Method of Regularized Stokeslets in 2D and 3D

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Stokes flow and Stokeslets

- Stokes flow
  - $Re = UL/\nu \ll 1$
  - Applications: motion of microorganisms, bubbles, MEMS devices etc.
- Two important properties: linearity and time-independence
- Stokeslet: the fundamental solution due to a Dirac-delta type point force. In 3D
- Consider a **regularized force** $\mathbf{F} = \mathbf{f_0}\phi_\epsilon \quad \rightarrow$ a **regularized Stokeslet**
  - ϕ_ϵ: a cutoff function, which has a finite radius of support
  - ϵ : a parameter which controls the spreading of a regularized force
- Main idea: for a given force distribution, velocity field can be found via linear superposition of regularized Stokeslets

$$\mathbf{u}(\mathbf{x}_0) = \frac{1}{8\pi\mu} \int_S \mathbf{f}(\mathbf{x}) \cdot \mathbf{S}^\epsilon(\mathbf{x}_0, \mathbf{x}) dS$$

Stokes equations

$$\mu \nabla^2 \mathbf{u} - \nabla p + \mathbf{F} = 0$$

$$\nabla \cdot \mathbf{u} = 0$$

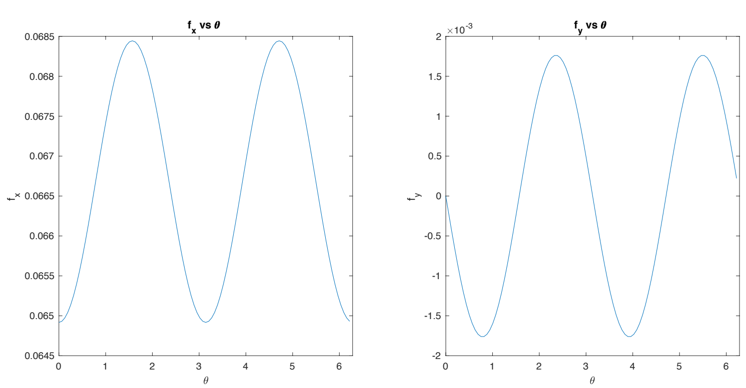
$$S_{ij}(\mathbf{x}_0, \mathbf{x}) = \frac{\delta_{ij}}{r} + \frac{r_i r_j}{r^3}$$

2D application I: flow past a cylinder

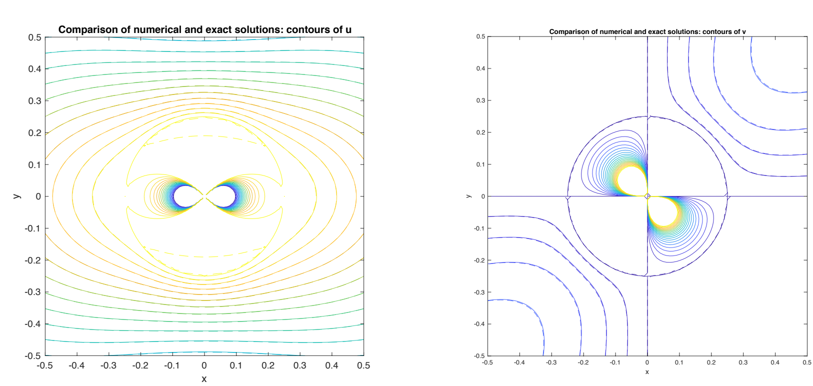
- Set-up: A cylinder translating at speed u = 1, v = 0
- N equally spaced discretization points
- Step 1: Find the forces from the velocity BC :
    - Solve the matrix equation $$\mathbf{u}(\mathbf{x}_i) = \sum_{j=1}^{N} M_{ij}(\mathbf{x}_1, \ldots, \mathbf{x}_N))\mathbf{f_j}$$
    - Note that matrix M (2N x 2N) is singular. Two ways to get a unique solution
        - additional constraint: total normal force add to zero
        - Or use GMRES with zero initial guess
- Step 2: find the velocity field elsewhere via superposition
- Compare with analytic solution (Stokes paradox)

Choose the 2D cutoff function:

