Dynamically orthogonal (DO) equations for solving the stochastic wave equation

Aaron Charous
5/12/2020

Karhunen–Loève Expansion

$u(x;\omega)$ stochastic field

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$u_i(x)$ modes

$\zeta_i(\omega)$ “coefficients”

$$u(x;\omega) \to \sum_{i=1}^{\infty} u_i(x)\zeta_i(\omega)$$

Karhunen–Loève Expansion

$u(x;\omega)$ stochastic field

$u_i(x)$ modes

$\zeta_i(\omega)$ "coefficients"

n number of modes

$$u(x;\omega) \to \sum_{i=1}^{\infty} u_i(x)\zeta_i(\omega)$$

$$\approx \sum_{i=1}^{n} u_i(x)\zeta_i(\omega)$$

Time dependence

Proper Orthogonal Decomposition

$$u(x, t; \omega) = \sum_{i=1}^{\infty} u_i(x)\zeta_i(t; \omega)$$

Papoulis. Probability, Random Variables and Stochastic Processes. McGraw-Hill, 1965.
J.L. Lumley. Stochastic Tools in Turbulence. Academic-Press, 1971.
R. Ghanem and P. Spanos. Stochastic finite elements: a Spectral Approach. Springer- Verlag, 1991.
Sapsis, T.P. and P.F.J. Lermusiaux. Dynamically orthogonal field equations for continuous stochastic dynamical systems. Physica D, 2009.

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Polynomial Chaos

$$u(x,t;\omega) = \sum_{i=1}^{\infty} u_i(x,t)\Phi_i(\eta(\omega))$$

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Dynamically Orthogonal Equations

$$u(x,t;\omega) = \sum_{i=1}^{\infty} u_i(x,t)\zeta_i(t;\omega) \qquad \left(u_i(\bullet,t), \frac{\partial u_j(\bullet,t)}{\partial t} \right) = 0 \ \forall i,j$$

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Uncertainty Evolution

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$$\frac{\partial u}{\partial t} = \mathscr{L}u$$

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DO vs MC

Dynamically Orthogonal Equations

Cost: $nG(N_x) + nH(N_\omega)$

Monte Carlo

Cost: $N_\omega G(N_x)$

$G(N_x)$ cost of solving PDE of size N_x

$H(N_\omega)$ cost of solving N_ω ODEs

DO vs MC

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Cost: $N_\omega G(N_x)$

$G(N_x)$ cost of solving PDE of size N_x

$H(N_\omega)$ cost of solving N_ω ODEs

Typically $G(N_x) \gg H(N_\omega)$

DO vs MC

Dynamically Orthogonal Equations

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DO vs MC

Dynamically Orthogonal Equations

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Monte Carlo

Cost: $N_\omega G(N_x)$

Cost Ratio: $O\left(\frac{n}{N_\omega}\right)$

Stochastic Acoustic Wave Equation

$$c^2 \rho \nabla \cdot \left(\frac{1}{\rho} \nabla p \right) - \alpha p_t + f = p_{tt}$$

Jensen, Finn B., et al. *Computational ocean acoustics*. Springer Science & Business Media, 2011.

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$$\phi=\log\rho\Rightarrow\rho\nabla\cdot\left(\frac{1}{\rho}\nabla p\right)=\nabla^2 p-\nabla\phi^T\nabla p$$

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$$\Psi = \begin{pmatrix} p \\ p_t \end{pmatrix}$$

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$$\Psi_t = \begin{pmatrix} 0 & 1 \\ c^2(\nabla^2 - \nabla \phi^T \nabla) & -\alpha \end{pmatrix} \Psi + \begin{pmatrix} 0 \\ f \end{pmatrix}$$

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$$u(x,t;\omega) = \sum_{i=1}^{\infty} u_i(x,t)\zeta_i(t;\omega)$$

$$\left(u_i(\bullet,t), \frac{\partial u_j(\bullet,t)}{\partial t}\right) = 0 \ \forall i,j$$

2-Mode Separable Example

$$p(x,0;\omega) = A(\omega)h\left(x+\frac{1}{6}\right) + B(\omega)h\left(x-\frac{1}{6}\right)$$

$$p_t(x,0;\omega) = cA(\omega)h'\left(x+\frac{1}{6}\right) - cB(\omega)h'\left(x-\frac{1}{6}\right)$$

$$A \sim \mathscr{U}\left(\frac{1}{2},\frac{3}{2}\right)$$

$$B \sim \text{Exp}\left(\frac{1}{2}\right)$$

h Nuttall window

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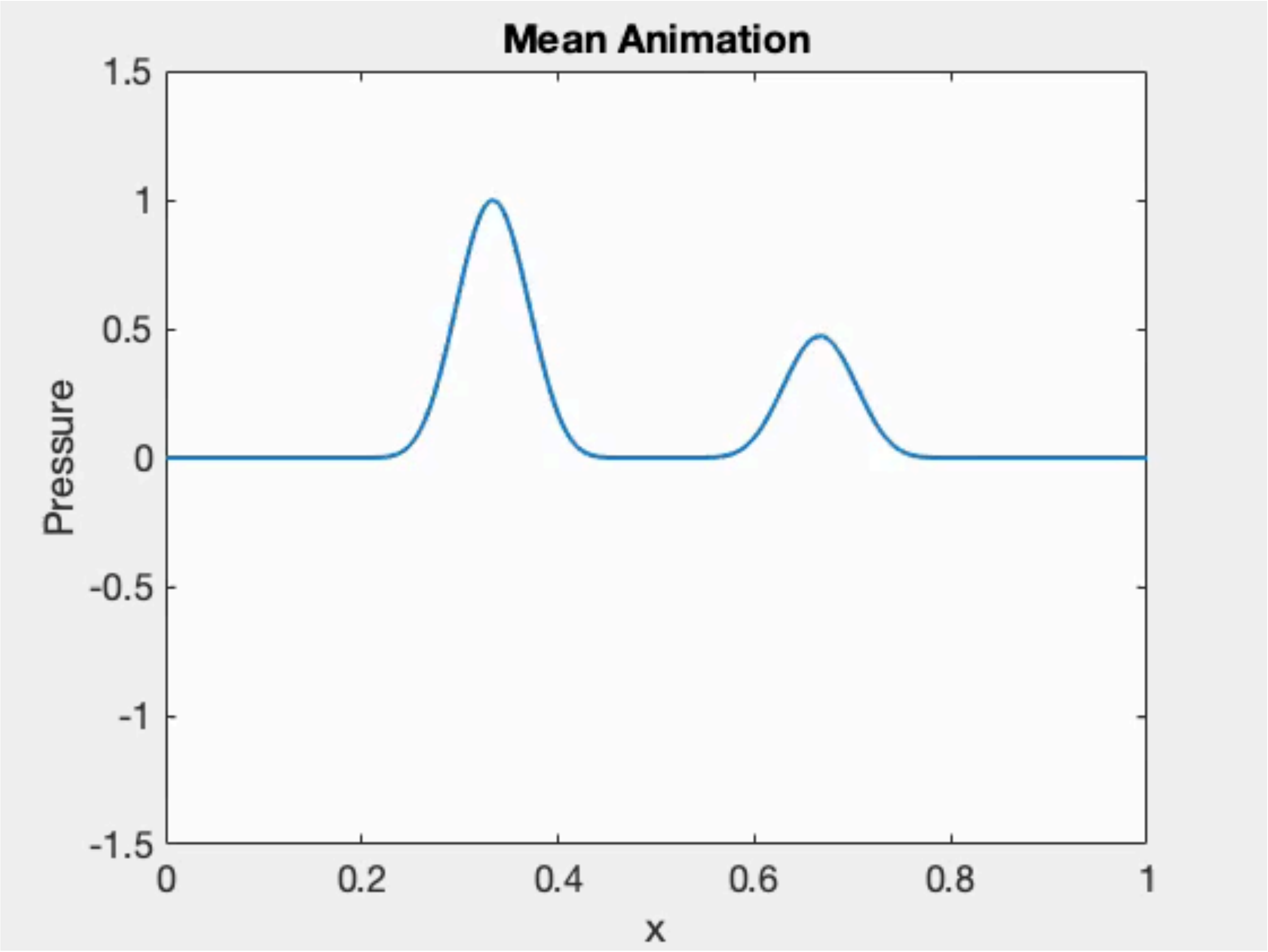
h Nuttall window



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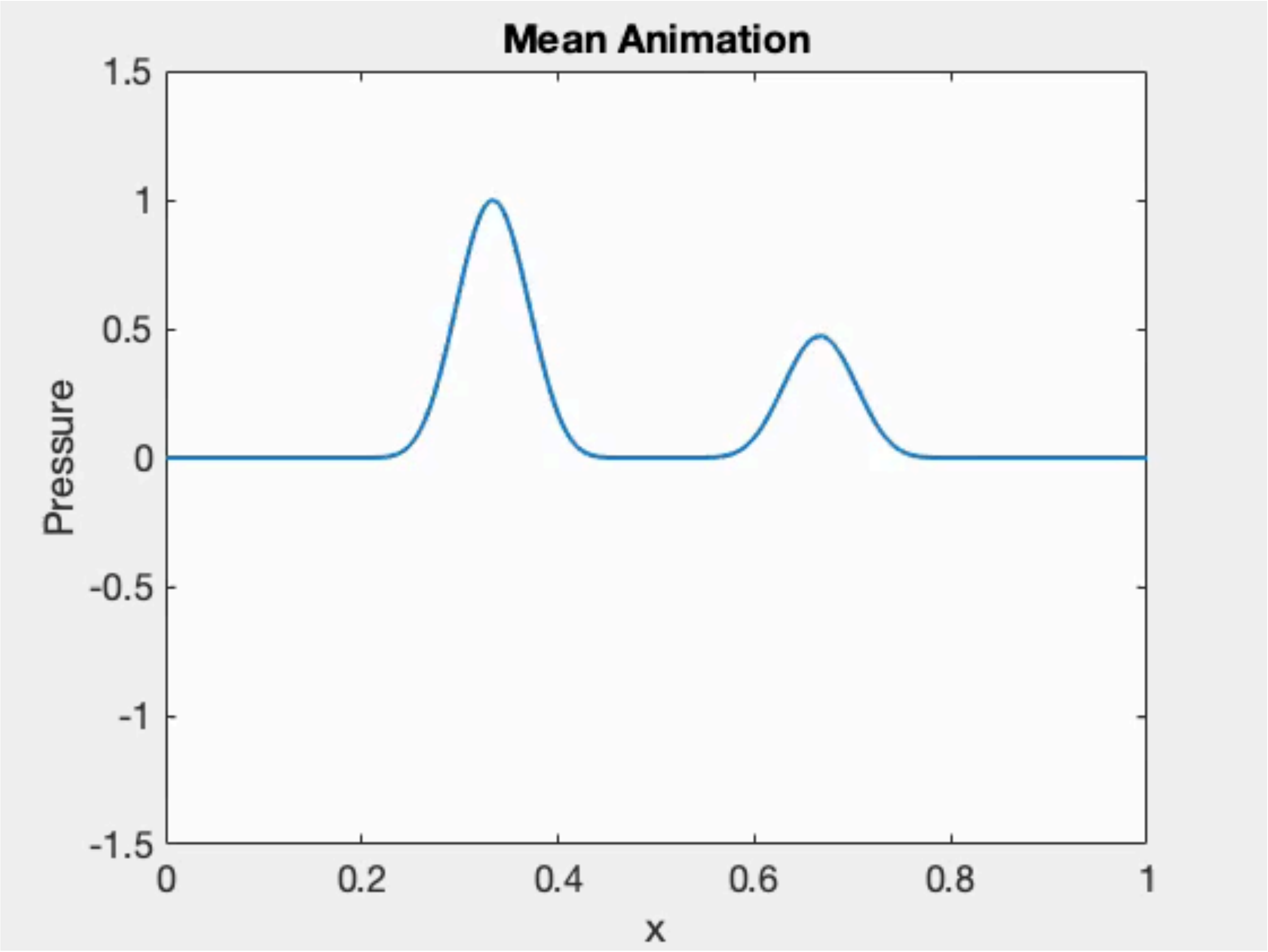

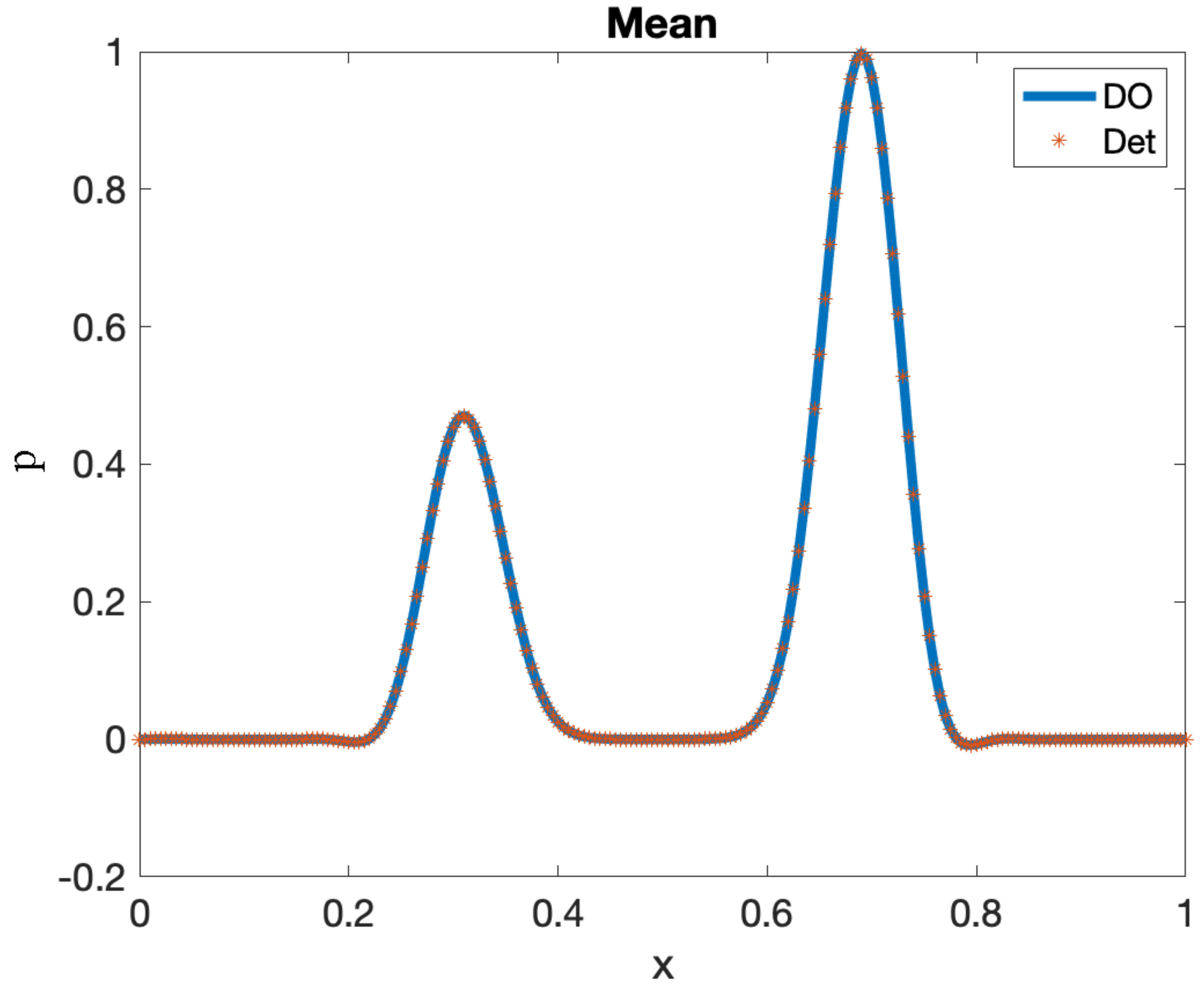

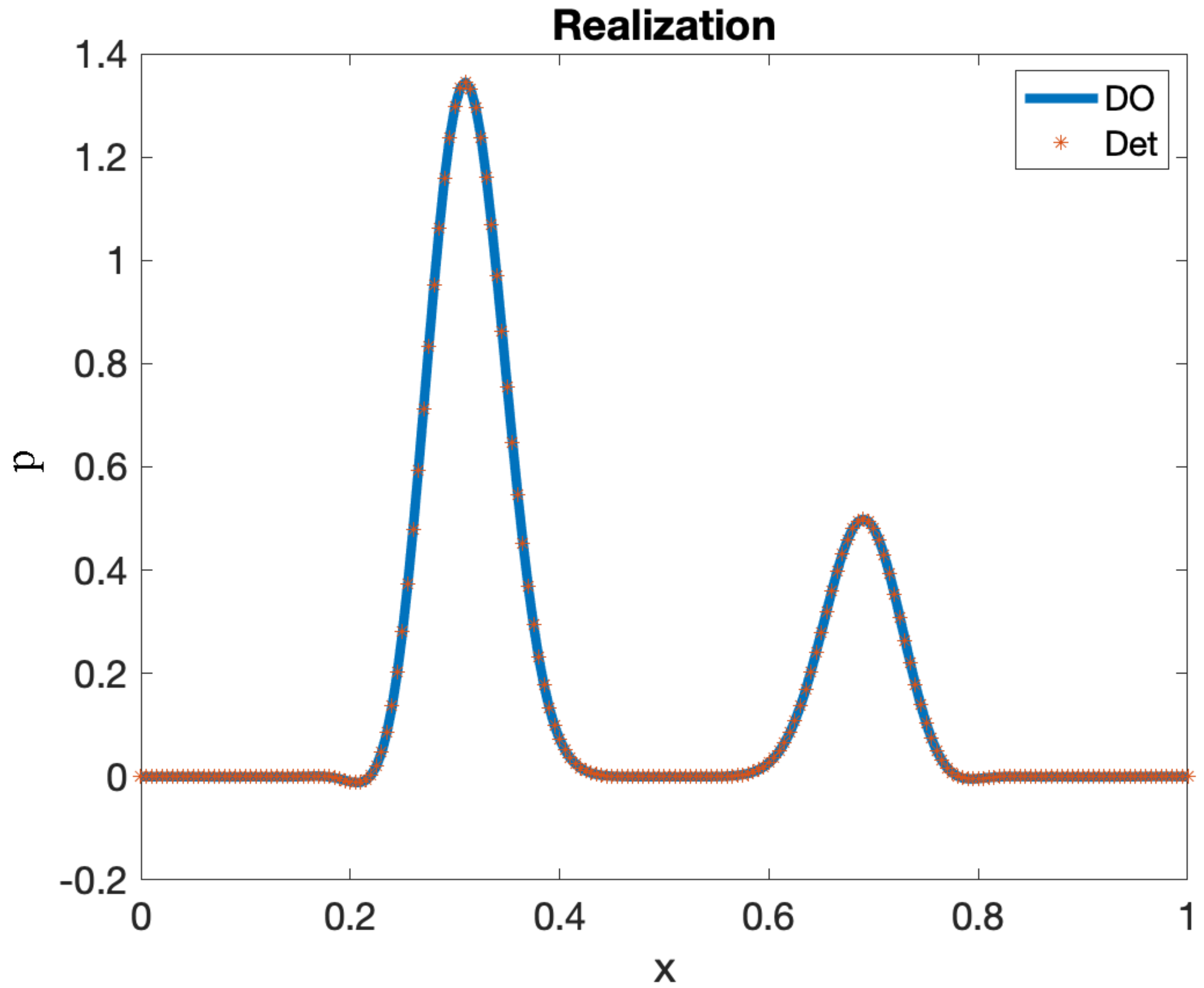

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$$A \sim \mathcal{U}\left(\frac{1}{2},\frac{3}{2}\right)$$

