Implementing Biogeochemical Equations in Stochastic Dynamically-Orthogonal Primitive Equations

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Some Definitions

- Primitive Equations

Cons. Mass $\nabla \cdot \mathbf{u} + \frac{\partial w}{\partial z} = 0,$

Cons. Horiz. Mom. $\frac{D\mathbf{u}}{Dt} + f\hat{k} \times \mathbf{u} = -\frac{1}{\rho_0}\nabla p + \mathbf{F},$

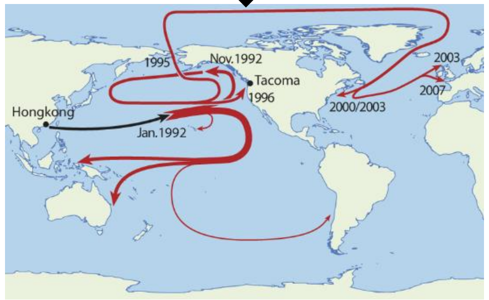
Cons. Vert. Mom. $\frac{\partial p}{\partial z} = -\rho g,$

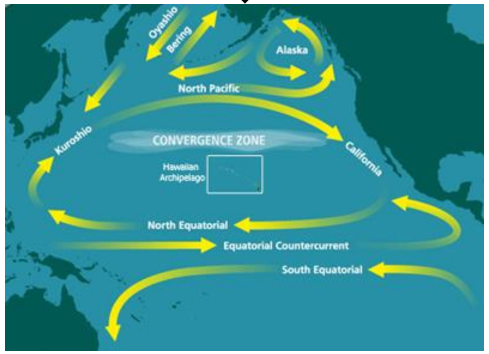
Cons. Heat $\frac{DT}{Dt} = F^T,$

Cons. Salt $\frac{DS}{Dt} = F^S,$

Eq. of State $\rho = \rho(z, T, S)$

- Tracers





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Contents

- What do ADR Equations and Marine Life have in common?
- Modelling what we don’t know
- How does uncertainty affect our forecasts?
- What’s next?

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Deterministic Biogeochemical ADR Equations

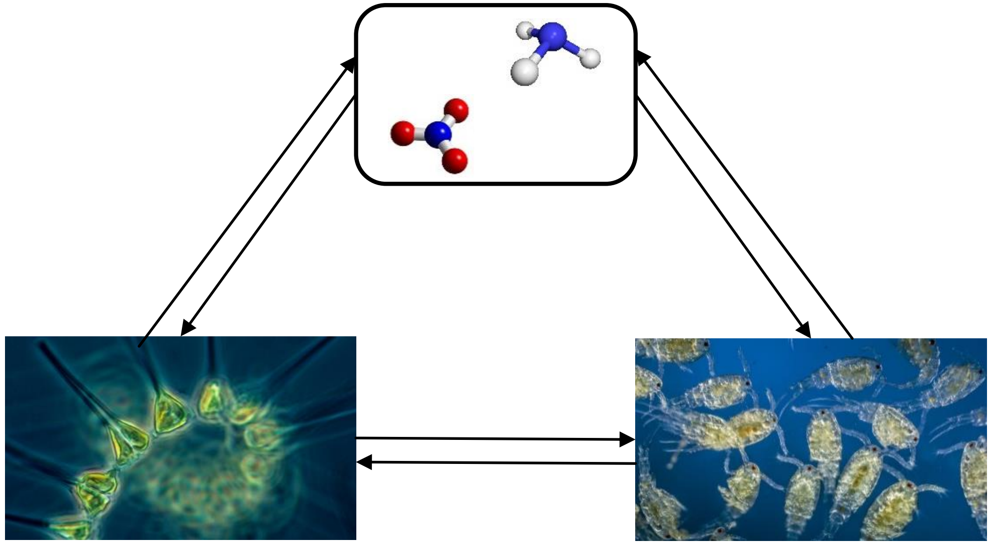
$$\frac{\partial \phi^i(\boldsymbol{x},t)}{\partial t} + \underbrace{\nabla.(\boldsymbol{u}(\boldsymbol{x},t)\phi^i(\boldsymbol{x},t))}_{Advection} - \underbrace{\mathcal{K}\nabla^2\phi^i(\boldsymbol{x},t)}_{Diffusion} = \underbrace{S^{\phi^i}(\phi^1,...,\phi^i,...,\phi^{N_\phi},\boldsymbol{x},t)}_{Reaction},$$