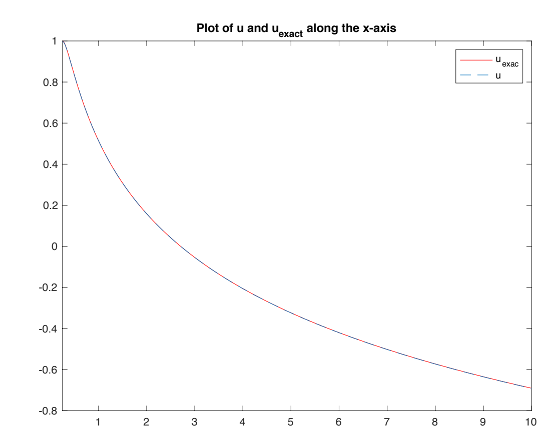
$$\phi_\epsilon(\mathbf{x}) = \frac{3\epsilon^3}{2\pi(|\mathbf{x}|^2 + \epsilon^2)^{5/2}}.$$



Force distribution (N=100)

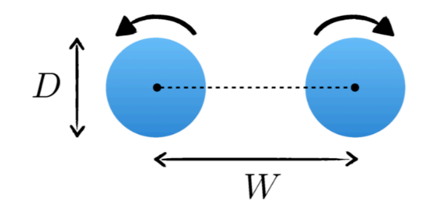


Velocity contours (solid line: exact solution; dashed line: numerical solution)

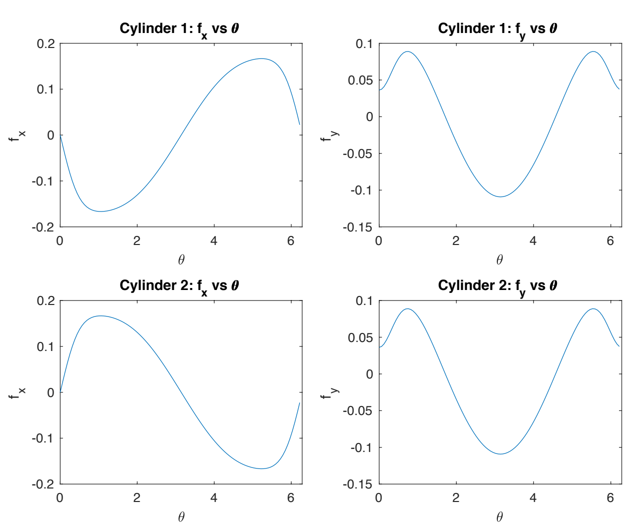


Velocity u along x-axis

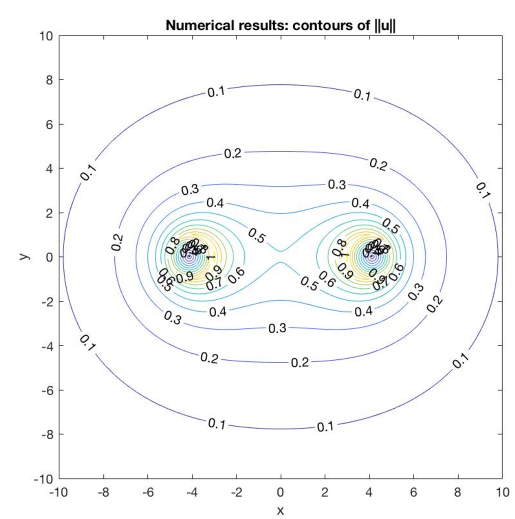
2D application II: a self-propelled cylinder pair



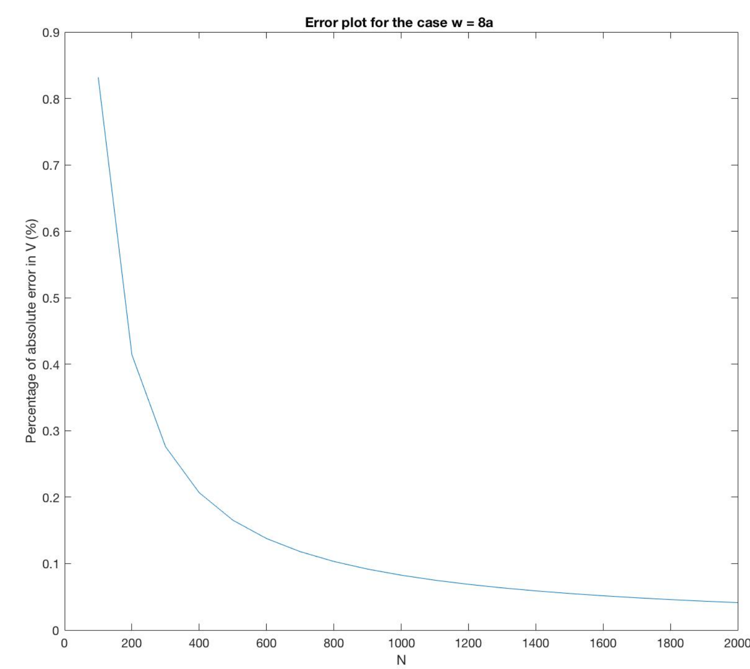
- Set-up:
  - Two identical cylinders with a fixed separation rotating about their axes
  - Self-propelled:
    - 3 additional scalar equations : $\Sigma F_x = 0 \quad \Sigma F_y = 0 \quad \Sigma \tau = 0$
    - 3 additional unknowns: overall translational velocity U and V and angular velocity Ω
- Find the forces on the cylinder through the velocity BC using $\mathbf{u}_{def} = \mathbf{Lf} - \mathbf{U} - \mathbf{\Omega} \times \mathbf{x}$
- Then find the velocity elsewhere
- Special case: counter-rotating at equal angular velocity
  - Analytic solution exists: Jeffery's solution
  - Expect $V = a^2\omega/w$



