Improving the Accuracy of Flowmaps

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2.29 project

What are flow maps?

The flow map is a quantity computed in ‘Post Processing’ after we have the velocity fields in a flow. **The flow map $\Phi_{t_1}^{t_2}(x)$ maps the position of a passive tracer in the flow at t_1 to its position at t_2**

The flow map can be used to compute Lagrangian characteristics of the flow. It contains the information of the integrated effect of velocity in a Lagrangian sense.

If we are looking at the time period $[0, T]$, **the flow map $\Phi_0^T(x)$ is called the forward flow map while $\Phi_T^0(x)$ is called the backward flow map**.

Computing the flow map

Path of the passive tracer:

$$\frac{dx(t)}{dt} = v(x(t), t)$$

1. Compute $\Phi_{i\Delta t}^{(i+1)\Delta t}$ and $\Phi_{(i+1)\Delta t}^{i\Delta t} \quad \forall \quad i \in 1 ... T/\Delta t$ using Forward Euler
2. Compose to get Φ_0^T and Φ_T^0

Improving the accuracy

I propose a way to improve the accuracy of the constituent flow maps $\Phi_{i\Delta t}^{(i+1)\Delta t}and\Phi_{(i+1)\Delta t}^{i\Delta t}$ using the fact that they are inverses of each other.

$$\Phi_0^{\Delta t}(x) = y$$
$$\Phi_{\Delta t}^{0}(y) = x$$

Improving the accuracy

$$\Phi_0^{\Delta t}(x) = \underbrace{x + v(x,0)\Delta t}_{\tilde{\Phi}_0^{\Delta t}(x)} + \frac{dv}{dt}(x,0)\frac{\Delta t^2}{2} + O(\Delta t^3)$$

$$\Phi_{\Delta t}^0(x) = \underbrace{x - v(x,\Delta t)\Delta t}_{\tilde{\Phi}_{\Delta t}^0(x)} + \frac{dv}{dt}(x,\Delta t)\frac{\Delta t^2}{2} + O(\Delta t^3)$$

We now have two approximations for the forward flowmap

$$\tilde{\Phi}_{0,f}^{\Delta t}(x) = x + v(x,0)\Delta t$$

$$\tilde{\Phi}_{0,b}^{\Delta t}(x - v(x,\Delta t)) = x$$

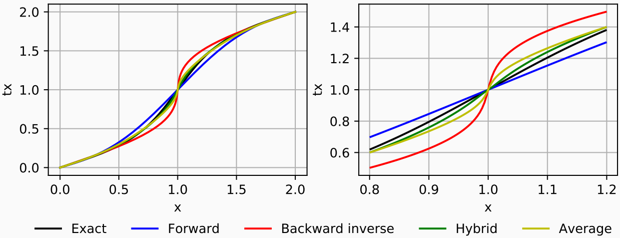
The idea is to get a higher order accurate scheme using these two approximations which have a different error. The idea is similar to Richardson extrapolation except that both the approximations have different error expressions.