$$\forall i = \{1,2,...,N_\phi\}$$

$$\frac{dN}{dt} = -G\frac{PN}{N+K_u} + \Xi P + \Gamma Z + R_m\gamma Z(1-\exp^{-\Lambda P})$$

$$\frac{dP}{dt} = G\frac{PN}{N+K_u} - \Xi P - R_m Z(1-\exp^{-\Lambda P})$$

$$\frac{dZ}{dt} = R_m(1-\gamma)Z(1-\exp^{-\Lambda P}) - \Gamma Z$$



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Introducing Uncertainty - DO

$$\frac{\partial \boldsymbol{u}(\boldsymbol{x},t;\omega)}{\partial t} = \mathcal{L}[\boldsymbol{u}(\boldsymbol{x},t;\omega),\boldsymbol{\alpha}(t;\omega),\boldsymbol{x},t;\omega]$$

$$\frac{d\boldsymbol{\alpha}(t;\omega)}{dt} = 0,$$

$$\boldsymbol{u}(\boldsymbol{x},t;\omega) = \bar{\boldsymbol{u}}(\boldsymbol{x},t) + \sum_{i=1}^{N_S} Y_i(t;\omega)\tilde{\boldsymbol{u}}_i(\boldsymbol{x},t)$$

$$\boldsymbol{V}_{\boldsymbol{N_S}} = span\{\tilde{\boldsymbol{u}}_i(\boldsymbol{x},t)\}_{i=1}^{N_S}$$

$$\frac{d\boldsymbol{V}_{\boldsymbol{N_S}}}{dt} \perp \boldsymbol{V}_{\boldsymbol{N_S}} \Leftrightarrow \left\langle \frac{\partial \tilde{\boldsymbol{u}}_i(\boldsymbol{x},t)}{\partial t}, \tilde{\boldsymbol{u}}_j(\boldsymbol{x},t) \right\rangle = 0 \quad \forall i,j \in \{1,...,N_S\}$$

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Introducing Uncertainty – Evolution Equations

$$\frac{\partial \bar{\boldsymbol{u}}(\boldsymbol{x},t)}{\partial t} = \mathbb{E}[\mathcal{L}[\boldsymbol{u}(\boldsymbol{x},t;\omega),\boldsymbol{\alpha}(t;\omega),\boldsymbol{x},t;\omega]]$$

$$\frac{\partial \tilde{\boldsymbol{u}}_i(\boldsymbol{x},t)}{\partial t} = \sum_{j=1}^{N_S} C^{-1}_{Y_iY_j}\Pi^{\perp}_{\boldsymbol{u}}[\mathbb{E}[Y_j(t;\omega)\mathcal{L}[\boldsymbol{u}(\boldsymbol{x},t;\omega),\boldsymbol{\alpha}(t;\omega),\boldsymbol{x},t;\omega]]]$$

$$\frac{\mathrm{d}Y_i(t;\omega)}{\mathrm{d}t} = \langle \mathcal{L}[\boldsymbol{u}(\boldsymbol{x},t;\omega),\boldsymbol{\alpha}(t;\omega),\boldsymbol{x},t;\omega] - \mathbb{E}[\mathcal{L}[\boldsymbol{u}(\boldsymbol{x},t;\omega),\boldsymbol{\alpha}(t;\omega),\boldsymbol{x},t;\omega]],\tilde{\boldsymbol{u}}_i(\boldsymbol{x},t)\rangle$$

Introducing Uncertainty – Stochastic ADR

$$\frac{\partial \phi^i(\boldsymbol{x},t;\omega)}{\partial t} + \nabla.(\mathbf{u}(\boldsymbol{x},t;\omega)\phi^i(\boldsymbol{x},t;\omega)) - \mathcal{K}\nabla^2\phi^i(\boldsymbol{x},t;\omega) = S^{\phi^i}(\boldsymbol{\phi}(\boldsymbol{x},t;\omega),\boldsymbol{\alpha}(t;\omega),\boldsymbol{x},t;\omega)$$