Forces on each cylinder (w = 3a)



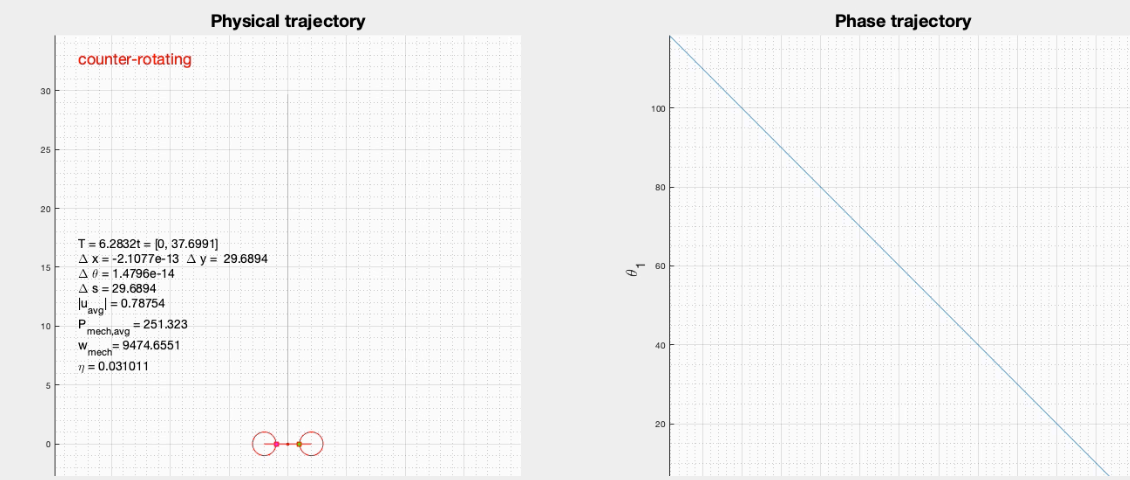
Velocity contours N= 200 (w = 8a)