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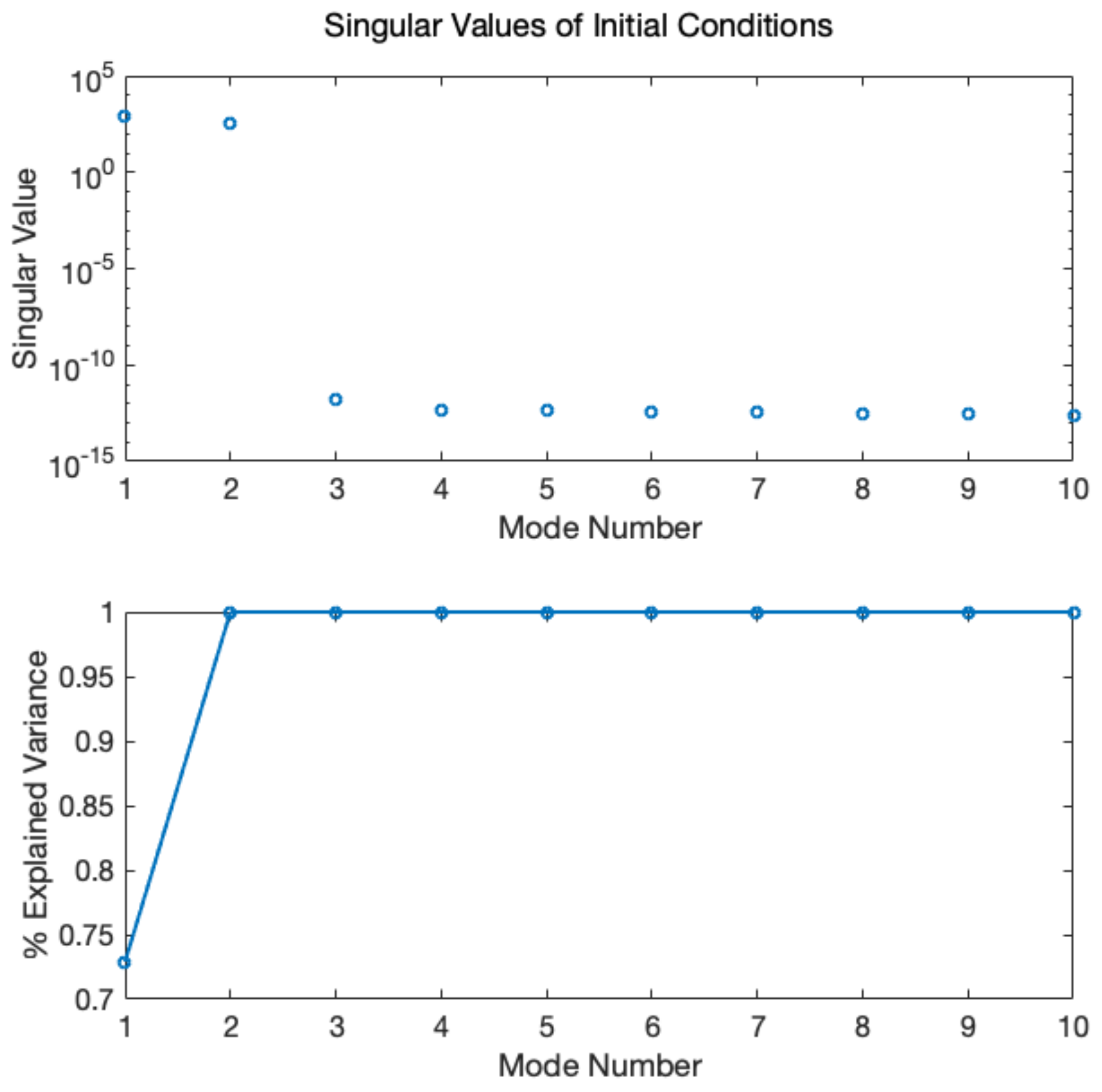

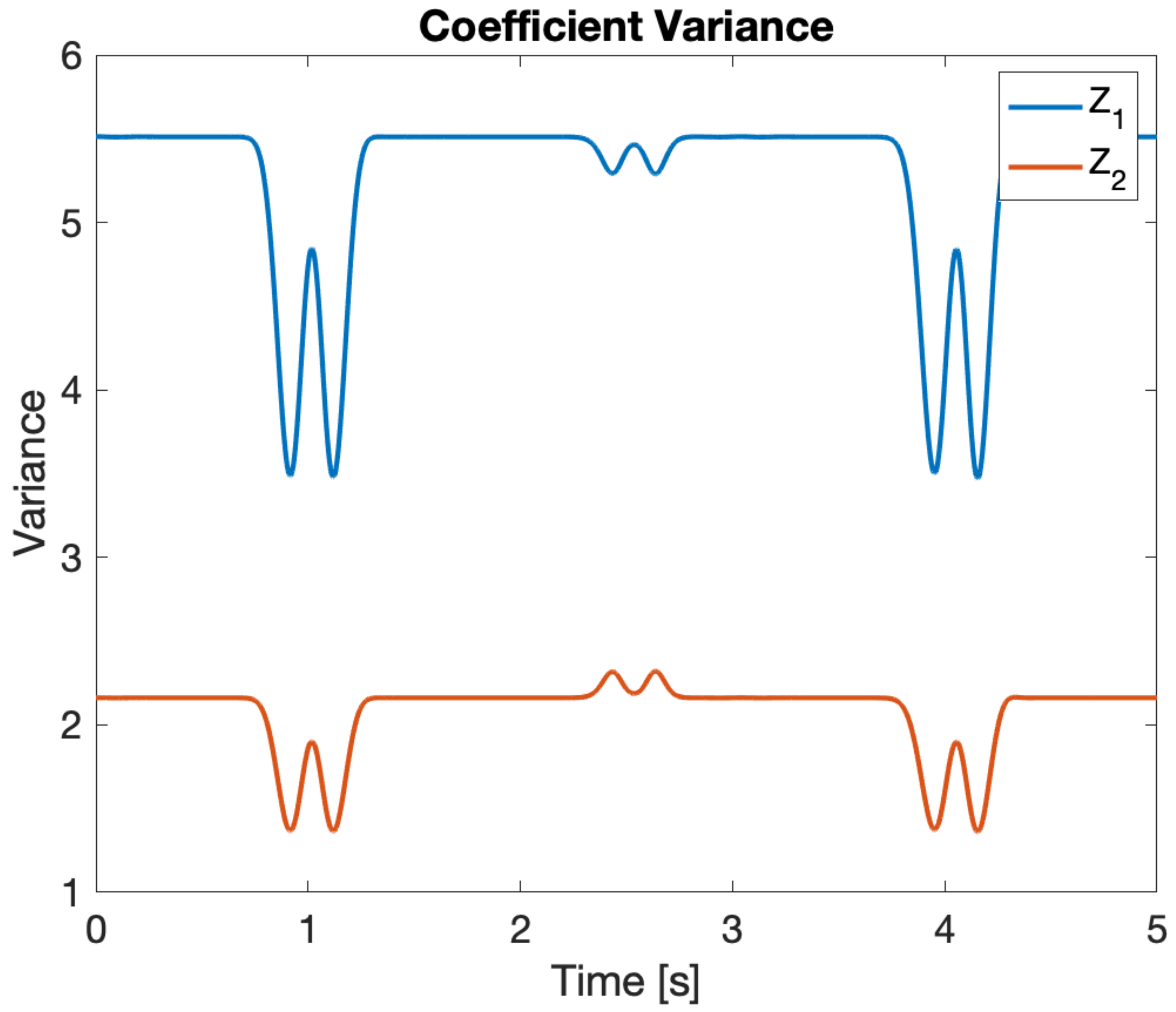

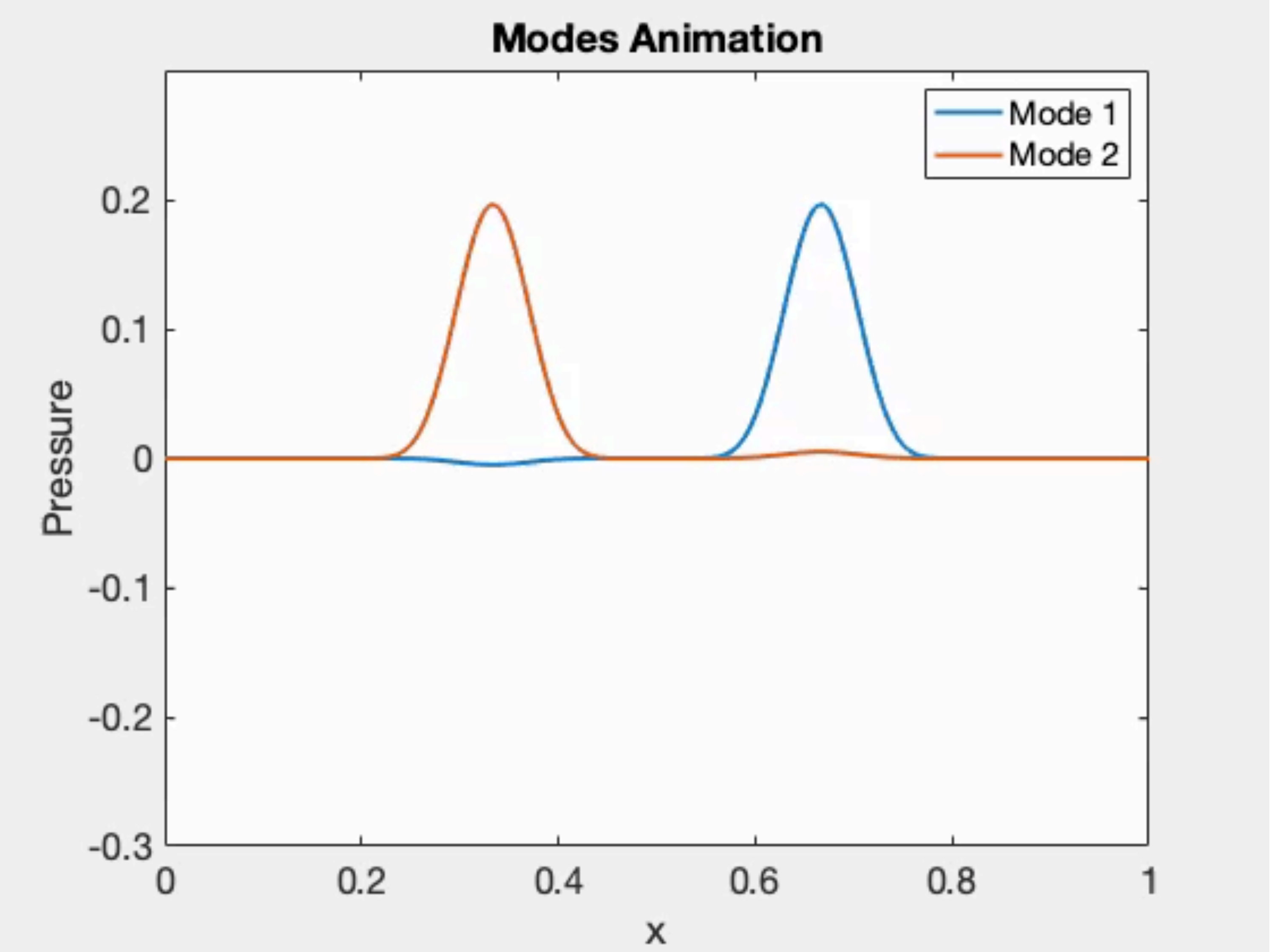

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$$p(x,0;\omega) = A(\omega)h\left(x+\frac{1}{6}\right) + B(\omega)h\left(x-\frac{1}{6}\right)$$

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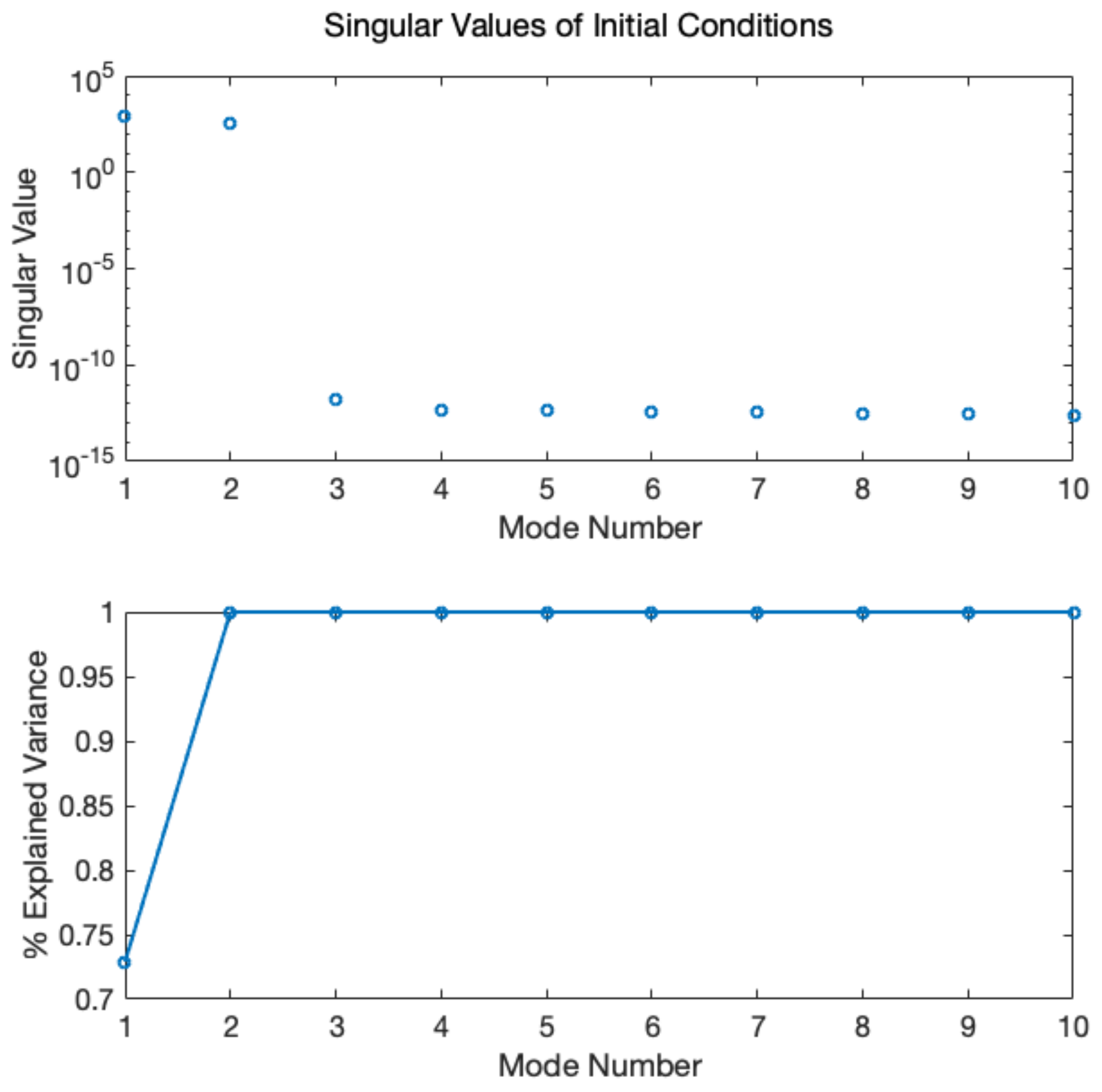