DO condition

$$\frac{\partial \bar{\boldsymbol{\phi}}}{\partial t} + \frac{dY_i}{dt}\tilde{\boldsymbol{\phi}}_i + Y_i\frac{\partial \tilde{\boldsymbol{\phi}}_i}{\partial t} = -\nabla.[(\bar{\boldsymbol{u}} + \tilde{\boldsymbol{u}}_j Y_j)(\bar{\boldsymbol{\phi}} + \tilde{\boldsymbol{\phi}}_i Y_i)] + \mathcal{K}\nabla^2(\bar{\boldsymbol{\phi}} + \tilde{\boldsymbol{\phi}}_i Y_i)$$

$$+\boldsymbol{S}^{\phi}\Big|_{\substack{\phi=\bar{\phi},\\ \alpha=\bar{\alpha}}} + \frac{\partial \boldsymbol{S}^{\phi}}{\partial \boldsymbol{\phi}}\Bigg|_{\substack{\phi=\bar{\phi},\\ \alpha=\bar{\alpha}}}\tilde{\boldsymbol{\phi}}_i Y_i + \frac{\partial \boldsymbol{S}^{\phi}}{\partial \alpha_i}\Bigg|_{\substack{\phi=\bar{\phi},\\ \alpha=\bar{\alpha}}} E_i^{\alpha}$$

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Introducing Uncertainty – Mean

$$
\begin{aligned}
\frac{\partial \bar{\phi}}{\partial t} + \underbrace{\mathbb{E}\left[\frac{dY_i}{dt}\right]}_{0} \tilde{\phi}_i + \underbrace{\mathbb{E}\left[Y_i\right]}_{0} \frac{\partial \tilde{\phi}_i}{\partial t} = \; & - \quad \nabla.(\bar{\boldsymbol{u}}\bar{\phi}) - \underbrace{\mathbb{E}[Y_i]}_{0} \nabla(\bar{\boldsymbol{u}}\tilde{\phi}_i) - \underbrace{\mathbb{E}[Y_j]}_{0}(\tilde{\boldsymbol{u}}_j\bar{\phi}) \\
& - \quad \underbrace{\mathbb{E}[Y_iY_j]}_{C_{Y_iY_j}} \nabla(\tilde{\boldsymbol{u}}_j\tilde{\phi}_i) + \mathcal{K}\nabla^2(\bar{\phi}) + \underbrace{\mathbb{E}[Y_i]}_{0} \mathcal{K}\nabla^2(\tilde{\phi}_i) \\
& + \quad \bar{\boldsymbol{S}}^{\phi} + \underbrace{\mathbb{E}[Y_i]}_{0} \frac{\partial \boldsymbol{S}^{\phi}}{\partial \phi} \tilde{\phi}_i + \underbrace{\mathbb{E}[E_i^{\alpha}]}_{0} \frac{\partial \boldsymbol{S}^{\phi}}{\partial \alpha_i}
\end{aligned}
$$

$$
\Downarrow
$$

$$
\frac{\partial \bar{\phi}}{\partial t} = -\nabla.(\bar{\boldsymbol{u}}\bar{\phi}) - C_{Y_iY_j} \nabla(\tilde{\boldsymbol{u}}_j\tilde{\phi}_i) + \mathcal{K}\nabla^2(\bar{\phi}) + \bar{\boldsymbol{S}}^{\phi}
$$

Introducing Uncertainty – Modes

$$\frac{\partial \tilde{\phi}_i}{\partial t} = \boldsymbol{Q}_i^{\phi} - \langle \boldsymbol{Q}_i, \tilde{\boldsymbol{\Phi}}_j \rangle \tilde{\phi}_j \qquad \boldsymbol{Q}_i = \begin{bmatrix} \boldsymbol{Q}_i^{u} \\ \boldsymbol{Q}_i^{\phi} \end{bmatrix}, \qquad \tilde{\boldsymbol{\Phi}}_i = \begin{bmatrix} \tilde{\boldsymbol{u}}_i \\ \tilde{\phi}_i \end{bmatrix}$$