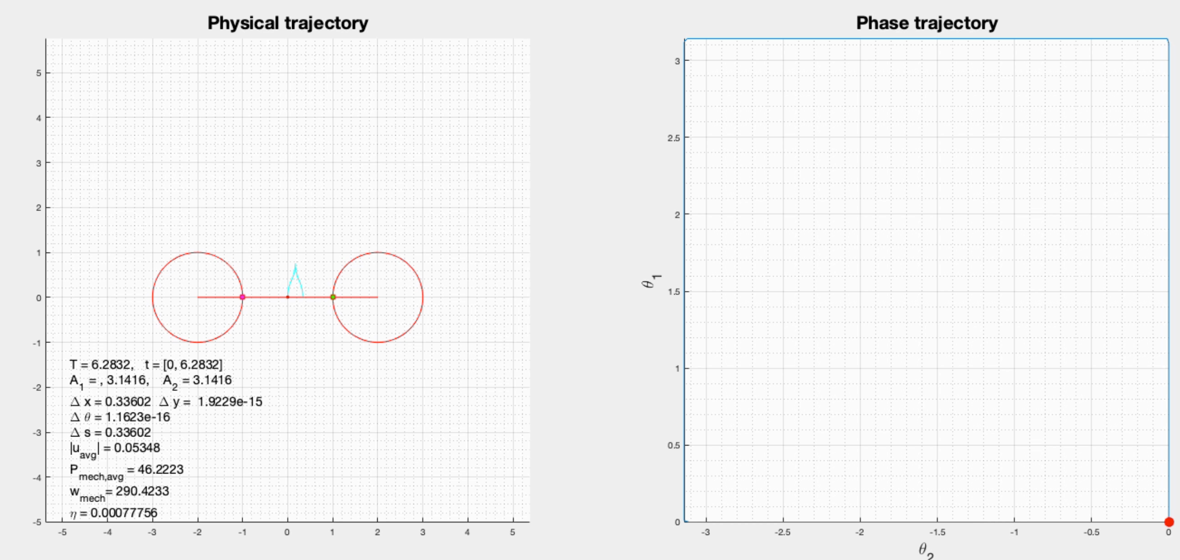
Error in V vs N for w = 8a

2D application II: a self-propelled cylinder pair

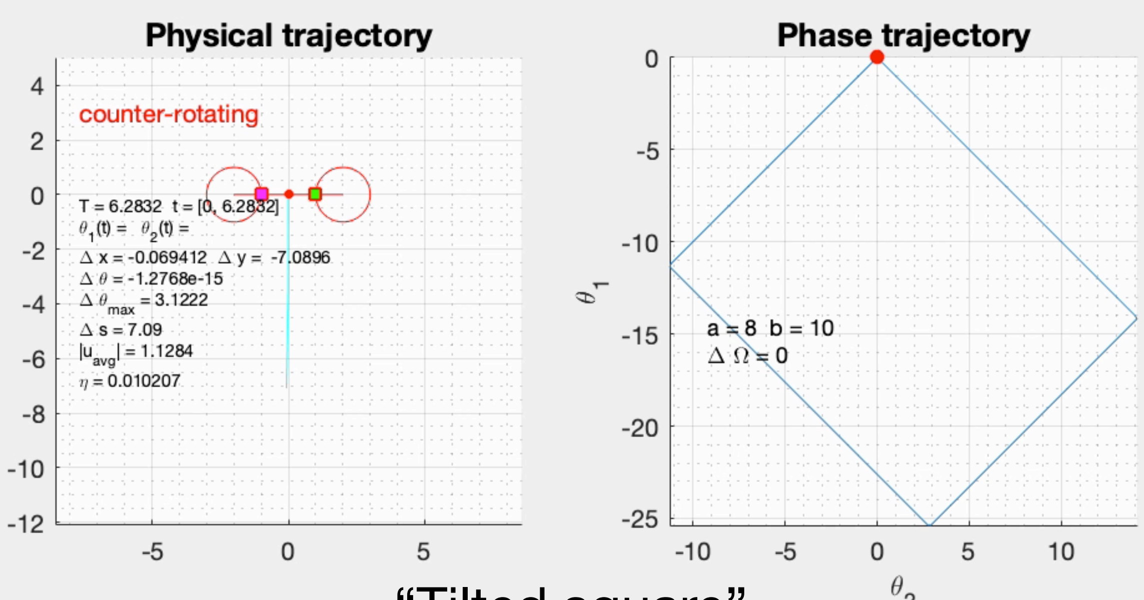
- Visualize the trajectory for arbitrary paths in the angle phase space
- Next step: optimize the efficiency of closed phase trajectories for a given input energy



