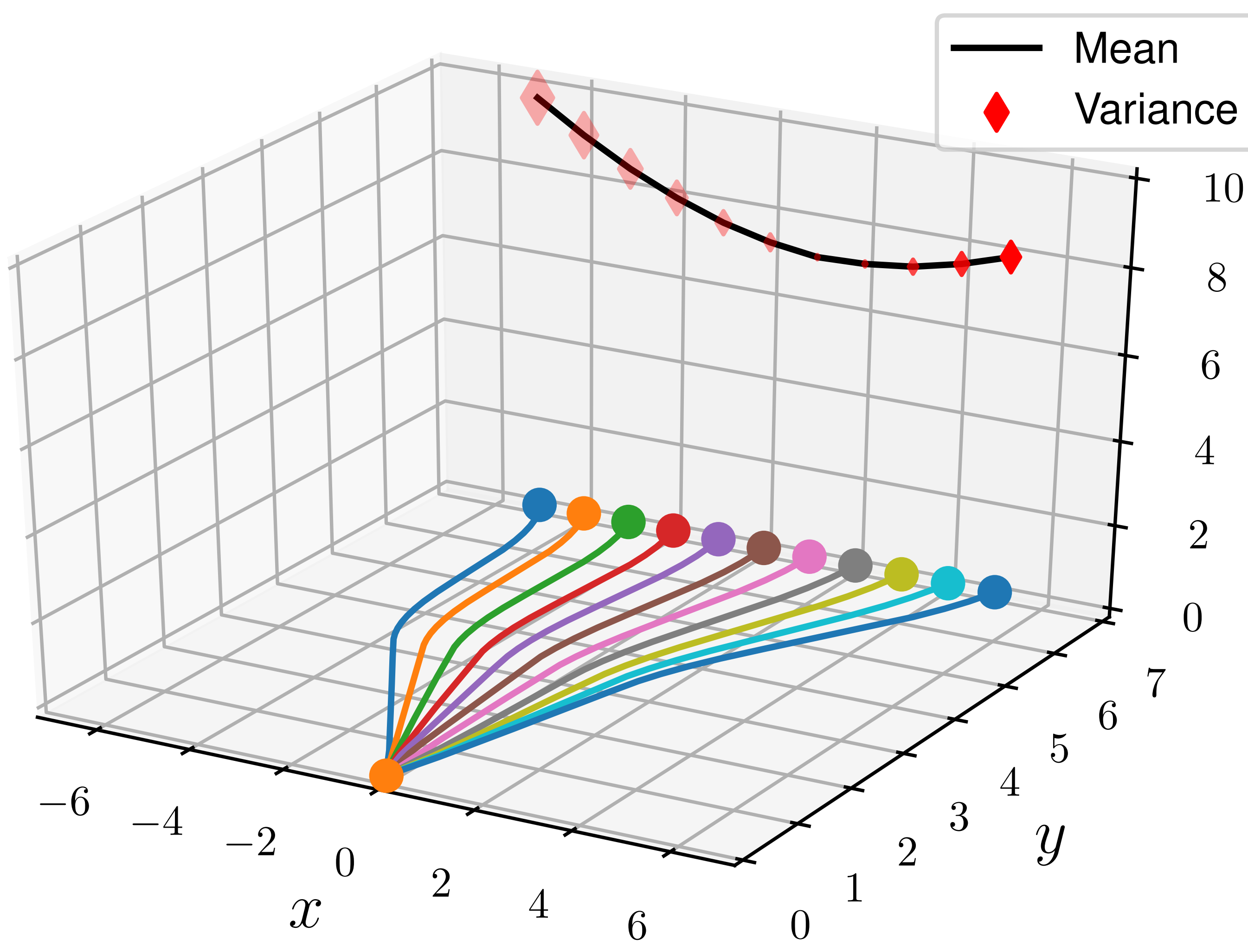
Time consuming if reassign computational domain every step, i.e. N=1

```
if i % N == 0:
        nodes = np.where((phi_flat<50) & (phi_flat>-0.1))[0]
        Dx_new = Dx[np.ix_(nodes,nodes)]
        Dy_new = Dy[np.ix_(nodes,nodes)]
```