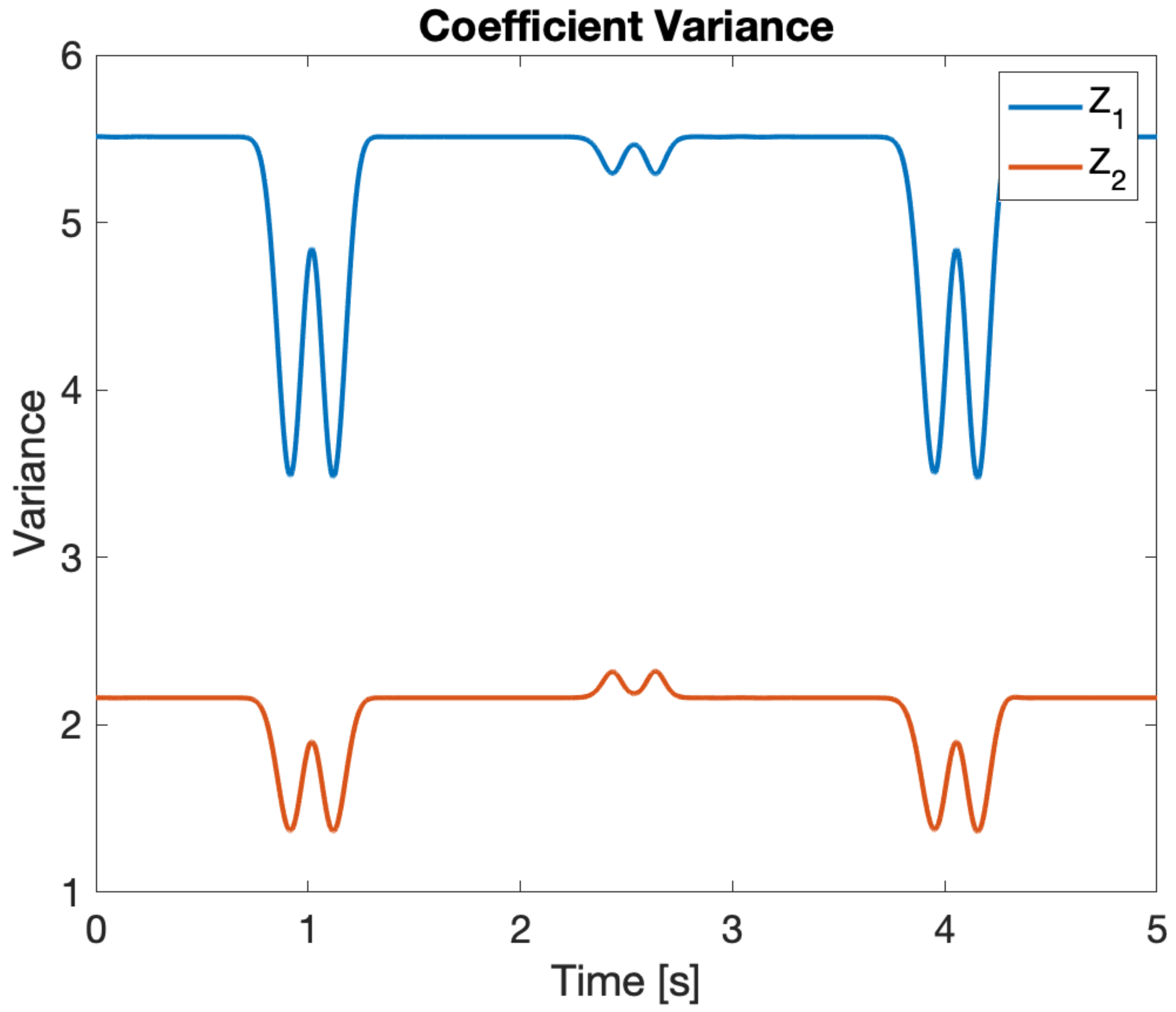

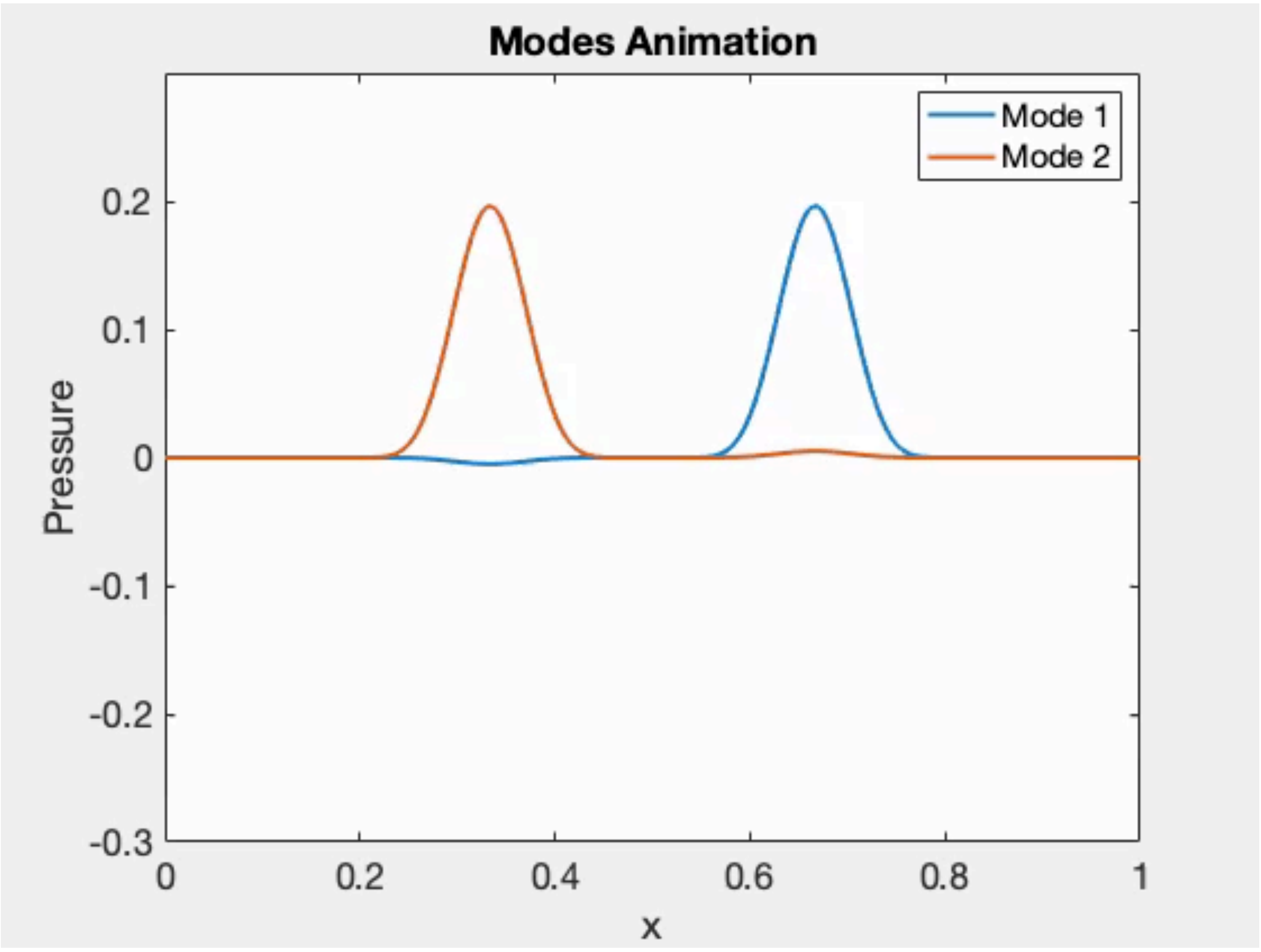

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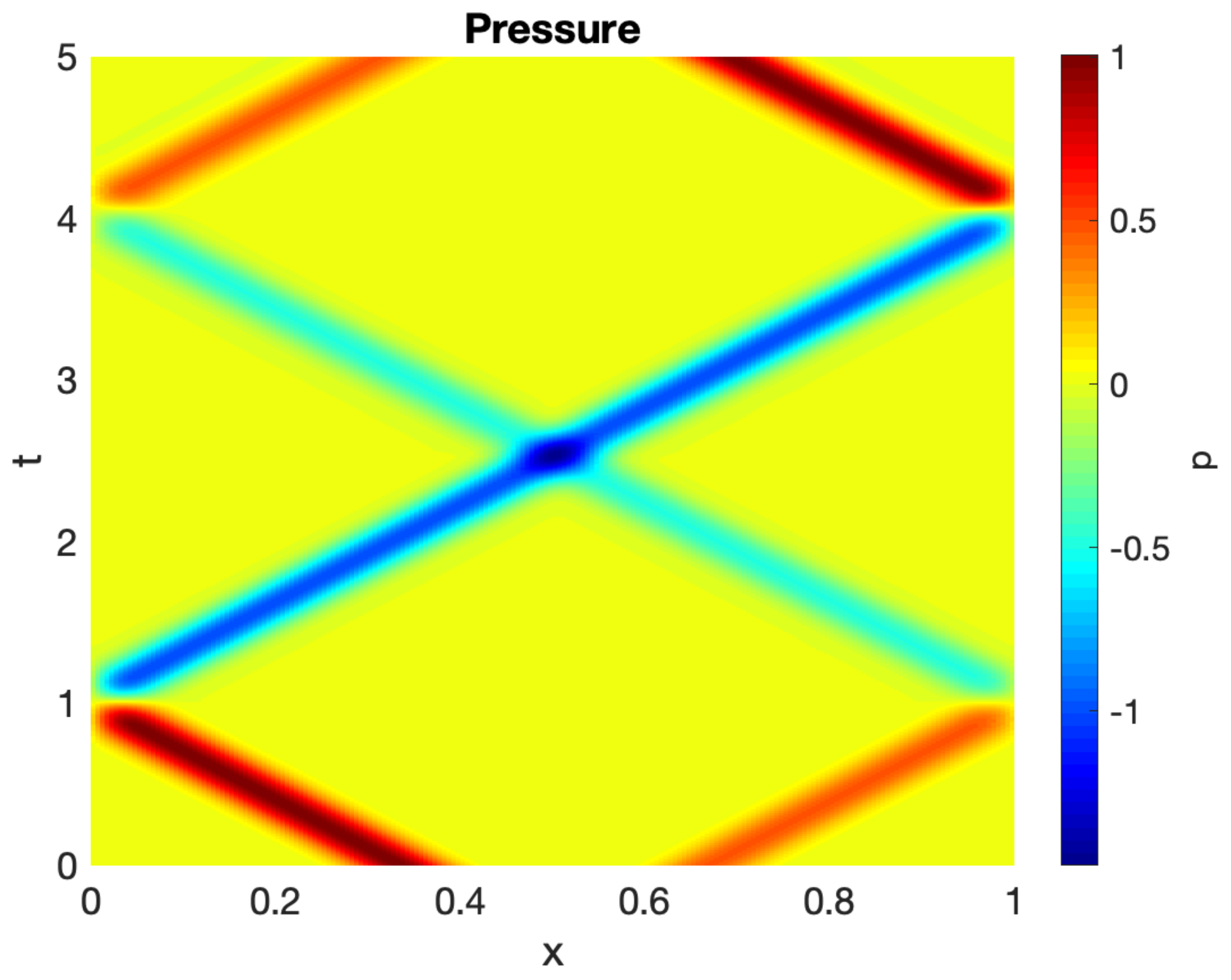

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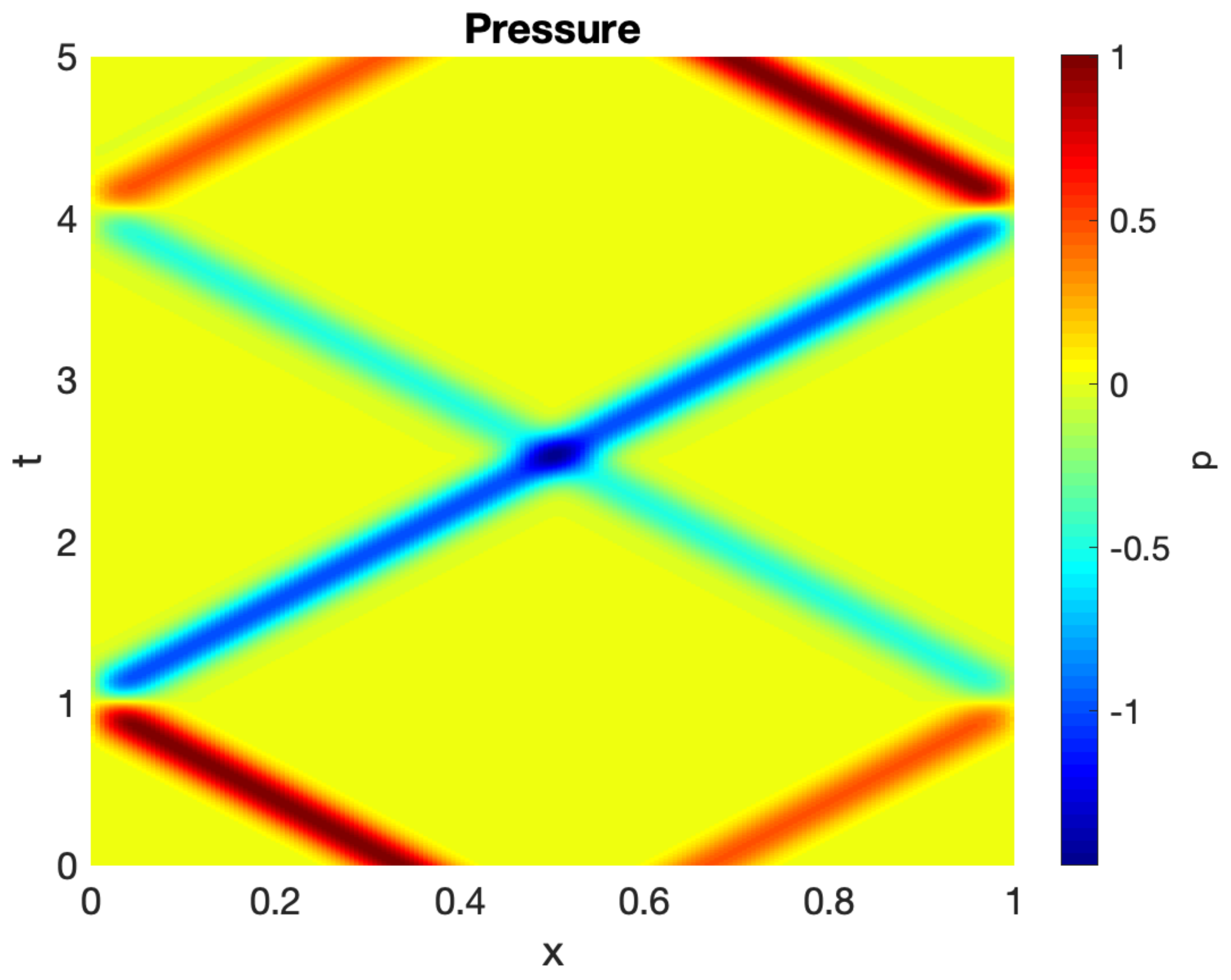