Improving the accuracy

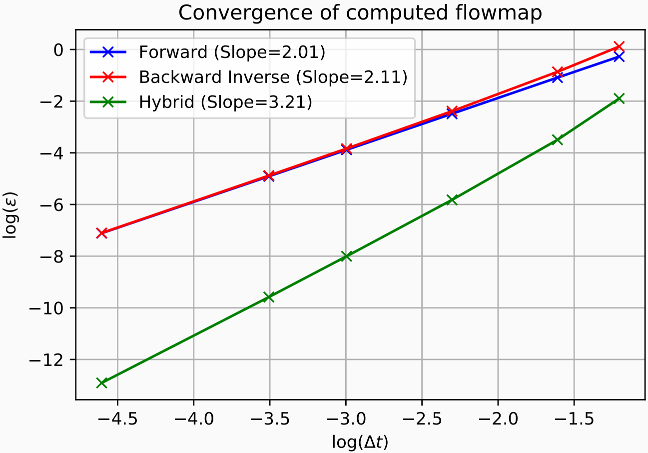
$$
\begin{aligned}
\Phi_0^{\Delta t}(x) - \tilde{\Phi}_{0,f}^{\Delta t}(x) =& \frac{dv}{dt}(x,0)\frac{\Delta t^2}{2} + O(\Delta t^3) \\
\Phi_0^{\Delta t}(x) - \tilde{\Phi}_{0,b}^{\Delta t}(x) =& - \frac{d\Phi_0^{\Delta t}}{dx}(x)\frac{dv}{dt}(x,0)\frac{\Delta t^2}{2} + O(\Delta t^3) \\
\tilde{\Phi}_{0,h}^{\Delta t}(x) =& \frac{\frac{d\Phi_0^{\Delta t}}{dx}(x)\tilde{\Phi}_{0,f}^{\Delta t} + \tilde{\Phi}_{0,b}^{\Delta t}}{1 + \frac{d\tilde{\Phi}_{0,b}^{\Delta t}}{dx}} \\
\Phi_0^{\Delta t}(x) - \tilde{\Phi}_{0,h}^{\Delta t}(x) =& O(\Delta t^3)
\end{aligned}
$$

Results

$$v(x,t) = -\sin(\pi x)(t + 0.5s)$$



Results



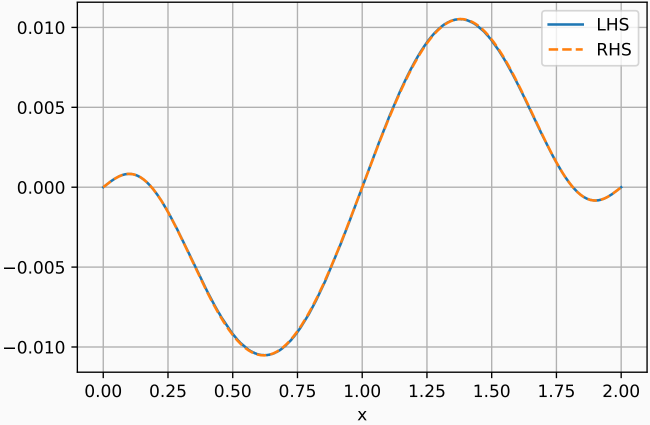
Convergence of computed flowmap
Forward (Slope=2.01)
Backward Inverse (Slope=2.11)
Hybrid (Slope=3.21)
0
−2
−4
−6
−8
−10
−12
log(ε)
−4.5
−4.0
−3.5
−3.0
−2.5
−2.0
−1.5
log(Δt)

Another approach

What we have demonstrated so far is a way to improve accuracy for flow maps that have been computed using a specific scheme. Also, the flow maps must span a short time interval for the approach to be valid. We now try to derive a relation that is independent of the scheme used to compute the flow maps:

$$
\begin{aligned}
\Phi_0^T(x) =& \tilde{\Phi}_0^T(x) + \epsilon_1(x) \\
\Phi_T^0(x) =& \tilde{\Phi}_T^0(x) + \epsilon_2(x) \\
x =& \Phi_0^T(\Phi_T^0(x)) \\
x =& \tilde{\Phi}_0^T(\tilde{\Phi}_T^0(x) + \epsilon_2(x)) + \epsilon_1(\tilde{\Phi}_T^0(x) + \epsilon_2(x)) \\
x - \tilde{\Phi}_0^T(\tilde{\Phi}_T^0(x)) \approx& \left.\frac{d\tilde{\Phi}_0^T}{dx}\right|_{\tilde{\Phi}_T^0(x)} \epsilon_2(x) + \epsilon_1(\tilde{\Phi}_T^0(x))
\end{aligned}
$$

Another approach



Another approach

We now have a relationship between the errors of the two flow maps that is independent of how the flow maps were computed. If we have another approximate relationship about the two errors using the methods used to compute them, we can use that in conjunction with the above relation to eliminate the error.