N	1	50	100	200
Time (s)	0.879	0.382	0.474	0.631

Arrival: Mean and Variance

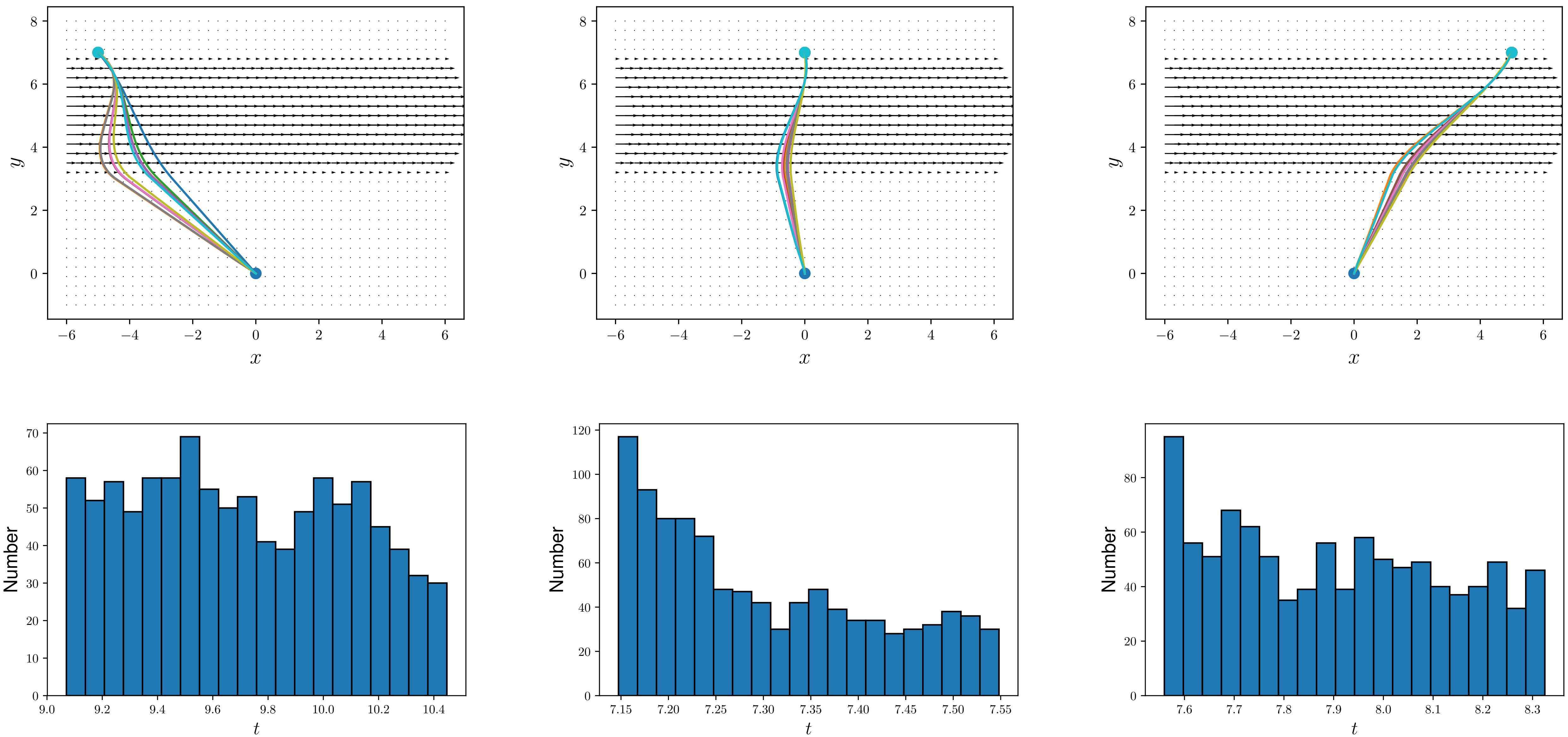
1000 realizations



Paths shown are for $u(y) = \bar{u}(y)$

y_f	Mean	Variance
(-5, 7)	9.71	0.148
(-4, 7)	9.03	0.108
(-3, 7)	8.44	0.077
(-2, 7)	7.95	0.059
(-1, 7)	7.57	0.030
(0, 7)	7.30	0.014
(1, 7)	7.15	0.001
(2, 7)	7.18	0.002
(3, 7)	7.31	0.011
(4, 7)	7.56	0.027
(5, 7)	7.90	0.052

Distribution and Optimal Paths



$\mathbf{y}_f = (-5,7)$ $\mathbf{y}_f = (0,7)$ $\mathbf{y}_f = (+5,7)$

Thank you for your listening!

Q & A

References:

1. Lolla, T., Lermusiaux, P. F., Ueckermann, M. P., & Haley, P. J. (2014). Time-optimal path planning in dynamic flows using level set equations: theory and schemes. *Ocean Dynamics*, *64*(10), 1373-1397.
2. Subramani, D. N., Wei, Q. J., & Lermusiaux, P. F. (2018). Stochastic time-optimal path-planning in uncertain, strong, and dynamic flows. *Computer Methods in Applied Mechanics and Engineering*, *333*, 218-237.
3. Bindu, V. R., & Nair, K. R. (2014, February). A fast narrow band level set formulation for shape extraction. In The Fifth International Conference on the Applications of Digital Information and Web Technologies (ICADIWT 2014) (pp. 137-142). IEEE.