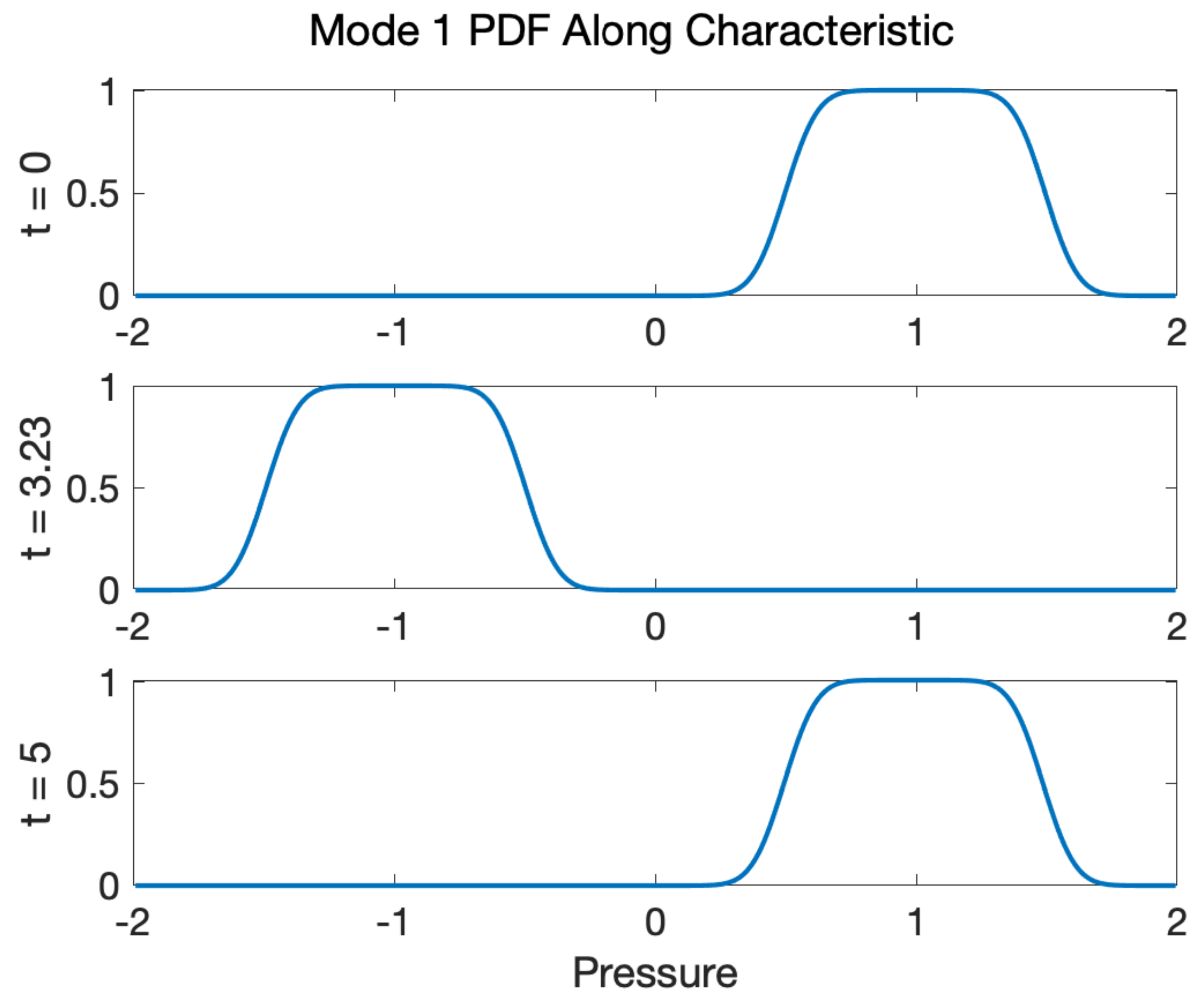

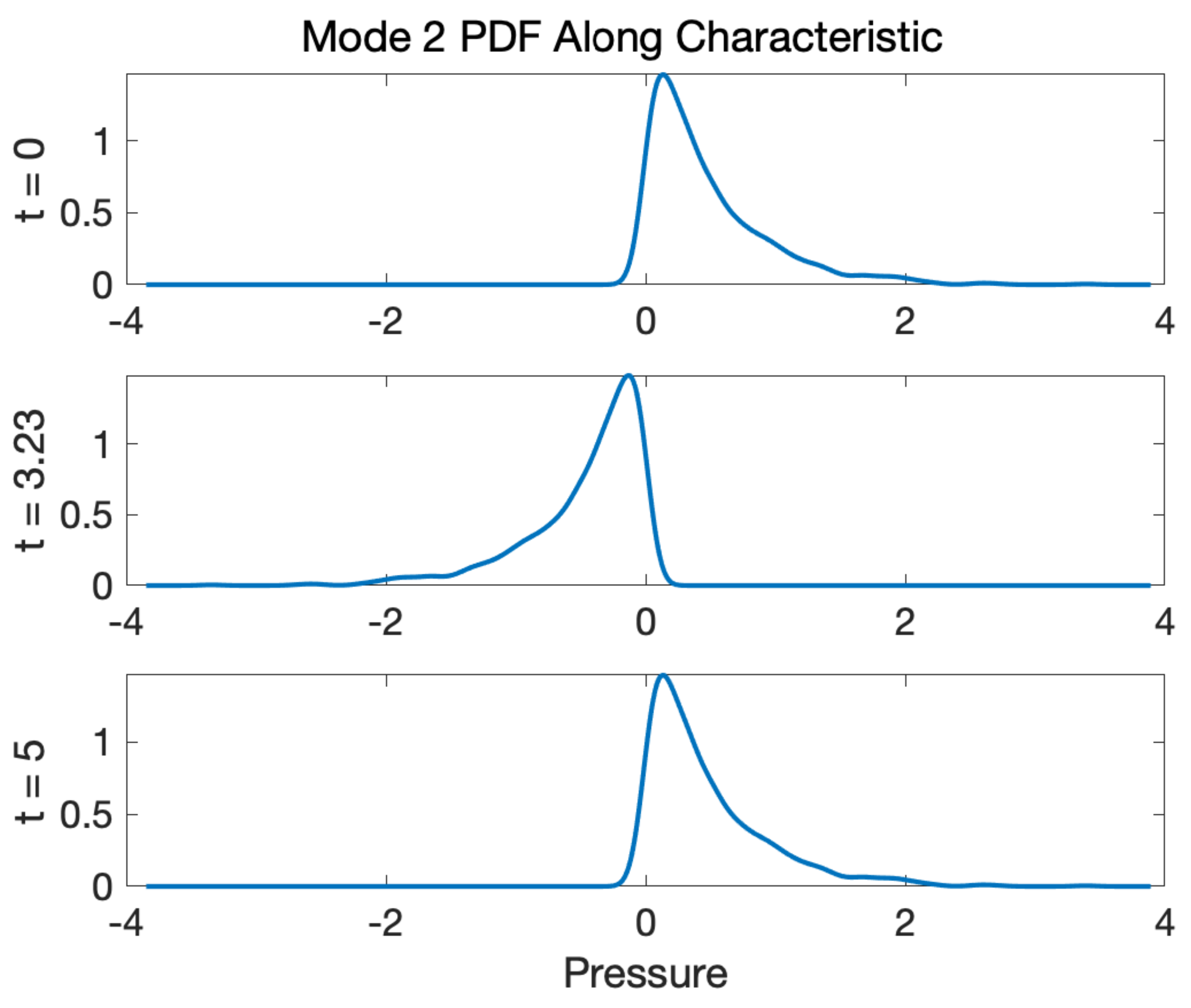

3-Mode Separable Example

$$p(x,0;\omega) = A(\omega)h\left(x+\frac{1}{6}\right) + B(\omega)h\left(x-\frac{1}{6}\right) + C(\omega)h\left(x+\frac{1}{24}\right)$$

$$p_t(x,0;\omega) = cA(\omega)h'\left(x+\frac{1}{6}\right) - cB(\omega)h'\left(x-\frac{1}{6}\right) - cC(\omega)h'\left(x+\frac{1}{24}\right)$$

h Nuttall window

$$A \sim \mathscr{U}\left(\frac{1}{2},\frac{3}{2}\right)$$

$$B \sim \text{Exp}\left(\frac{1}{2}\right)$$

$$C = A + (B - \mathbb{E}B)^2$$

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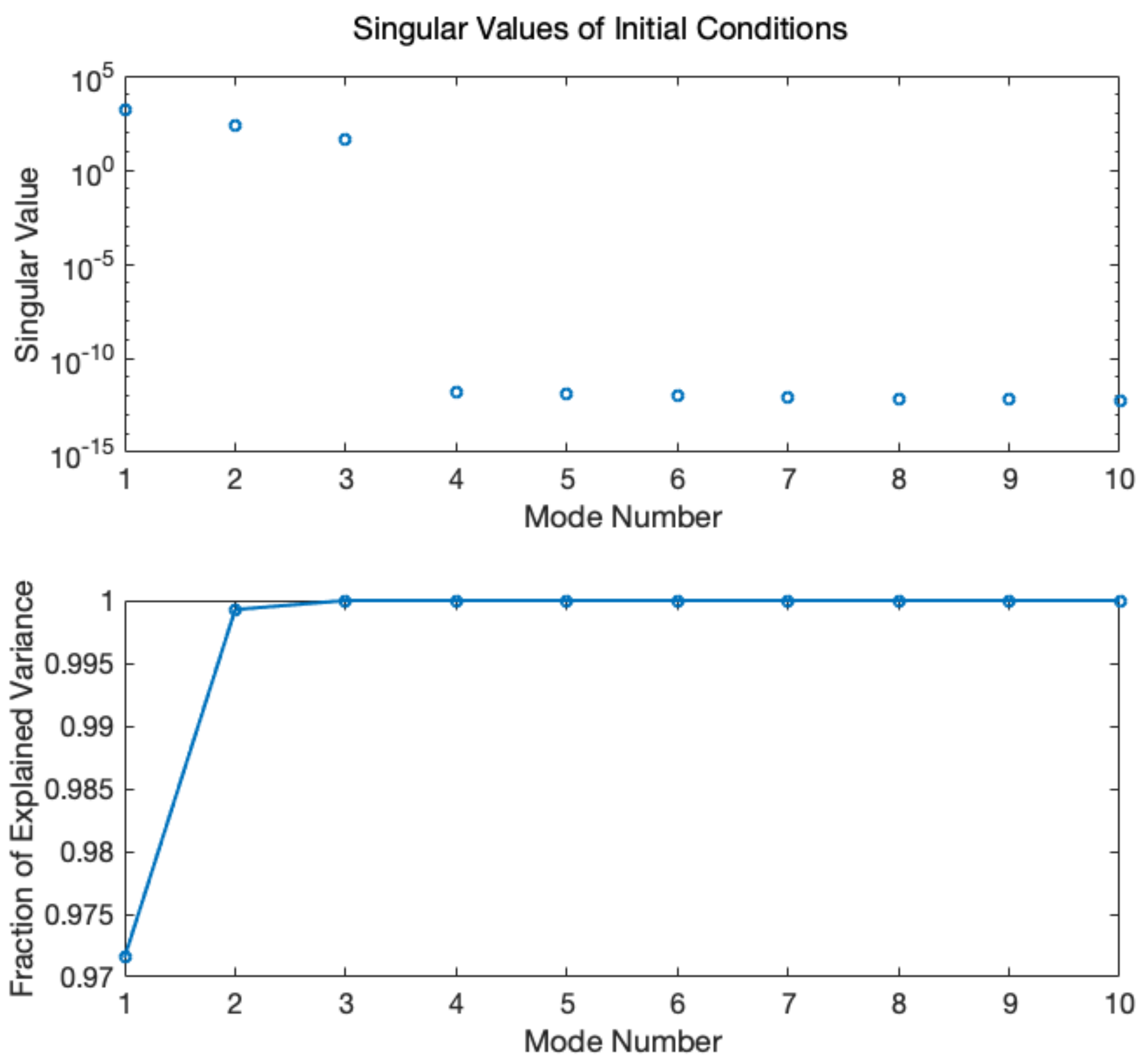

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$$p(x,0;\omega) = A(\omega)h\left(x+\frac{1}{6}\right) + B(\omega)h\left(x-\frac{1}{6}\right) + C(\omega)h\left(x+\frac{1}{24}\right) \qquad A \sim \mathscr{U}\left(\frac{1}{2},\frac{3}{2}\right)$$

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h Nuttall window

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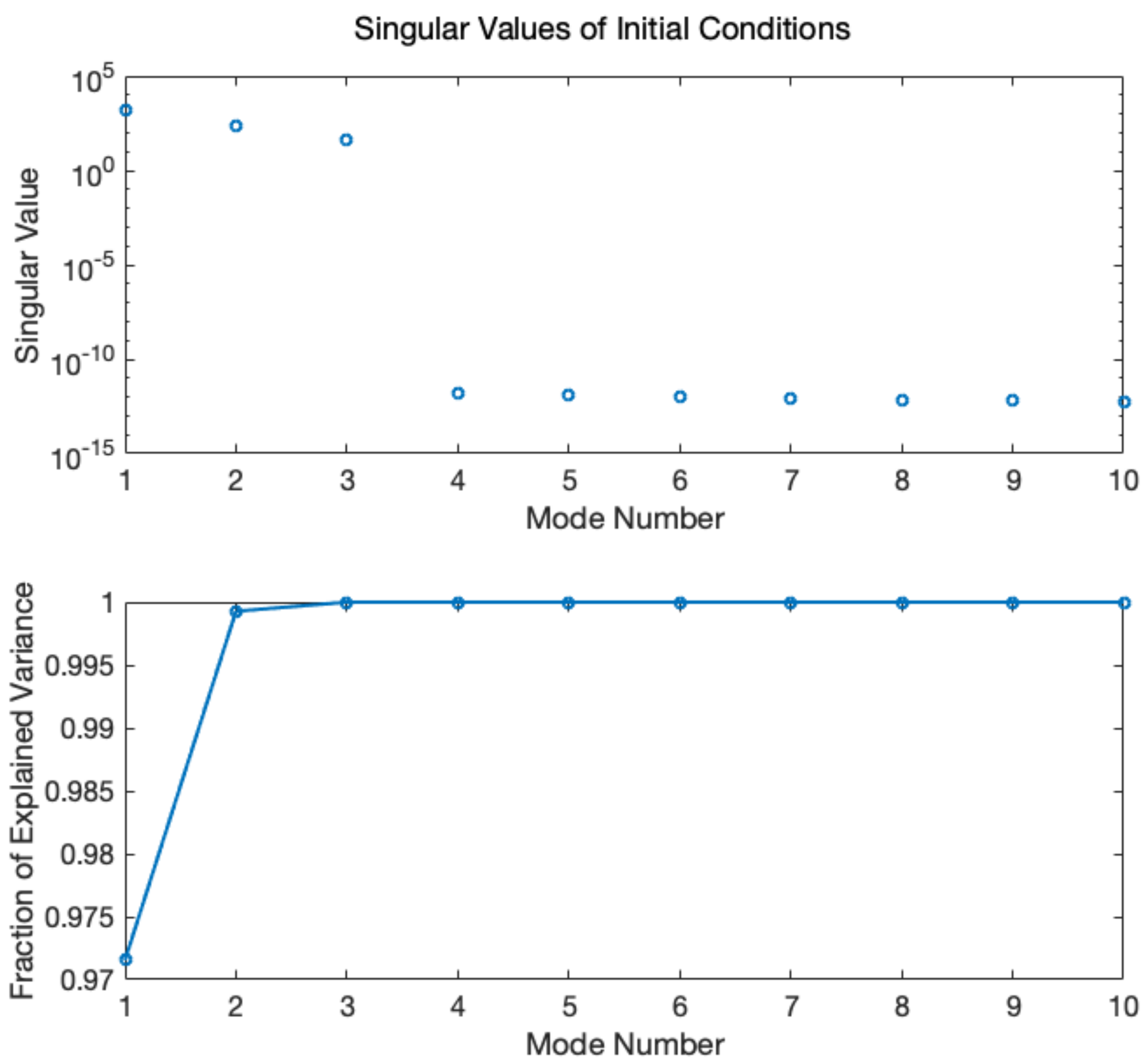

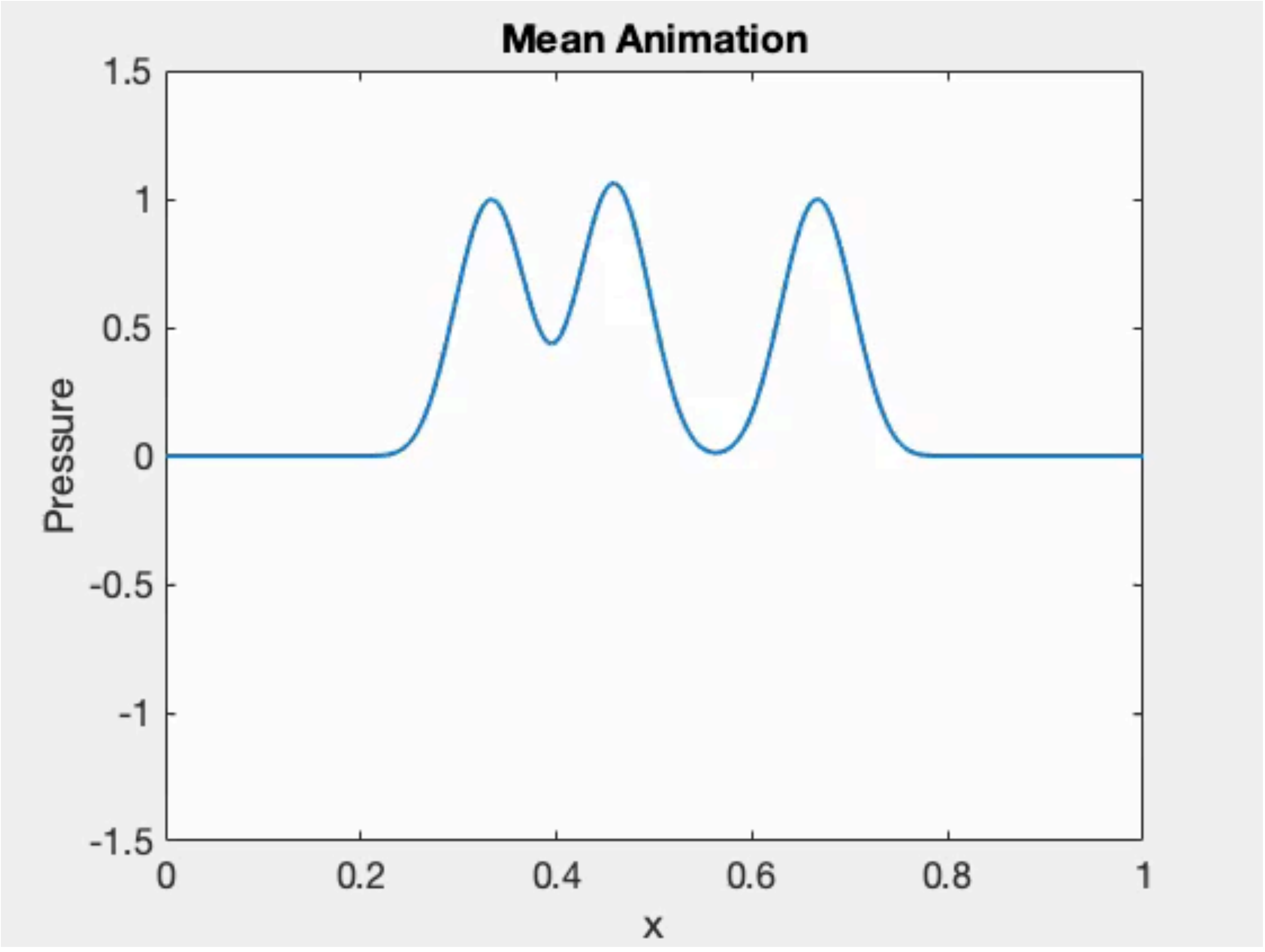

3-Mode Separable Example

$$p(x,0;\omega) = A(\omega)h\left(x+\frac{1}{6}\right) + B(\omega)h\left(x-\frac{1}{6}\right) + C(\omega)h\left(x+\frac{1}{24}\right) \qquad A \sim \mathscr{U}\left(\frac{1}{2},\frac{3}{2}\right)$$