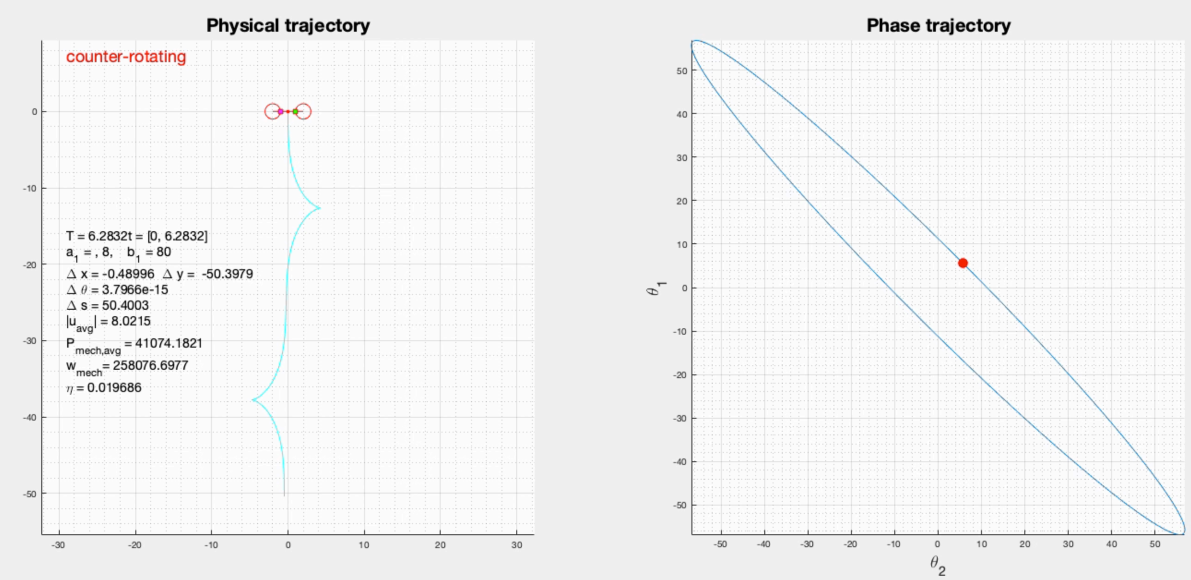
Counter-rotating



"Purcell's strokes" (one cycle)



"Tilted square"



"Tilted ellipse"

Method of Regularized Stokeslets in 3D: the original method

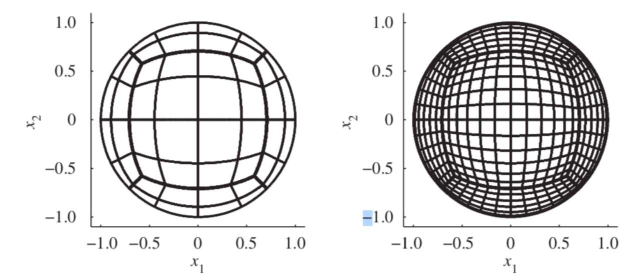
- Similar to 2D, starting from the boundary integral equation, $\mathbf{u}(\mathbf{x}_0) = \frac{1}{8\pi\mu}\int_S \mathbf{f}(\mathbf{x}) \cdot \mathbf{S}^\epsilon(\mathbf{x}_0, \mathbf{x}) dS$
- Cortez et al (2005) derived that:
  - for N Stokeslets located along the surface of the solid body D, the fluid velocity can be approximate with

$$\mathbf{u}(\mathbf{x}_0) = \frac{1}{8\pi\mu}\sum_{n=1}^{N} w_n \mathbf{f}_n \cdot \mathbf{S}^\epsilon(\mathbf{x}_0, \mathbf{x}_n) \quad (1)$$

  - Where $\mathbf{S}^\epsilon$ is the regularized Green's function (regularized Stokeslet)
    $\mathbf{f}_n$ the Stokeslet strength, force per unit area or length
    w_n the quadrature weight of the nth particle
- Here, we use the 3D cutoff function $\phi_\epsilon(\mathbf{x}) = \frac{15\epsilon^4}{8\pi(||\mathbf{x}||^2 + \epsilon^2)^{7/2}}$