$$\begin{aligned} \boldsymbol{Q}_i^{\phi} = -\frac{\partial \tilde{\phi}_i}{\partial t} - \nabla(\bar{\boldsymbol{u}}\tilde{\phi}_i) - C_{Y_iY_j}C_{Y_jY_k}(\tilde{\boldsymbol{u}}_k\bar{\phi}) - C^{-1}_{Y_iY_j}C_{Y_kY_jY_n}\nabla(\tilde{\boldsymbol{u}}_k\tilde{\phi}_n) + \mathcal{K}\nabla^2(\tilde{\phi}_i) \\ + \frac{\partial \boldsymbol{S}^{\phi}}{\partial \phi}\tilde{\phi}_i + C^{-1}_{Y_iY_j}C_{E_n^{\alpha}Y_j}\frac{\partial \boldsymbol{S}^{\phi}}{\partial \alpha_n} \end{aligned}$$

$$\begin{aligned} \boldsymbol{Q}_i^{u} \;=\; & -\nabla.(\bar{\boldsymbol{u}}\tilde{\boldsymbol{u}}_i) - \nabla.(\tilde{\boldsymbol{u}}_i\bar{\boldsymbol{u}}) - C^{-1}_{Y_iY_j}C_{Y_nY_jY_m}\nabla.(\tilde{\boldsymbol{u}}_n\tilde{\boldsymbol{u}}_m) - \nabla\tilde{P}_i + \bar{\Lambda}\nabla^2(\tilde{\boldsymbol{u}}_i) \\ & + C_{E^{\Lambda}Y_j}C^{-1}_{Y_iY_j}\nabla^2\bar{\boldsymbol{u}} + C^{-1}_{Y_iY_j}C_{E^{\Lambda}Y_nY_j}\nabla^2\tilde{\boldsymbol{u}}_n \end{aligned}$$

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Introducing Uncertainty – Stochastic Coefficients

$$\frac{dY_i}{dt} = \langle \boldsymbol{F}_m, \tilde{\boldsymbol{\Phi}}_i \rangle Y_m - \langle \nabla.(\tilde{\boldsymbol{u}}_n \tilde{\boldsymbol{u}}_m), \tilde{\boldsymbol{u}}_i \rangle (Y_m Y_n - C_{Y_m Y_n})$$

$$-\langle \nabla.(\tilde{\boldsymbol{u}}_n \tilde{\phi}_m), \tilde{\phi}_i \rangle (Y_m Y_n - C_{Y_m Y_n}) + \left\langle \frac{\partial \boldsymbol{S}^{\phi}}{\partial \alpha_m}, \tilde{\phi}_i \right\rangle E_m^{\alpha}$$

$$\boldsymbol{F}_m = \begin{bmatrix} \boldsymbol{F}_m^{\boldsymbol{u}} \\ \boldsymbol{F}_m^{\phi} \end{bmatrix} \qquad \begin{aligned} \boldsymbol{F}_{\boldsymbol{m}}^{\boldsymbol{u}} &= \bar{\Lambda} \nabla^2 (\tilde{\boldsymbol{u}}_m) - \nabla.(\tilde{\boldsymbol{u}}_m \bar{\boldsymbol{u}}) - \nabla.(\bar{\boldsymbol{u}} \tilde{\boldsymbol{u}}_m) - \nabla \tilde{P}_m \\ \boldsymbol{F}_m^{\phi} &= -\nabla.(\tilde{\boldsymbol{u}}_m \bar{\phi}) - \nabla.(\bar{\boldsymbol{u}} \tilde{\phi}_m) + \mathcal{K} \nabla^2 \tilde{\phi}_m + \frac{\partial \boldsymbol{S}^{\phi}}{\partial \phi} \tilde{\phi}_m \end{aligned}$$