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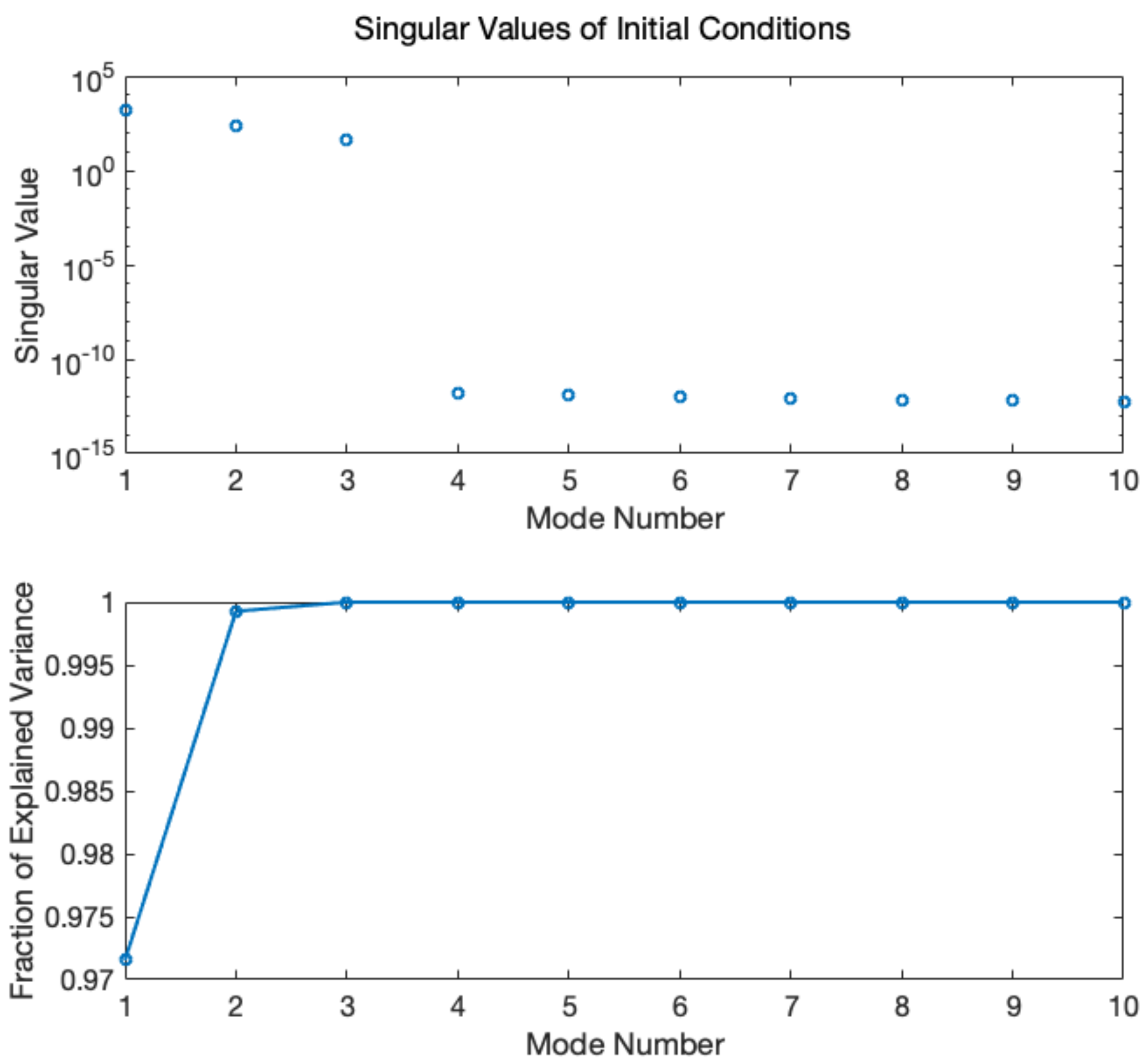
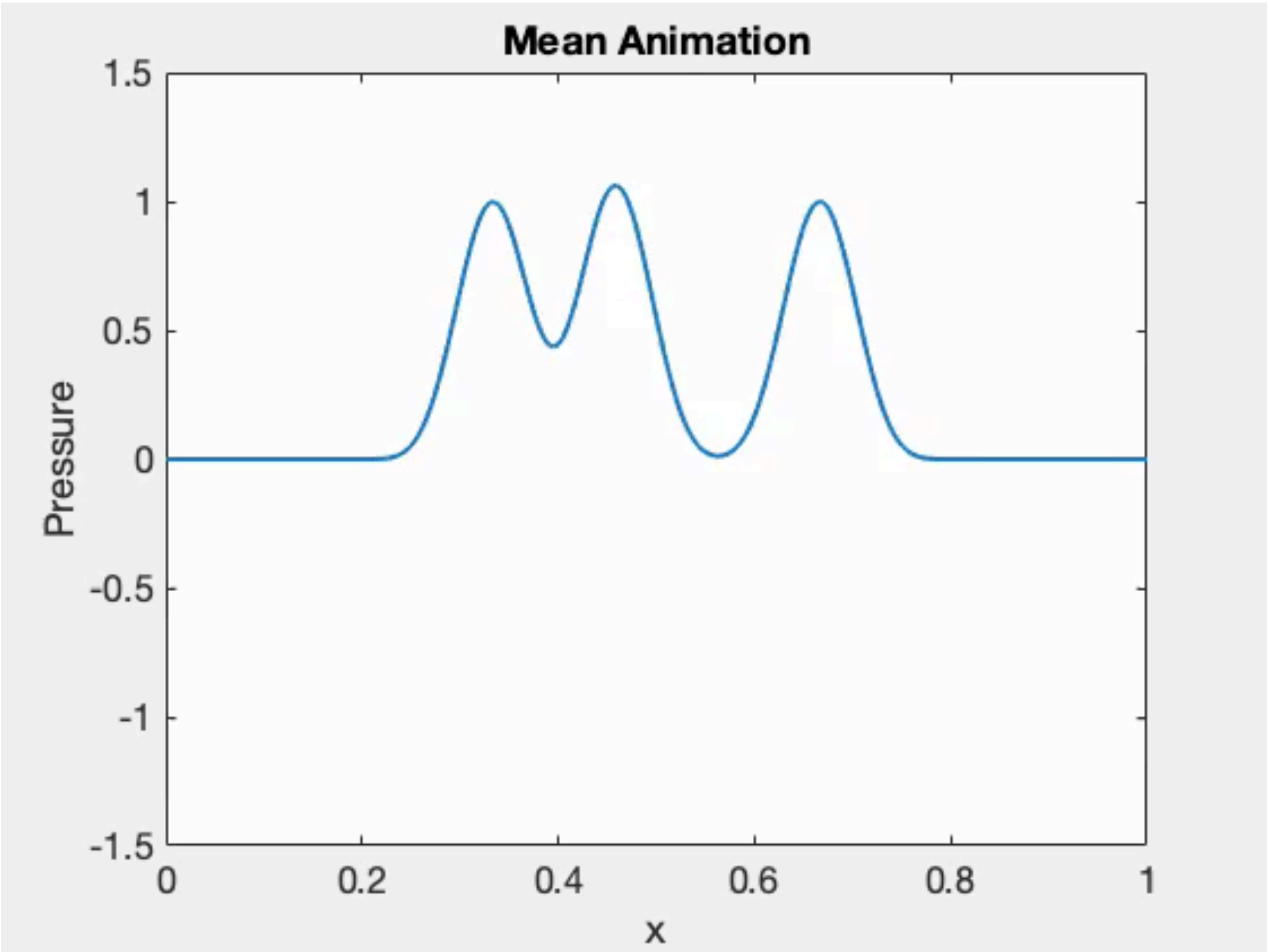
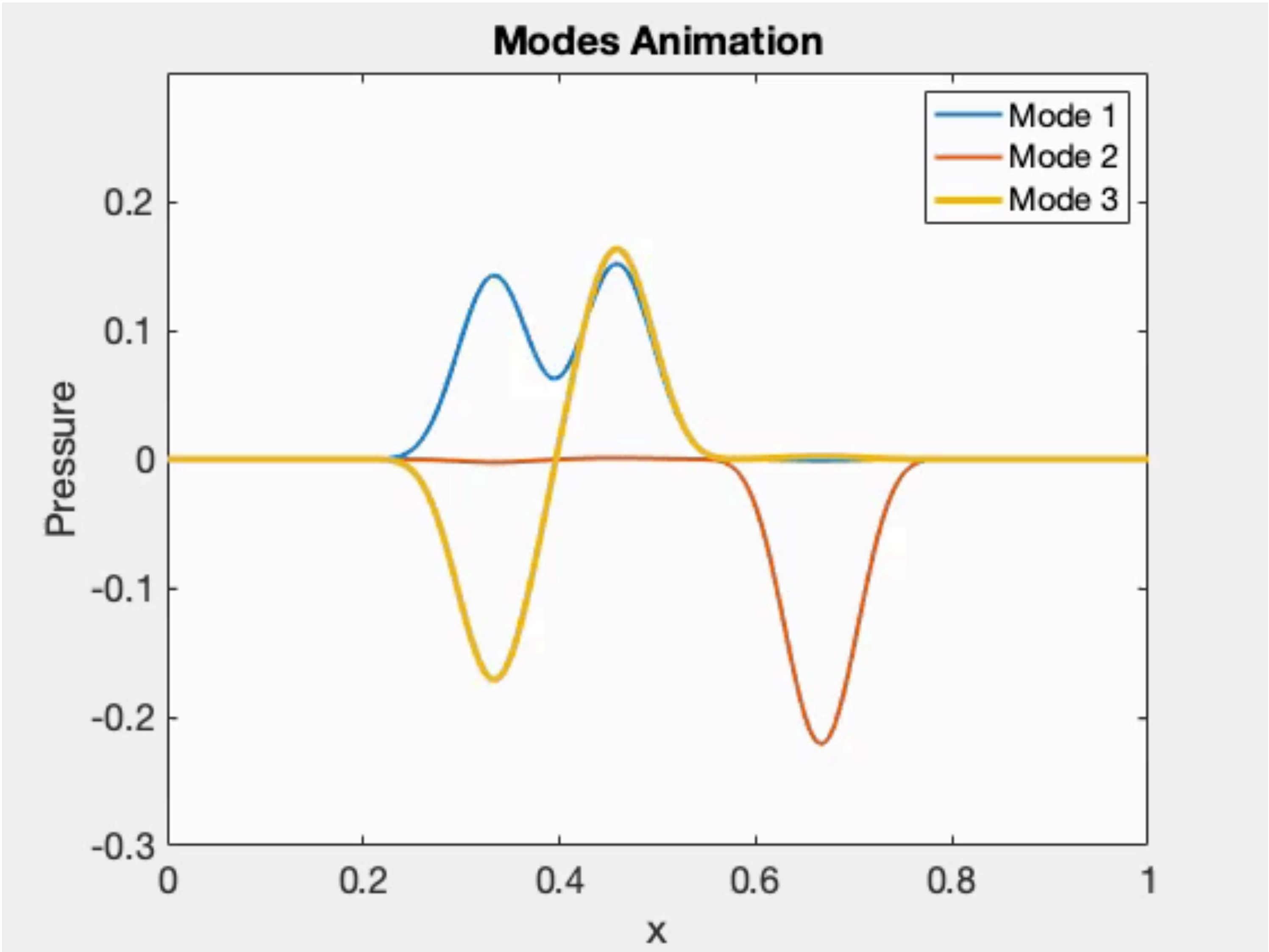
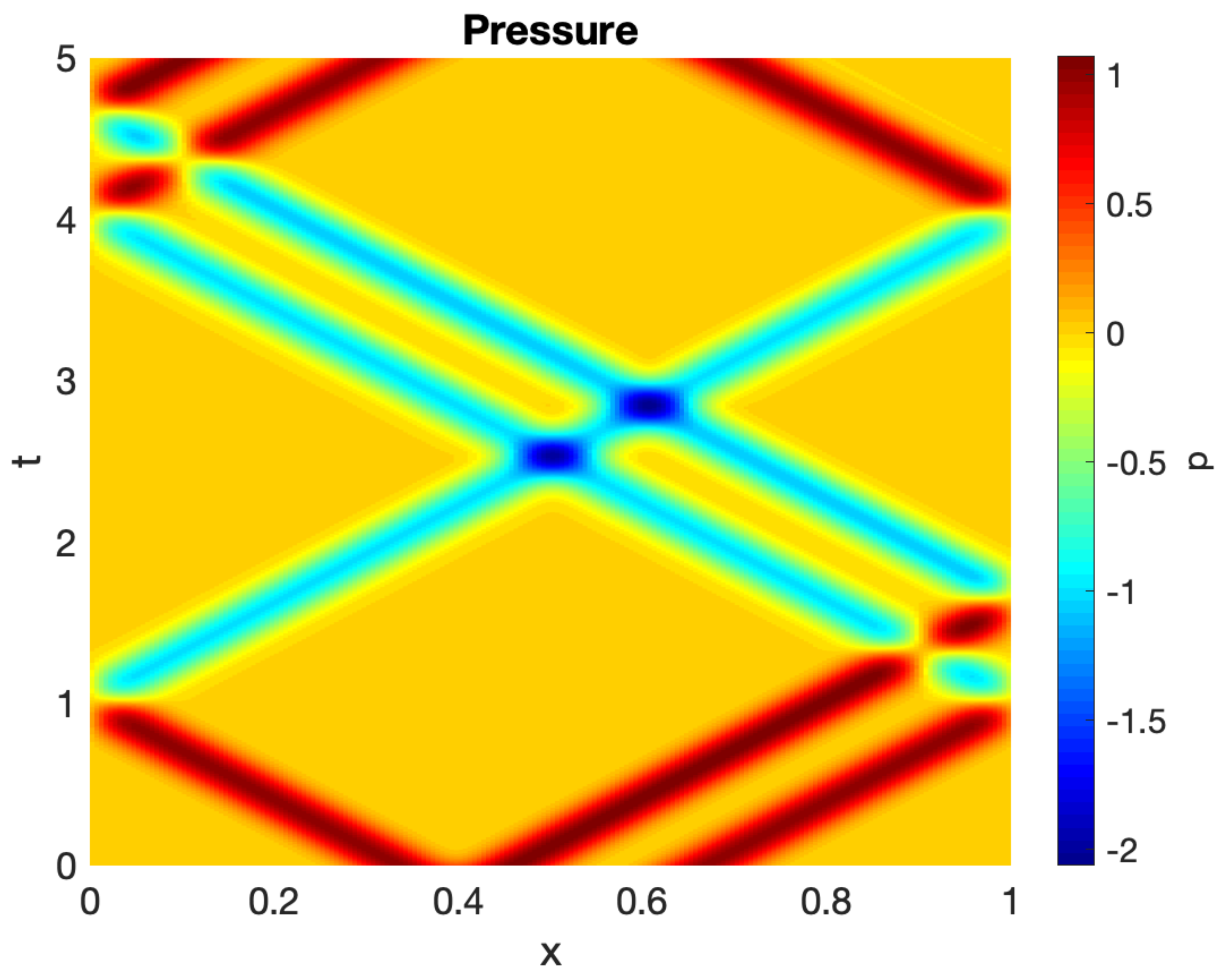
h Nuttall window $\qquad C = A + (B - \mathbf{E}B)^2$







3-Mode Separable Example



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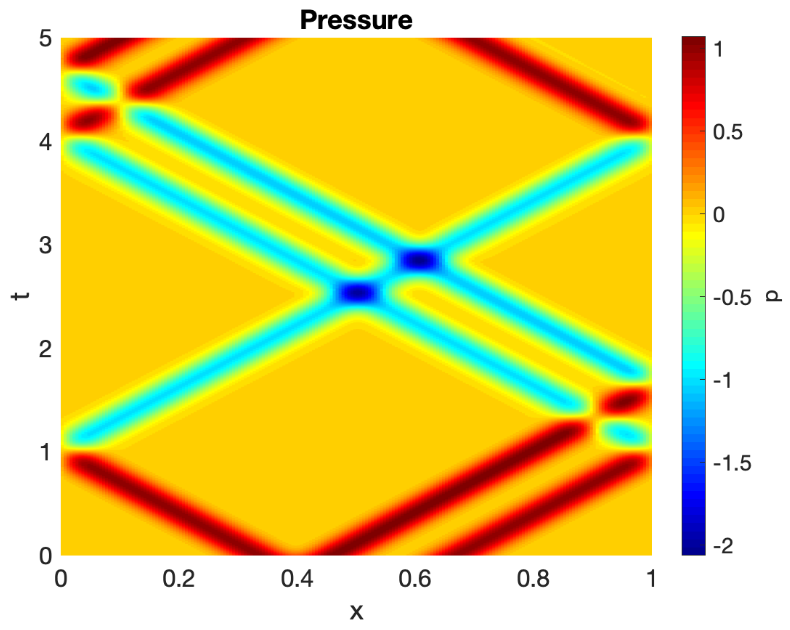