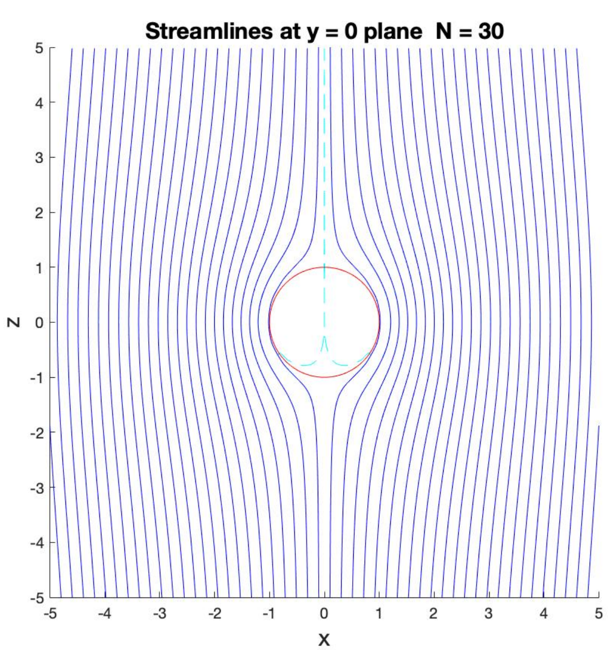
3D problem: flow past a sphere

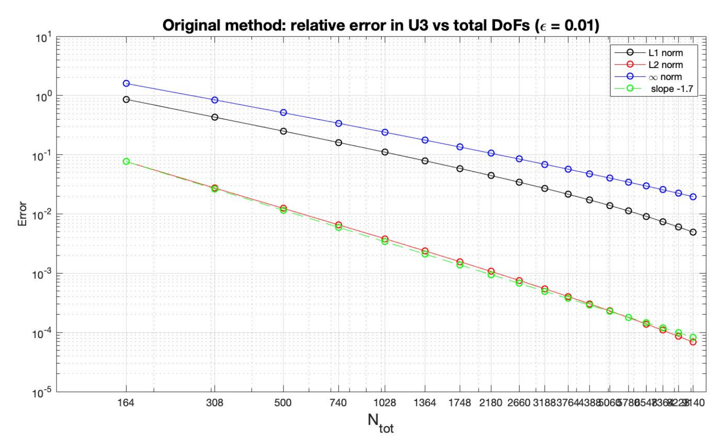
- The original approach: $$\mathbf{u}(\mathbf{x}_0) = \frac{1}{8\pi\mu}\sum_{n=1}^{N} w_n \mathbf{f}_n \cdot \mathbf{S}^{\epsilon}(\mathbf{x}_0, \mathbf{x}_n) \quad (1)$$
- Discretize the surface of a unit sphere using a six-patch structured grid.
    - Projecting the 6 faces of cube onto the surface of the sphere
    - Each face of the cube is discretized with an N x N uniform grid.
- evaluate the integral using trapezoidal rule.

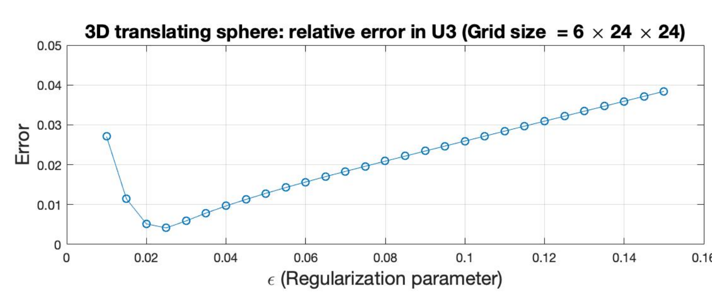


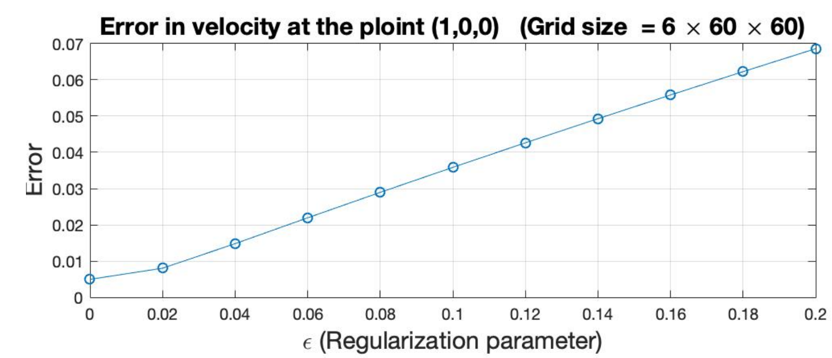
(Smith 2009)

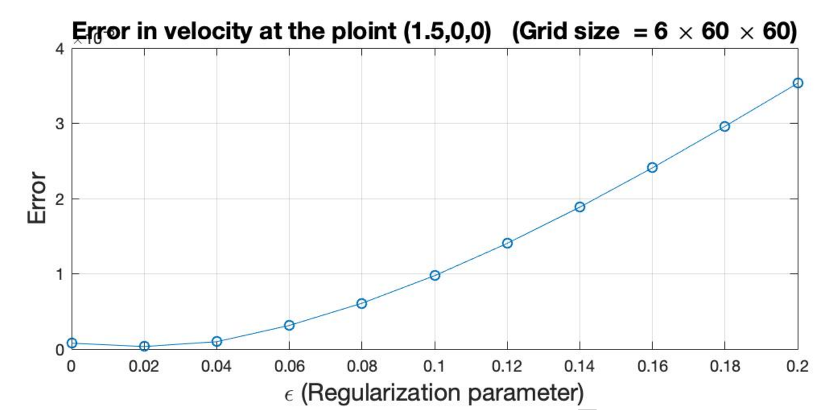
- Results:











Error curves

A boundary element regularized Stokeslet method

- Smith (2009): the original approach is equivalent to a constant-force boundary element method where low-order quadrature in which the abscissae are identified with the collocation points.