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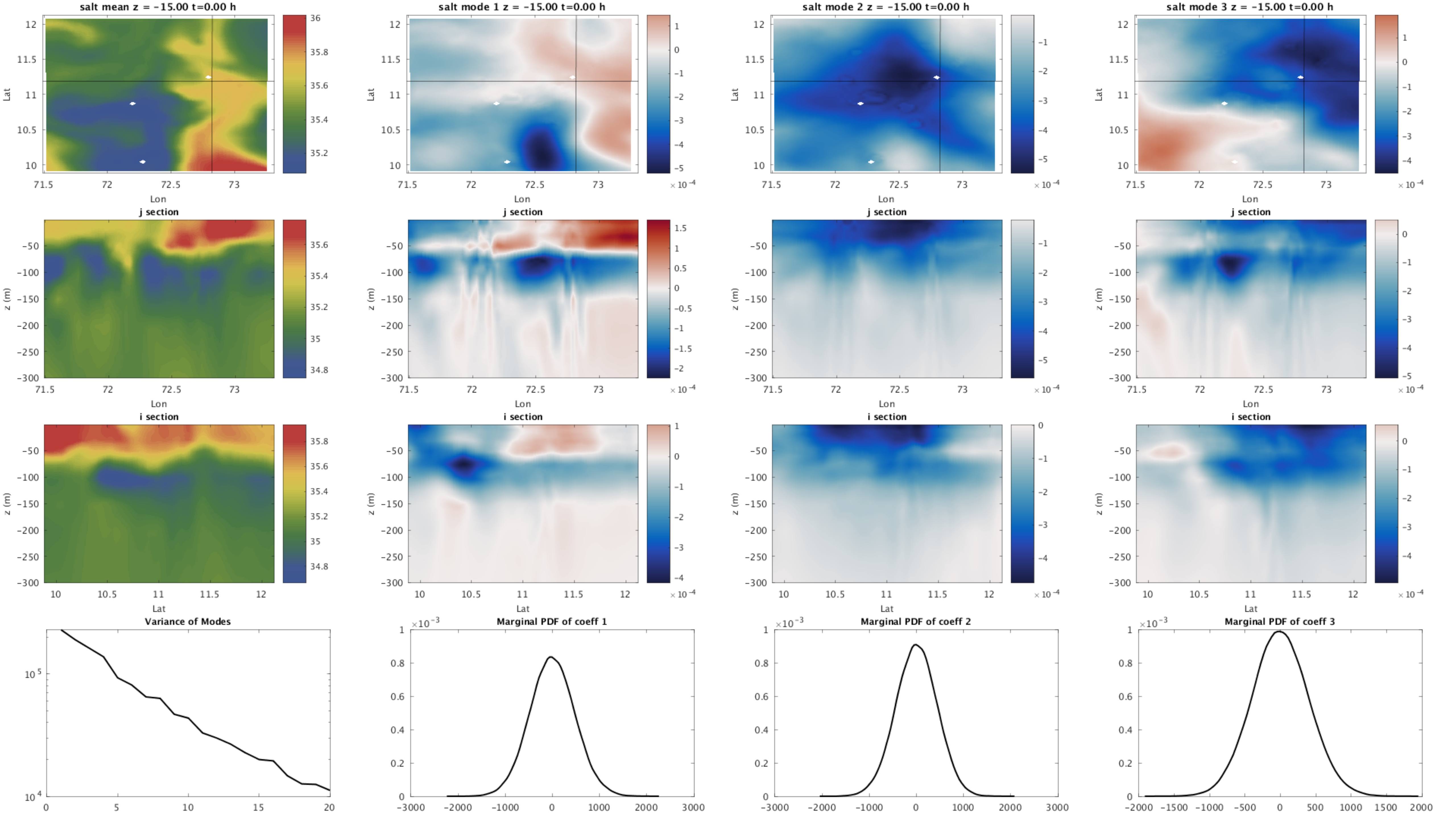
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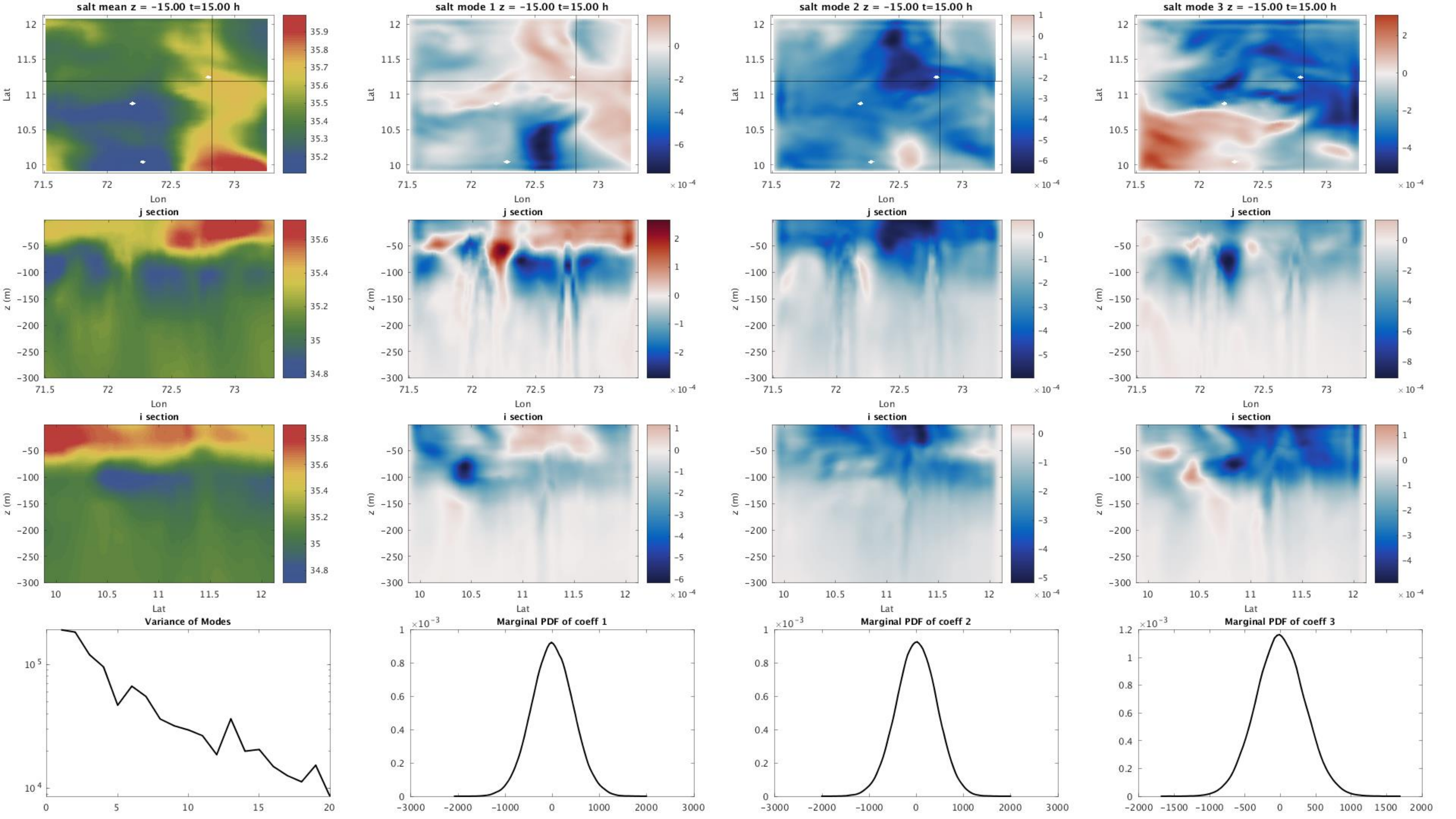
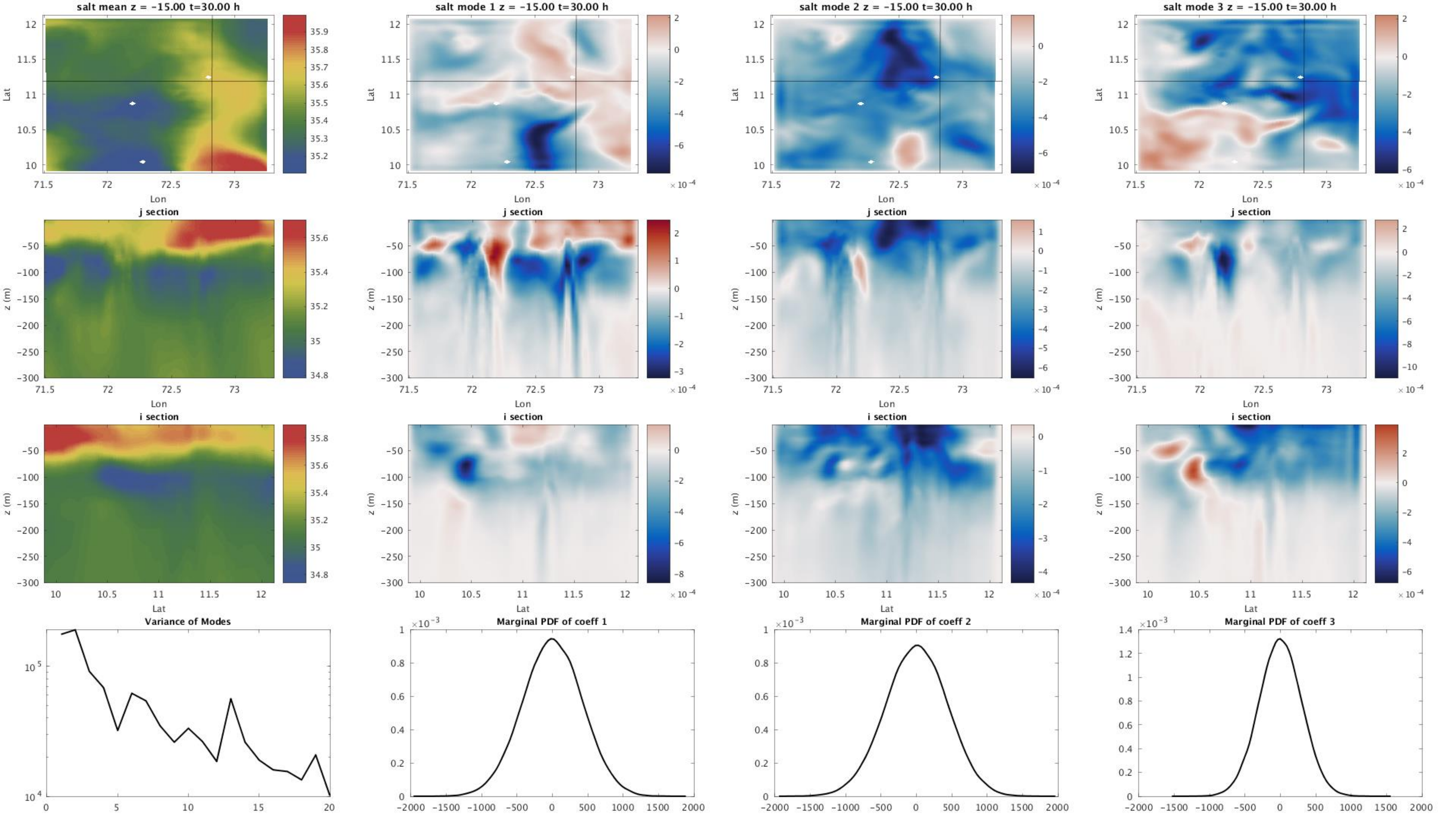
salt mode 1 z = −15.00 t=0.00 h
salt mode 2 z = −15.00 t=0.00 h
salt mode 3 z = −15.00 t=0.00 h
Lat
Lon
j section
i section
z (m)
Variance of Modes
Marginal PDF of coeff 1
Marginal PDF of coeff 2
Marginal PDF of coeff 3



salt mean z = −15.00 t=15.00 h
salt mode 1 z = −15.00 t=15.00 h
salt mode 2 z = −15.00 t=15.00 h
salt mode 3 z = −15.00 t=15.00 h
Lat
Lon
j section
i section
z (m)
×10⁻⁴
Variance of Modes
Marginal PDF of coeff 1
Marginal PDF of coeff 2
Marginal PDF of coeff 3
×10⁻³



salt mean z = -15.00 t=30.00 h
salt mode 1 z = -15.00 t=30.00 h
salt mode 2 z = -15.00 t=30.00 h
salt mode 3 z = -15.00 t=30.00 h
Lat
Lon
j section
i section
z (m)
Variance of Modes
Marginal PDF of coeff 1
Marginal PDF of coeff 2
Marginal PDF of coeff 3

What's next?

- Implementing Stochastic Biological Reaction Terms and investigate their effect on the forecasts
- Stochastic Initial Conditions
- Stochastic Boundary Conditions

Questions?

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