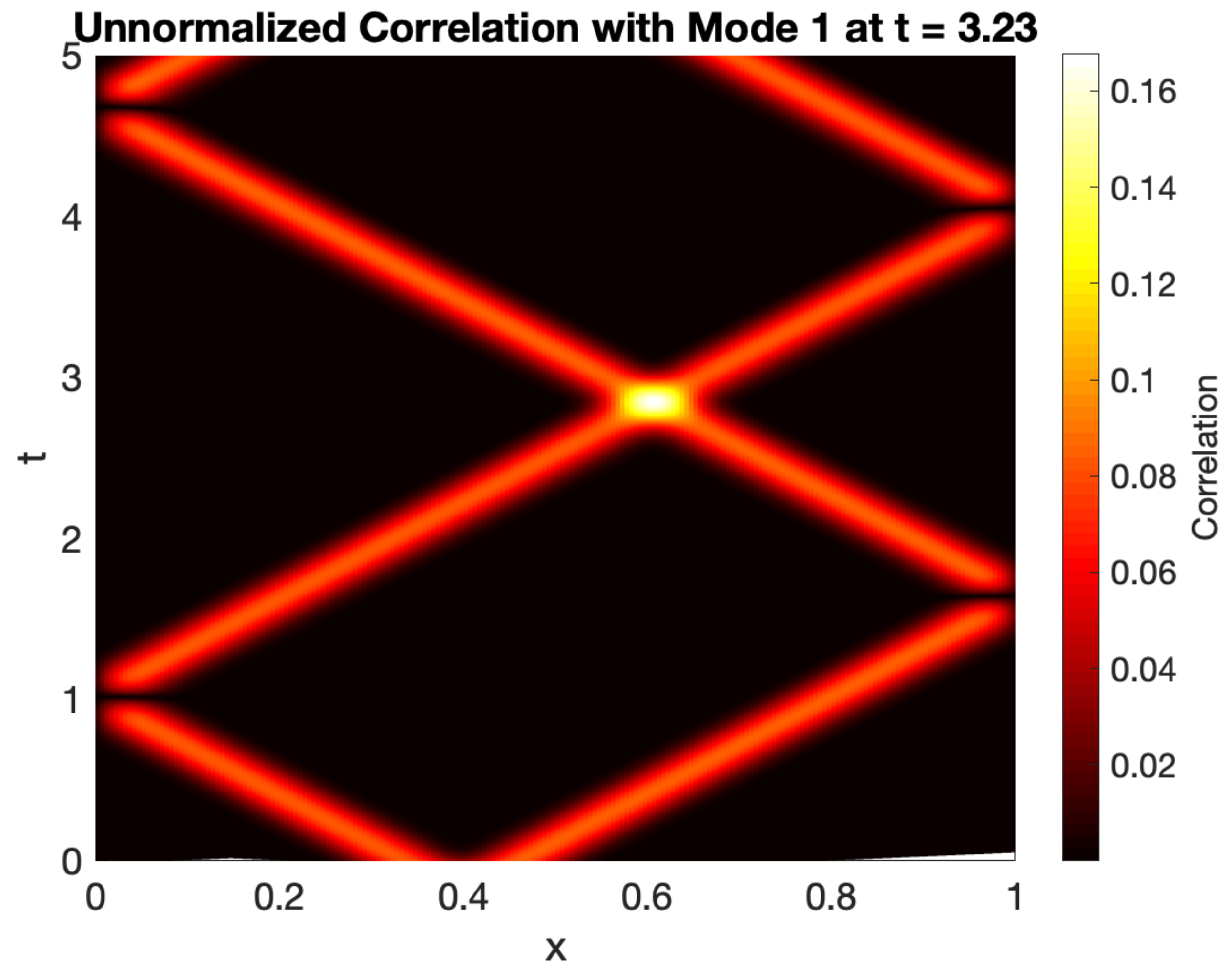

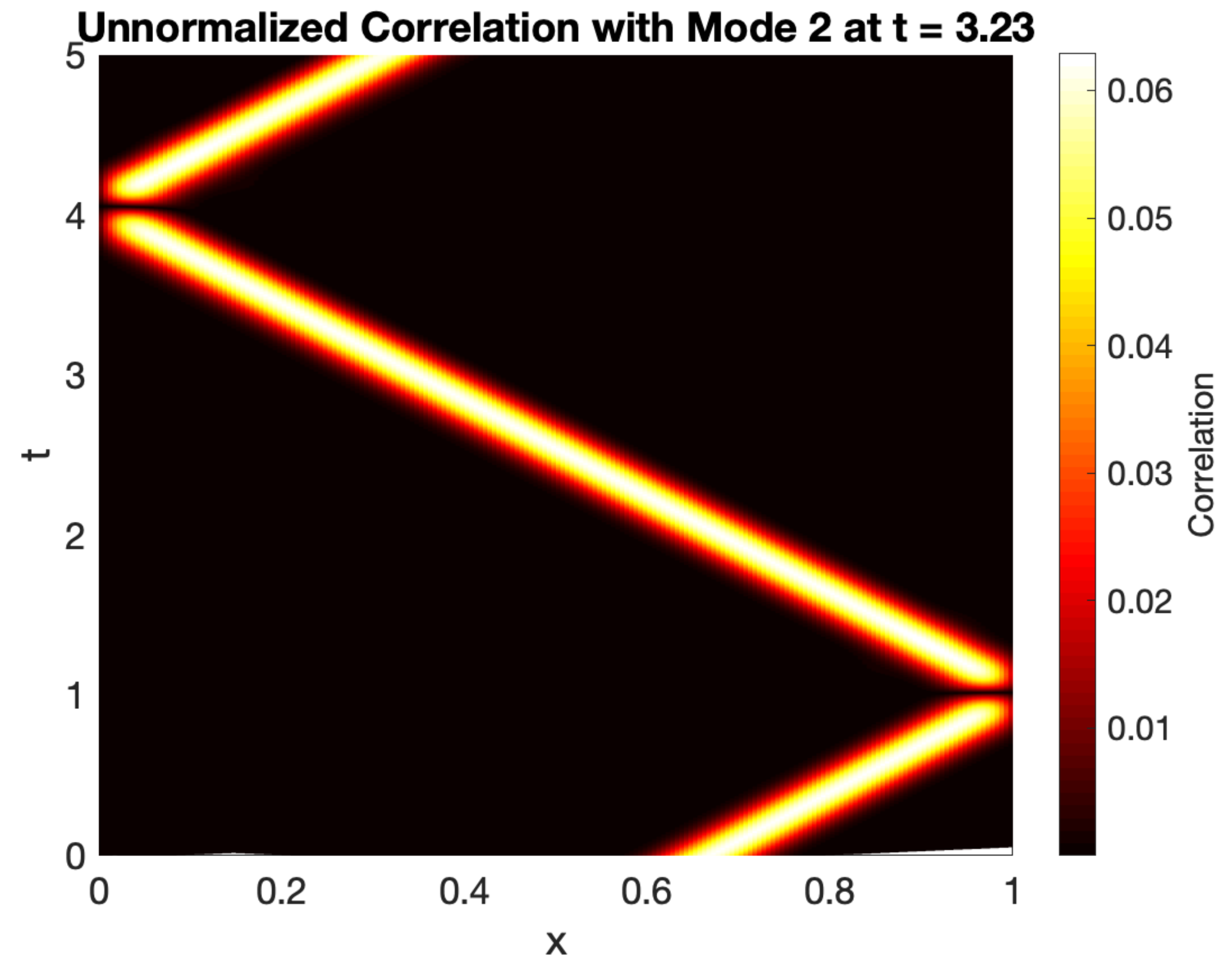

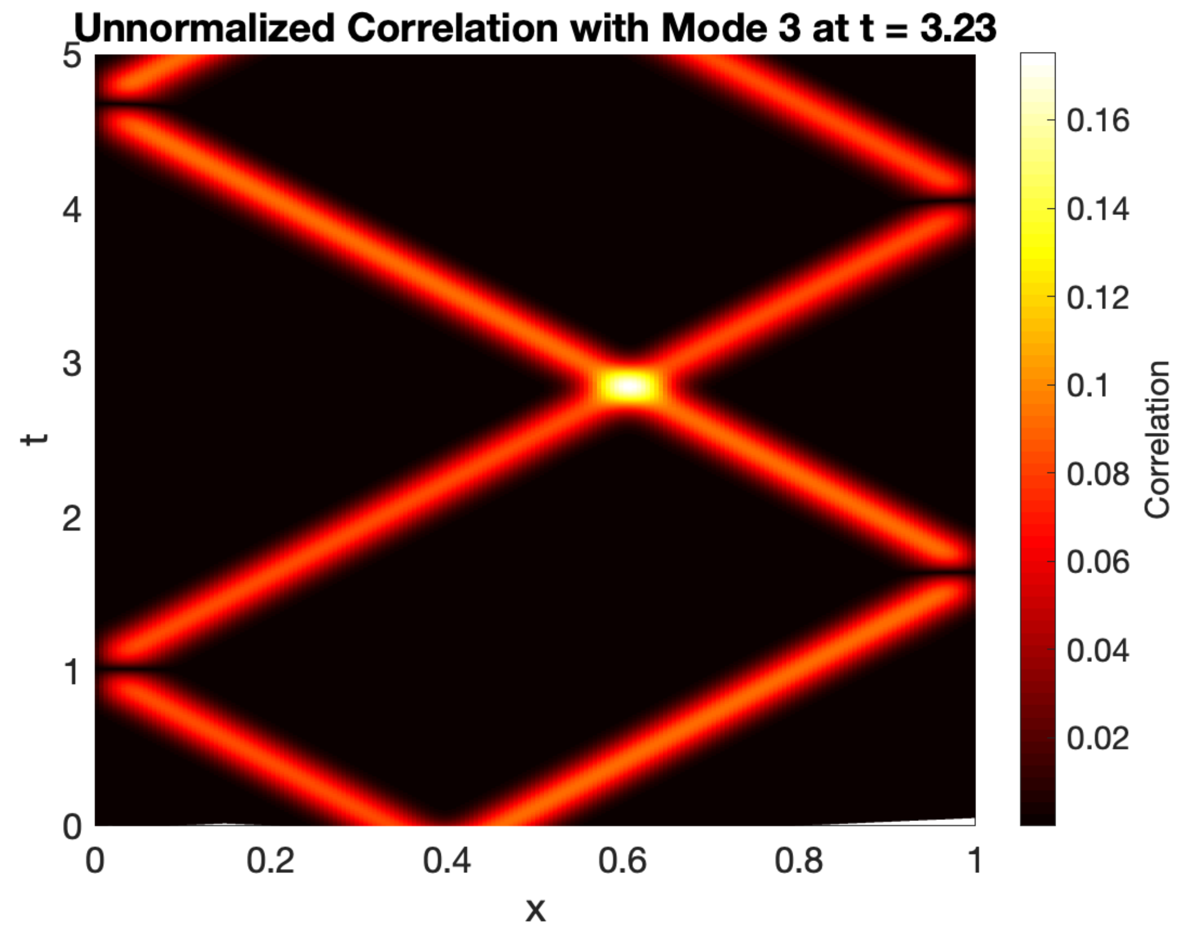

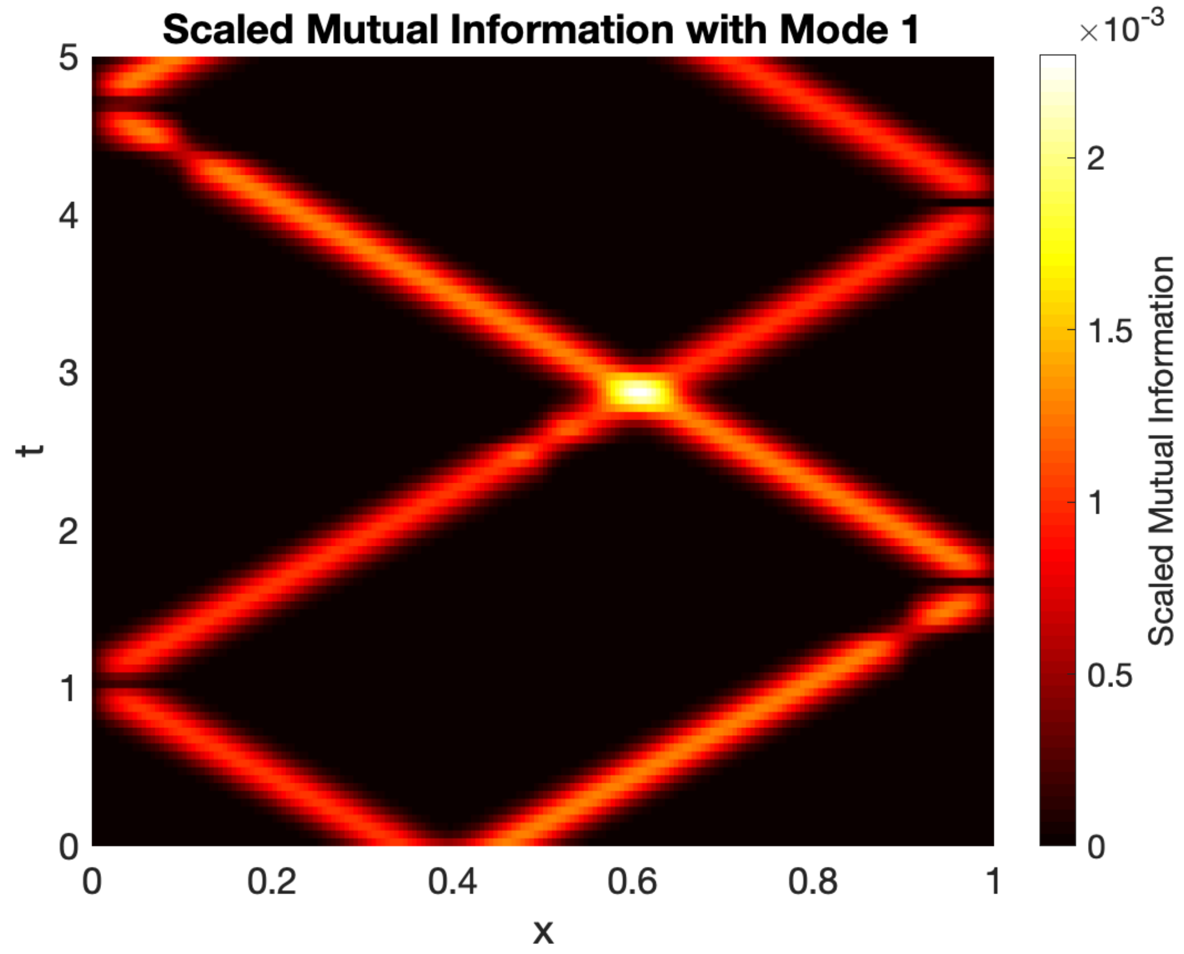

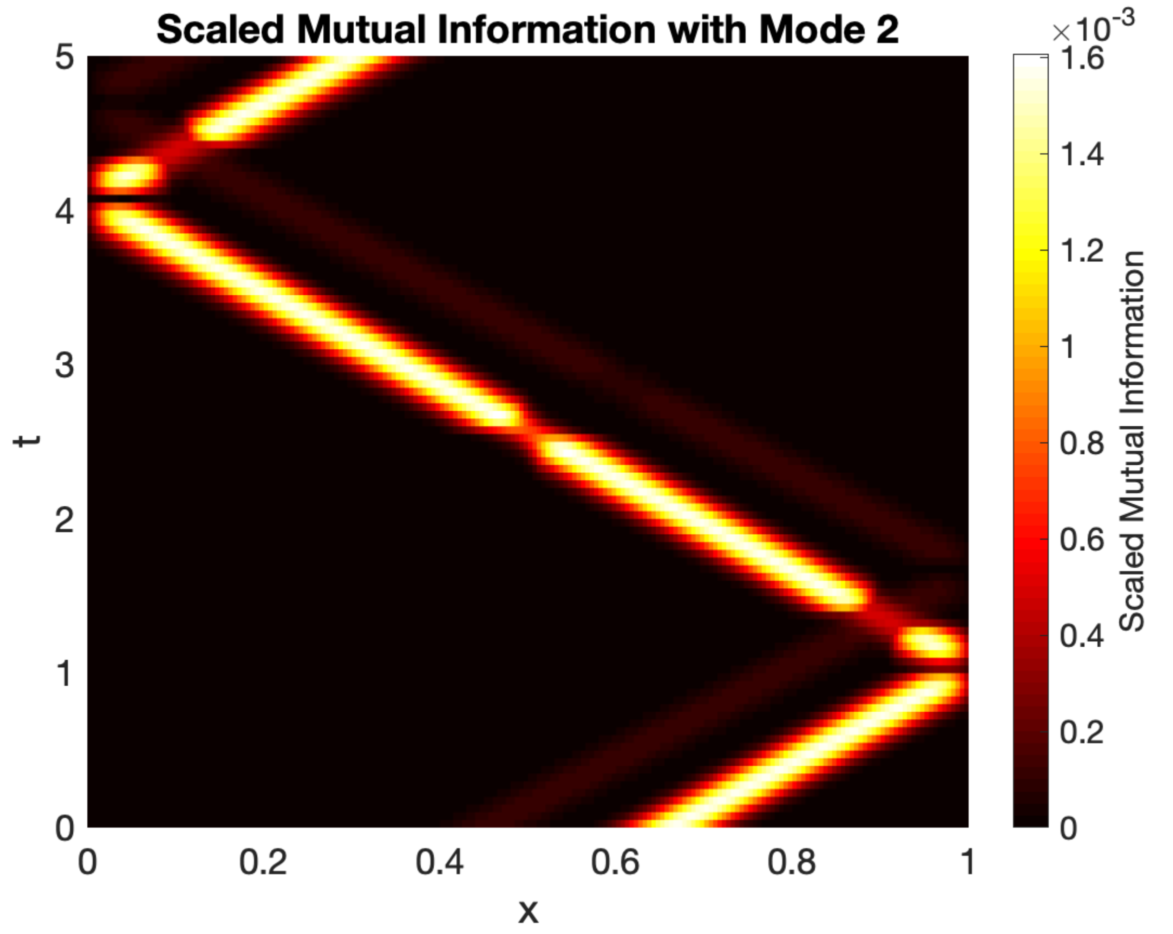

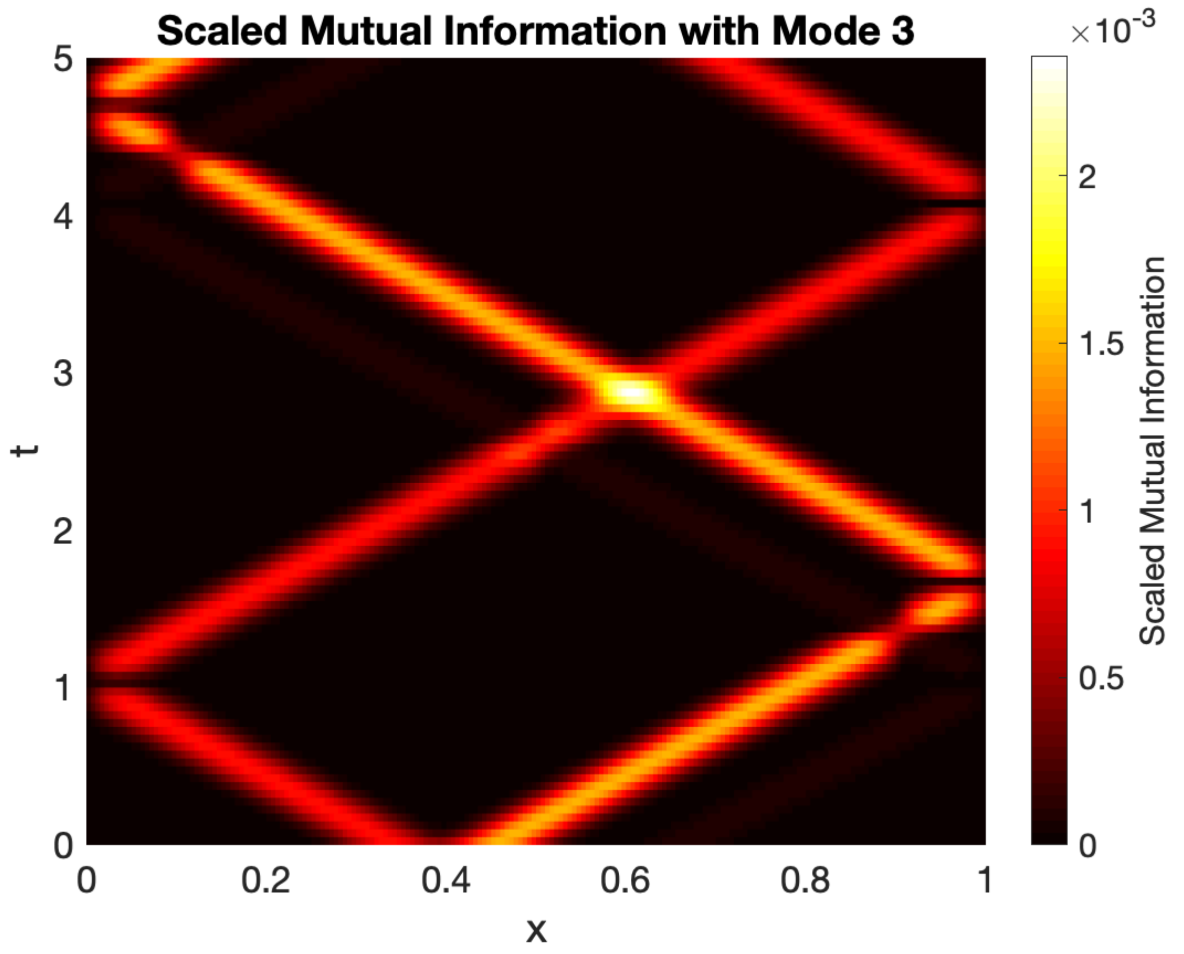

3-Mode Inseparable Example

$$p(x,0;\omega) = A(\omega)h\left(x + \frac{1}{6} + D(\omega)\right) + B(\omega)h\left(x - \frac{1}{6} + E(\omega)\right) + C(\omega)h\left(x + \frac{1}{24} + F(\omega)\right)$$