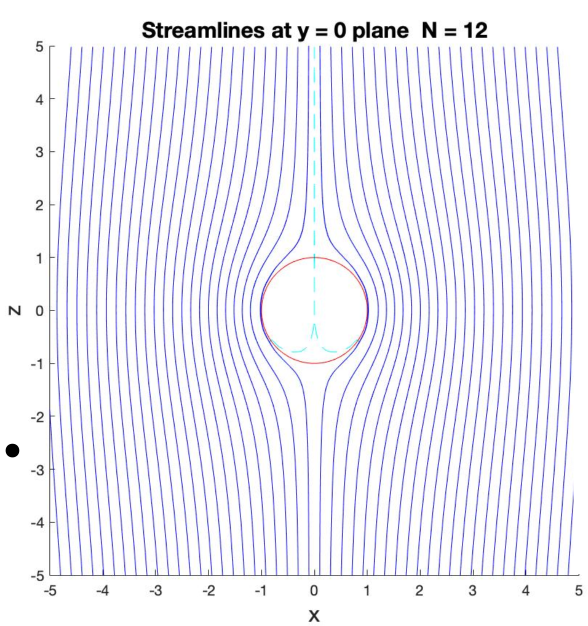
- Observation: The kernel $\mathbf{S}^{\epsilon}$ varies rapidly close to $\mathbf{x}_n$

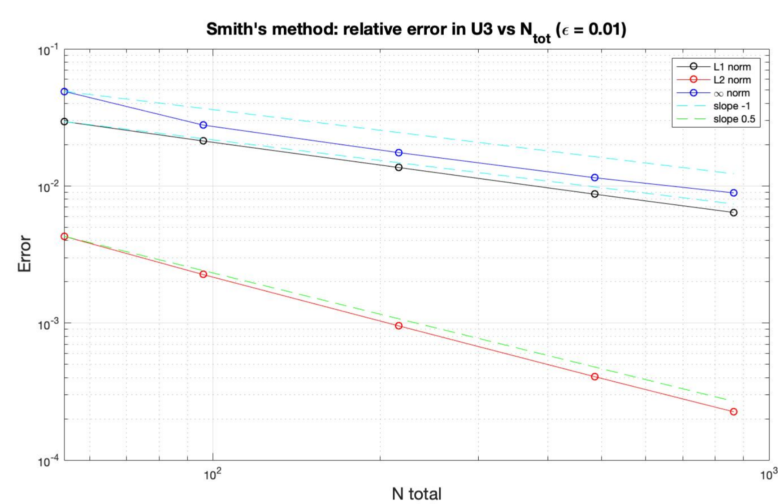
$$\mathbf{u}(x_0) = \frac{1}{8\pi\mu} \sum_{n=1}^{N} w_n \mathbf{f}_n \cdot \mathbf{S}^{\epsilon}(\mathbf{x_0}, \mathbf{x_n}) \quad (1)$$

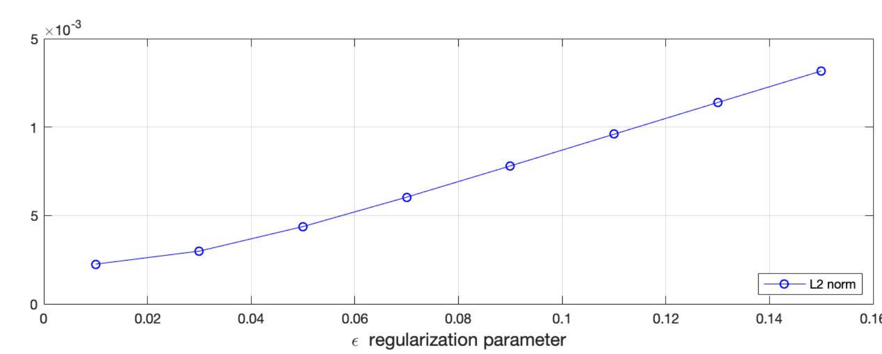
- Thus, we **decouple** the force and the integration of the kernel.

$$\mathbf{u}(\mathbf{x}_0) = \frac{1}{8\pi\mu} \sum_{n=1}^{N} \mathbf{f}_n \cdot \int_{S_n} \mathbf{S}^{\epsilon}(\mathbf{x_0}, \mathbf{x}) dS \quad (2)$$