$$p_t(x,0;\omega) = cA(\omega)h'\left(x + \frac{1}{6} + D(\omega)\right) - cB(\omega)h'\left(x - \frac{1}{6} + E(\omega)\right) - cC(\omega)h'\left(x + \frac{1}{24} + F(\omega)\right)$$

h Nuttall window

$$A \sim \mathcal{U}\left(\frac{1}{2},\frac{3}{2}\right)$$

$$B \sim \text{Exp}\left(\frac{1}{2}\right)$$

$$C = A + (B - \mathbb{E}B)^2$$

$$E, F, G \sim \mathcal{N}\left(0,\frac{1}{1024}\right)$$

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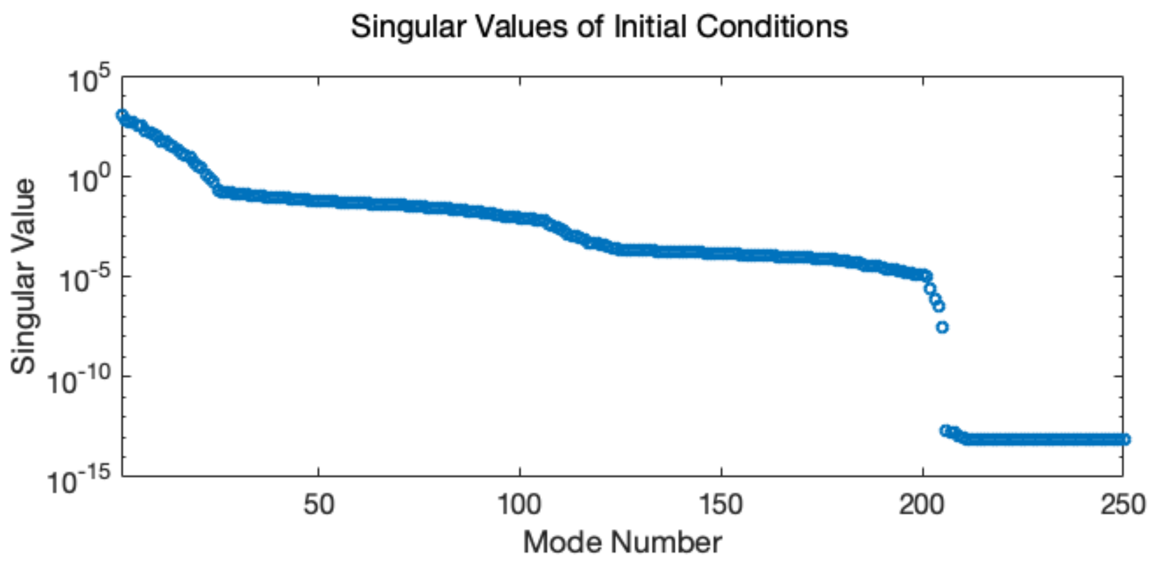
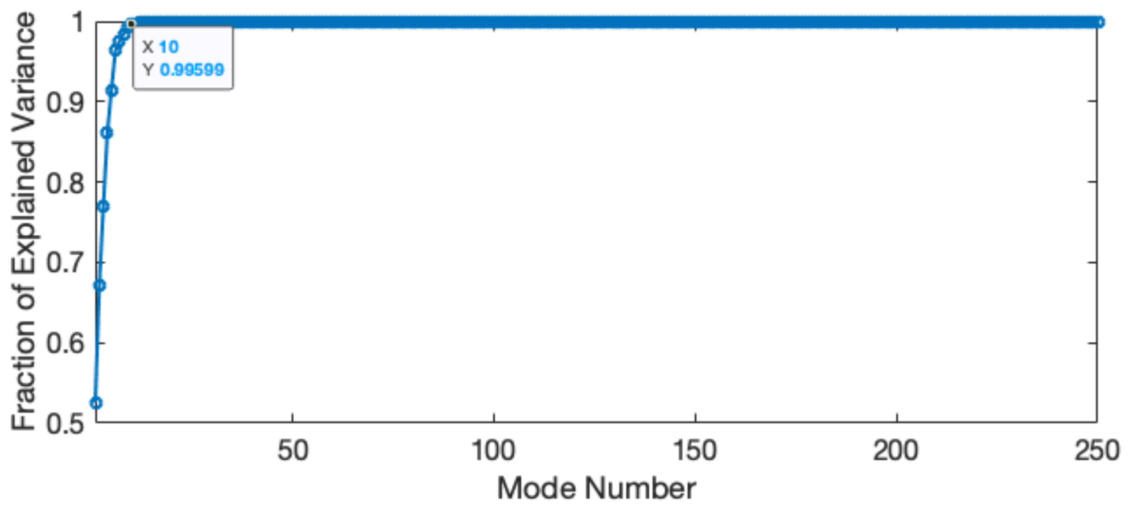
h Nuttall window





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$$B \sim \text{Exp}\left(\frac{1}{2}\right)$$

$$C = A + (B - \mathbf{E}B)^2$$

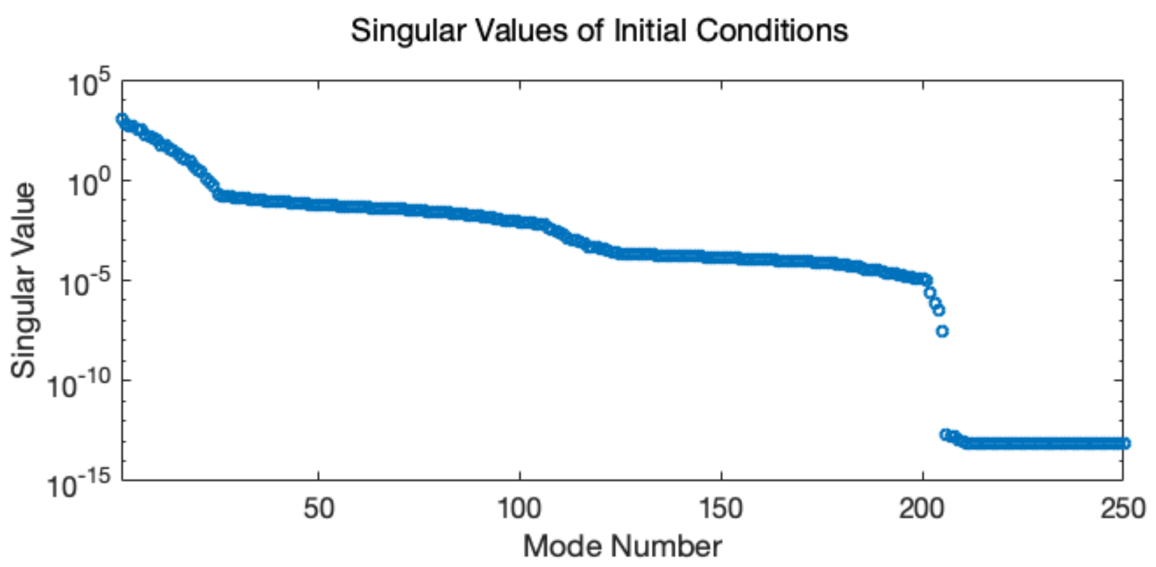
$$E, F, G \sim \mathcal{N}\left(0, \frac{1}{1024}\right)$$

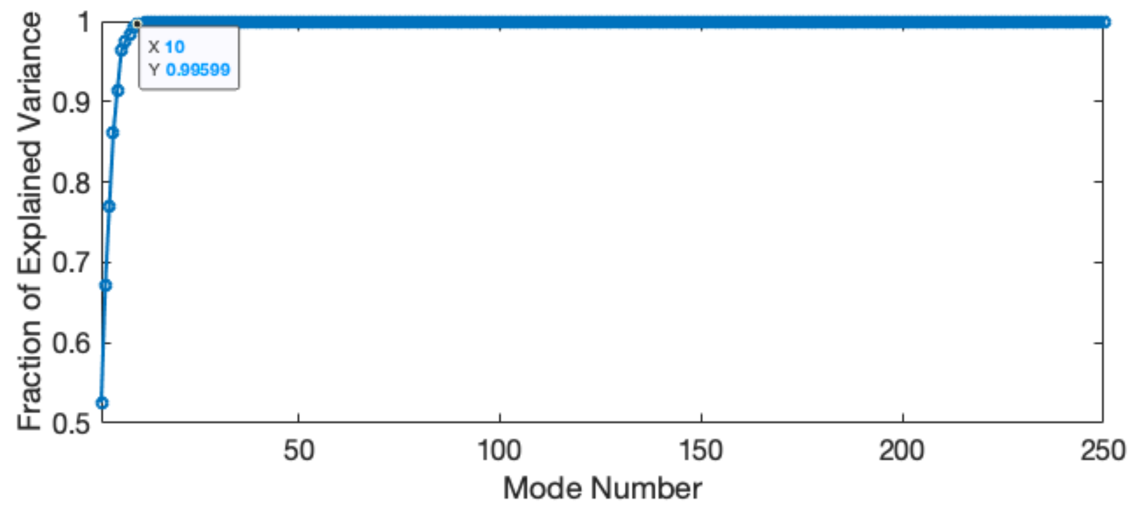
3-Mode Inseparable Example

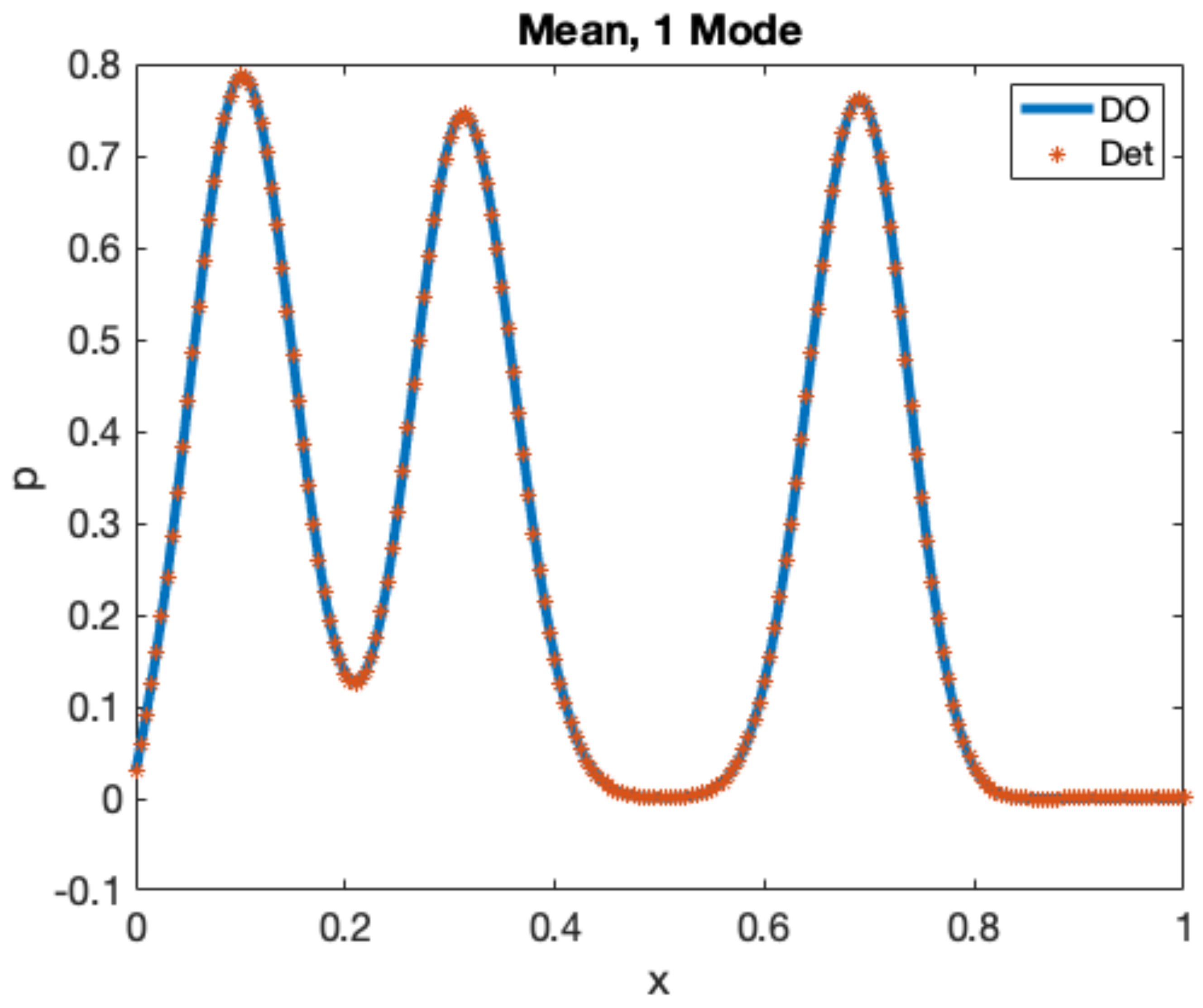
$$p(x,0;\omega) = A(\omega)h\left(x + \frac{1}{6} + D(\omega)\right) + B(\omega)h\left(x - \frac{1}{6} + E(\omega)\right) + C(\omega)h\left(x + \frac{1}{24} + F(\omega)\right)$$

$$p_t(x,0;\omega) = cA(\omega)h'\left(x + \frac{1}{6} + D(\omega)\right) - cB(\omega)h'\left(x - \frac{1}{6} + E(\omega)\right) - cC(\omega)h'\left(x + \frac{1}{24} + F(\omega)\right)$$

h Nuttall window







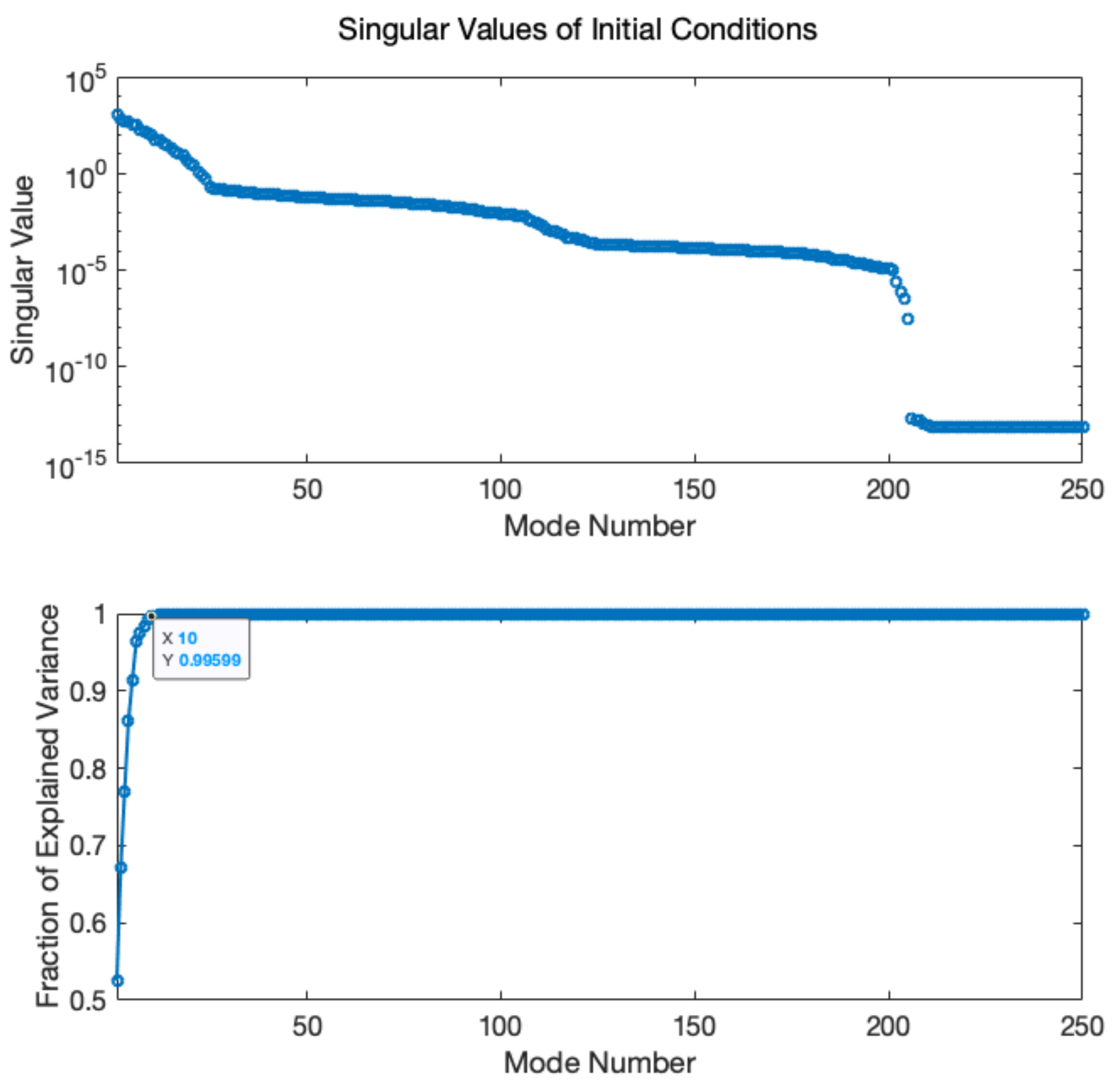
$$A \sim \mathcal{U}\left(\frac{1}{2}, \frac{3}{2}\right)$$