- Note:
    - The choice of constant-force element is purely for simplicity: higher order methods maybe used
    - The integral maybe evaluated using any appropriate methods. E.g. Gauss-Legendre quadrature (12 x 12 points for ‘near-singular’ cases ; 4 x 4 points otherwise)







Error curves

Comparison of the two approaches

- Limiting factor: matrix setup and storage cost:
    - e.g. halving the element width → 16 times # matrix elements
- A method that gives accurate results with a relatively coarse mesh is beneficial.
- The boundary element approach is better — achieves high accuracy with fewer DoFs

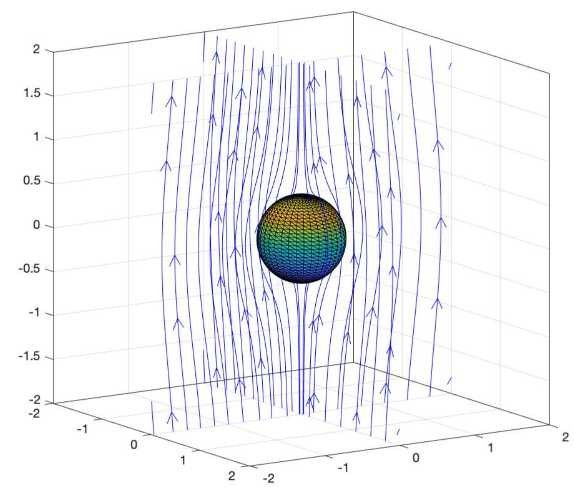
N	DOF	matrix entries	no. kernel evaluations	rel. % error in drag
Original method				
$6 \times 12 \times 12$	2592	6.72×10^6	6.72×10^6	12.6
$6 \times 24 \times 24$	10368	1.07×10^8	1.07×10^8	2.76
$6 \times 36 \times 36$	23328	5.44×10^8	5.44×10^8	0.849
$6 \times 48 \times 48$	41472	1.72×10^9	1.72×10^9	0.265
Boundary element method				
$6 \times 3 \times 3$	162	26244	482112	0.827
$6 \times 4 \times 4$	288	82944	1.44×10^6	0.626
$6 \times 6 \times 6$	648	419904	6.97×10^6	0.431
$6 \times 9 \times 9$	1458	2.12×10^6	3.48×10^7	0.320
$6 \times 12 \times 12$	2592	6.72×10^6	1.08×10^8	0.279

(Smith 2009)

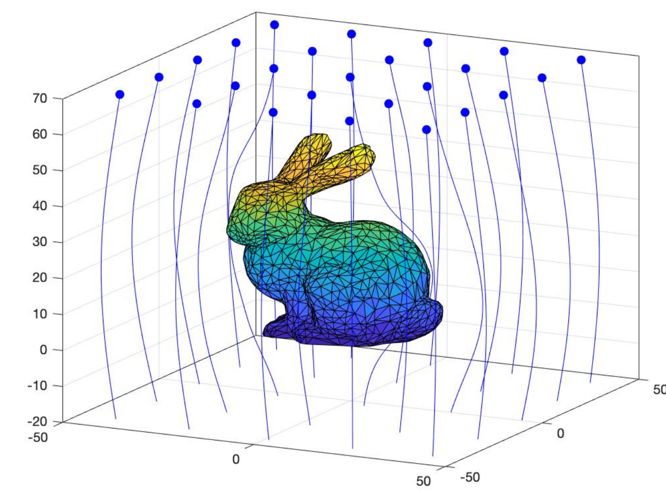
Summary

- We have reviewed and implemented the method of regularized Stokeslets in 2D and 3D for a few cases.

- We can also use triangular meshes and solve for flows for other shapes or geometries.



Triangular mesh for the sphere



Stokes flow past a Stanford bunny

- Next step:

    - we can extend the current approach using non-constant force elements, where the forces are approximated by a set of basis functions.

    - We will investigate other 3D swimming problems in Stokes flow, e.g. a self-propelled cylinder pair, necklace-shaped swimmer etc.

References

- Smith, D. J. 2009 A boundary element regularized Stokeslet method applied to cilia- and flagella-driven flow. *Proc. R. Soc. A* **465**, 3605-3626.

- Cortez, R., Fauci, L. & Medovikov, A. 2005 The method of regularized Stokeslets in three dimensions: Analysis, validation, and application to helical swimming. *Physics of fluids* **17,** 1-14.

- Cortez, R. 2001 The method of regularized Stokeslets. *SIAM J. Sci. Comput*.**23**, 1204-1225.