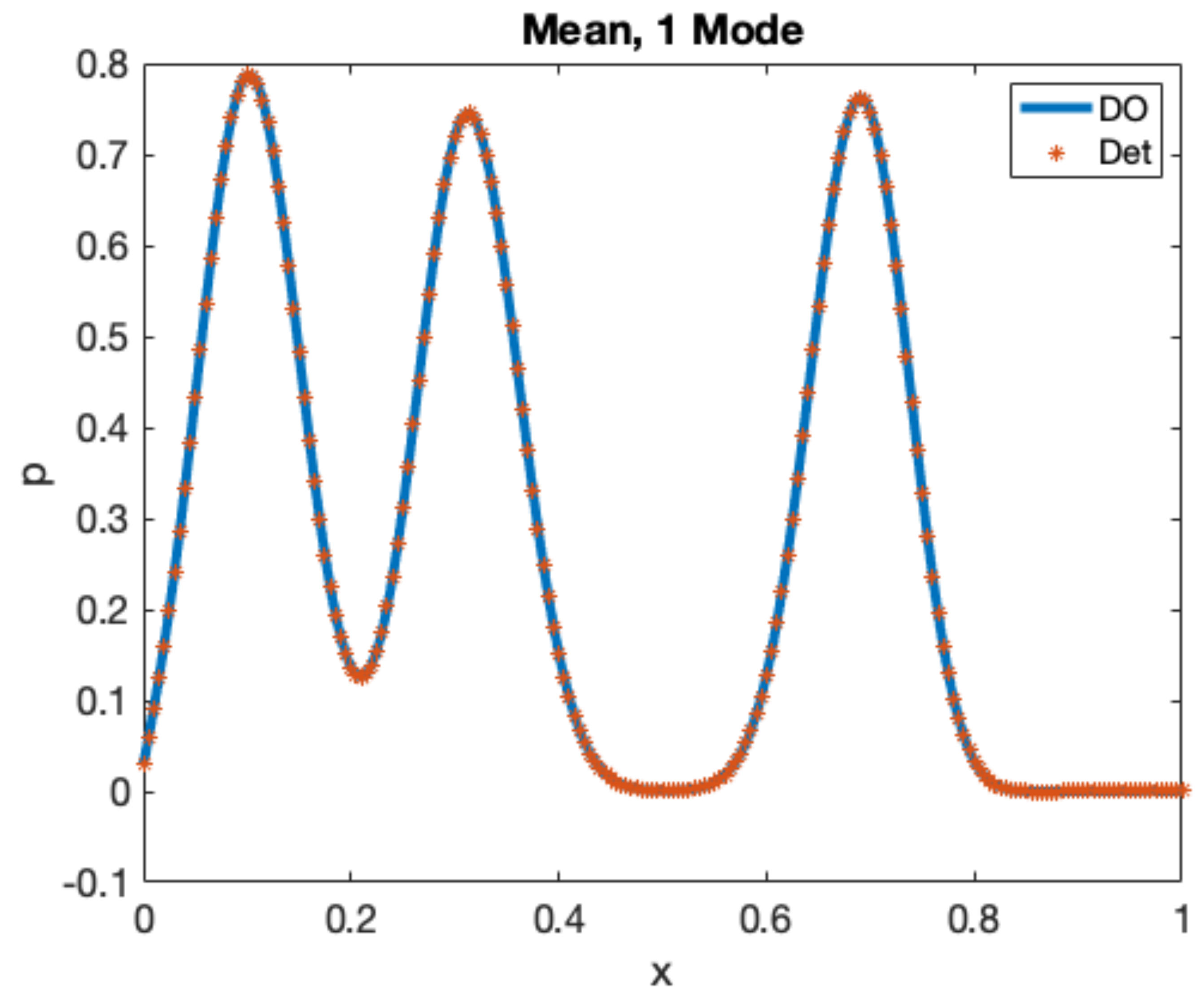
$$B \sim \text{Exp}\left(\frac{1}{2}\right)$$

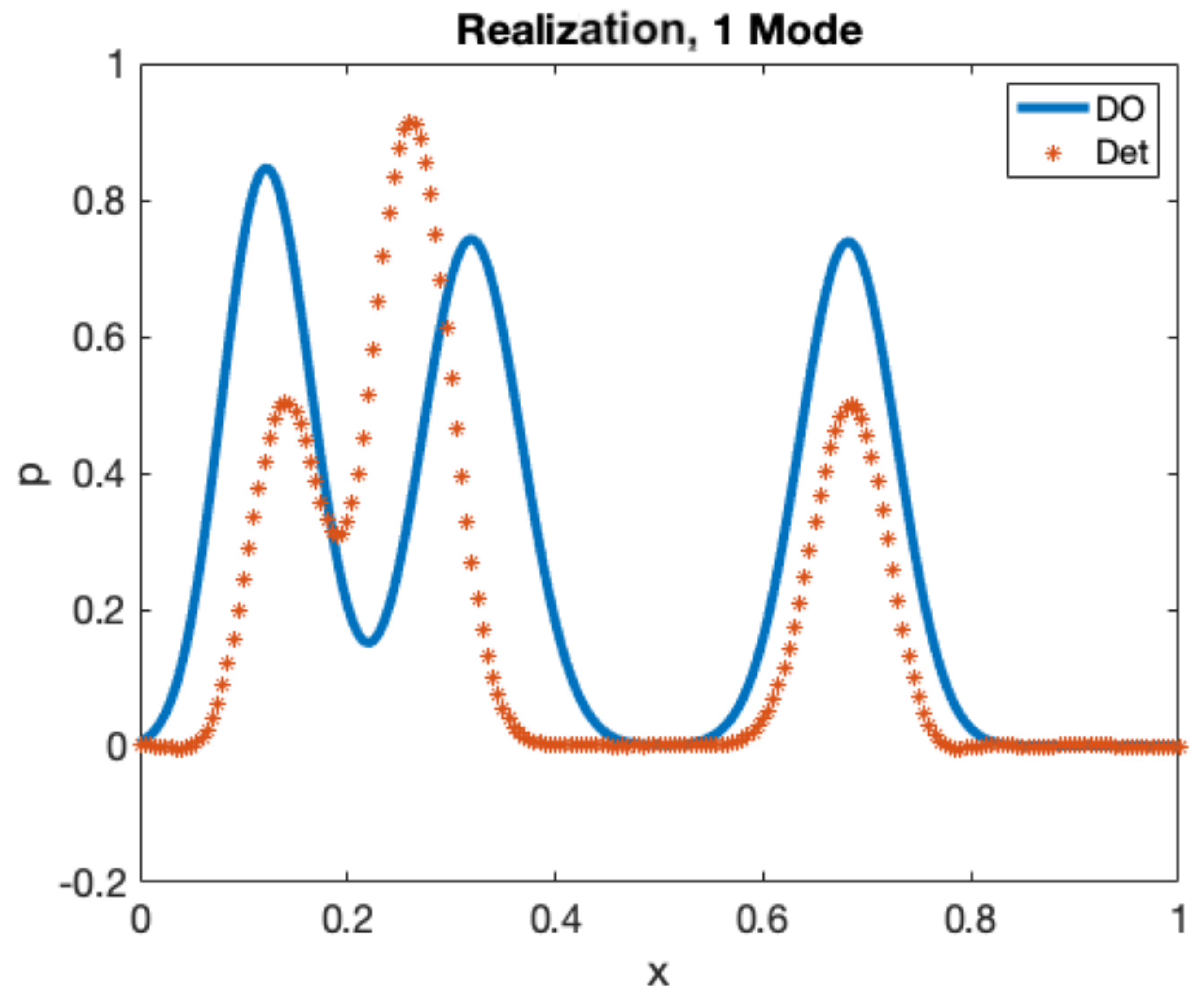
$$C = A + (B - \mathbb{E}B)^2$$

$$E, F, G \sim \mathcal{N}\left(0, \frac{1}{1024}\right)$$

3-Mode Inseparable Example







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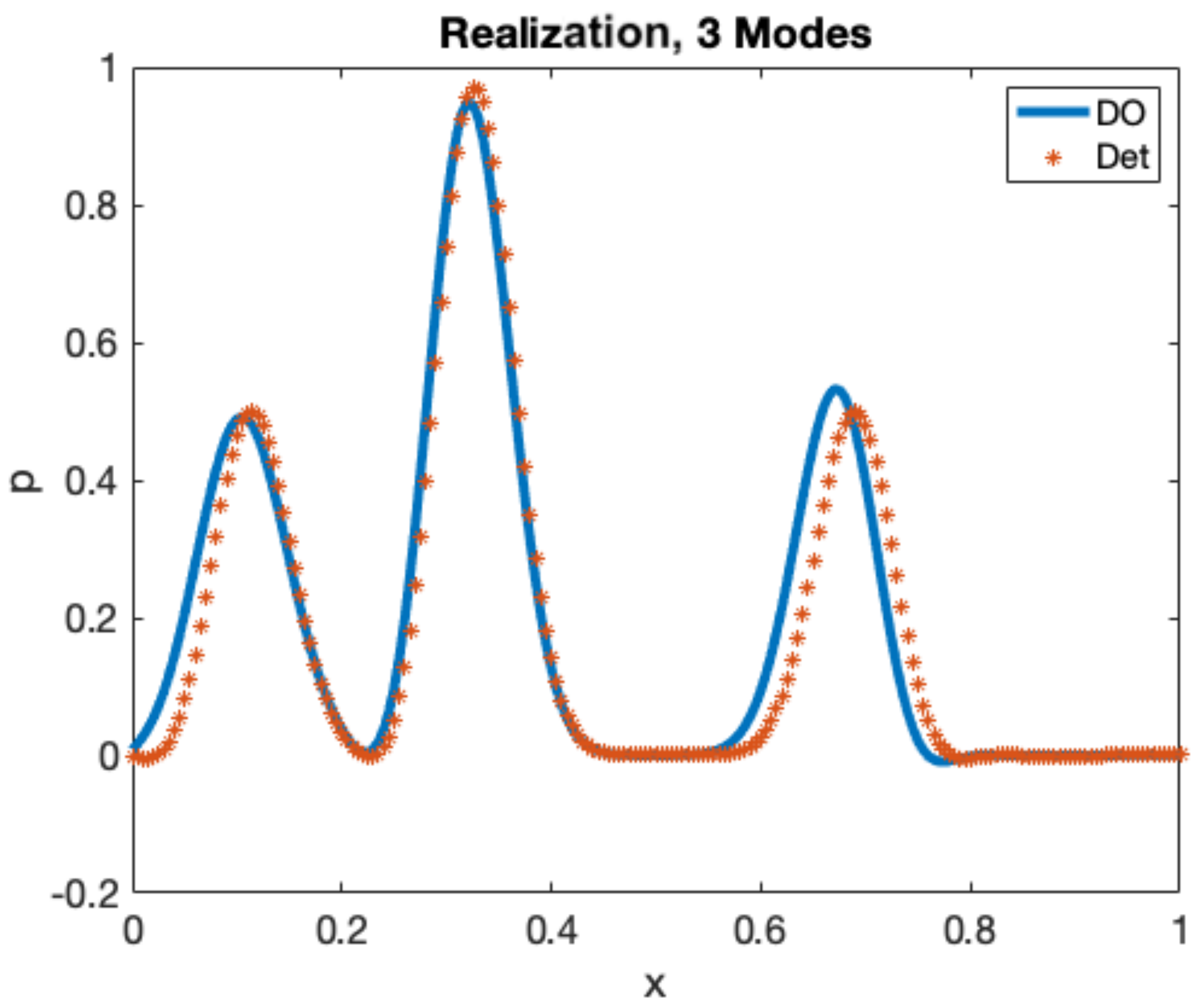

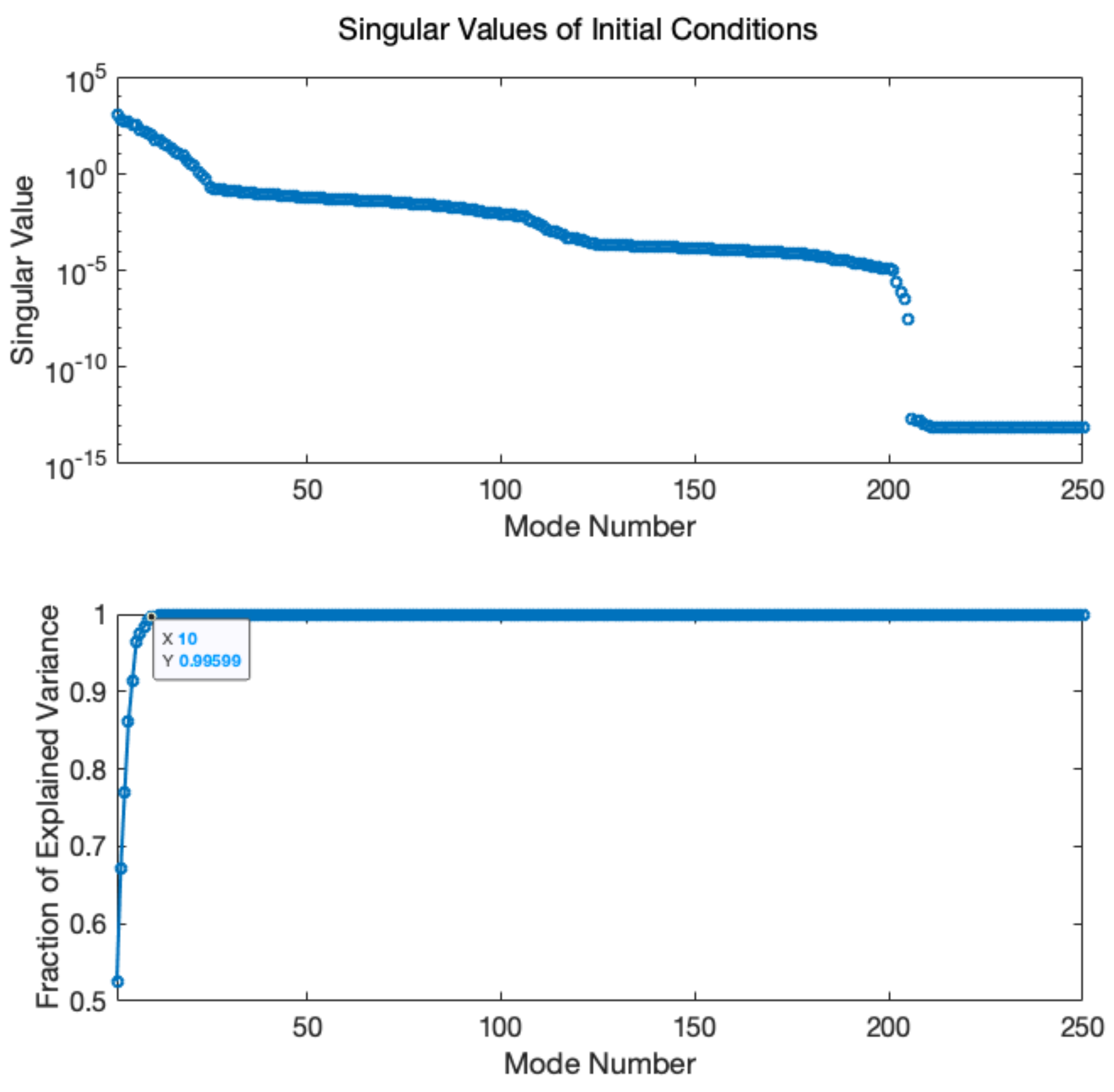

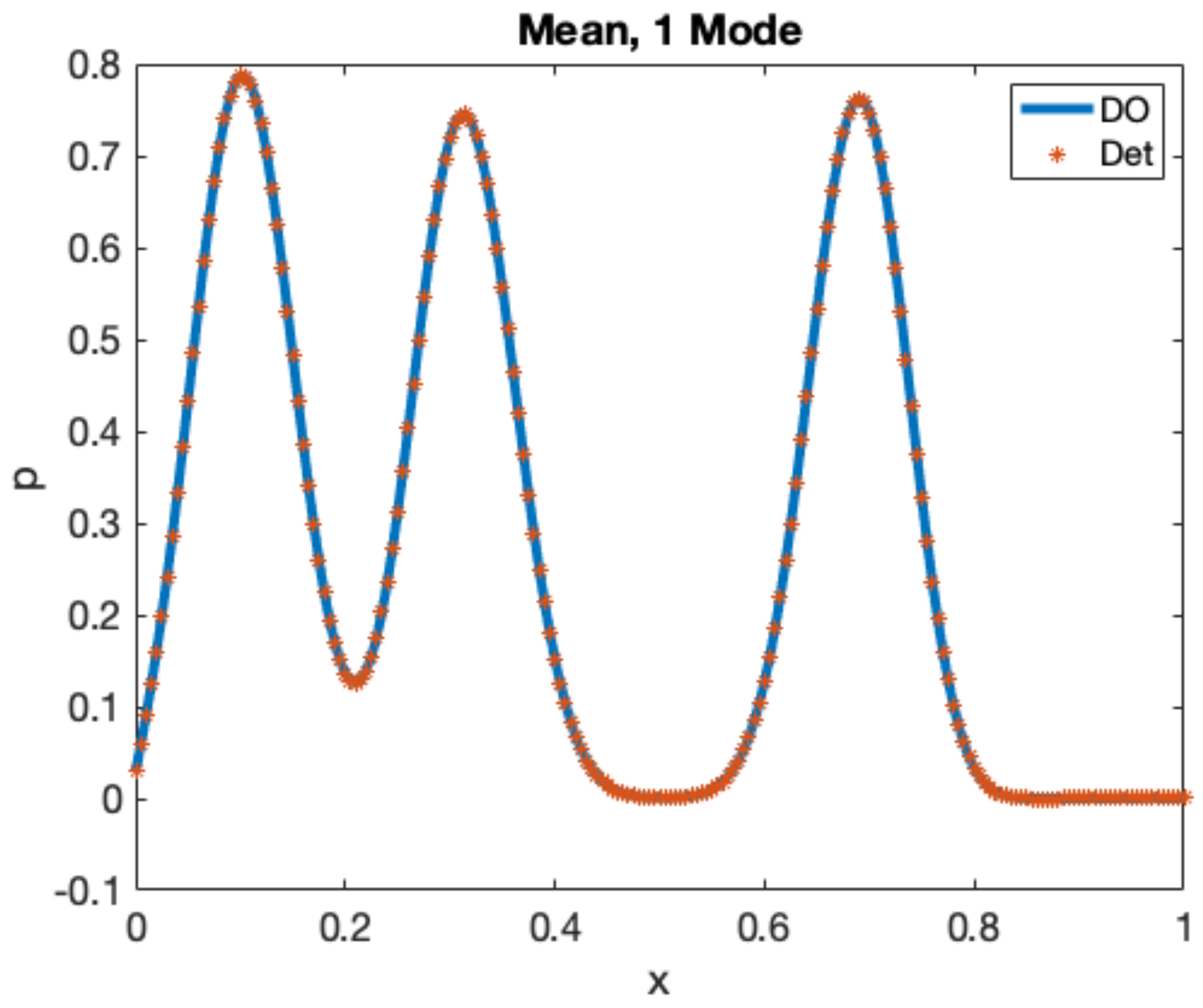

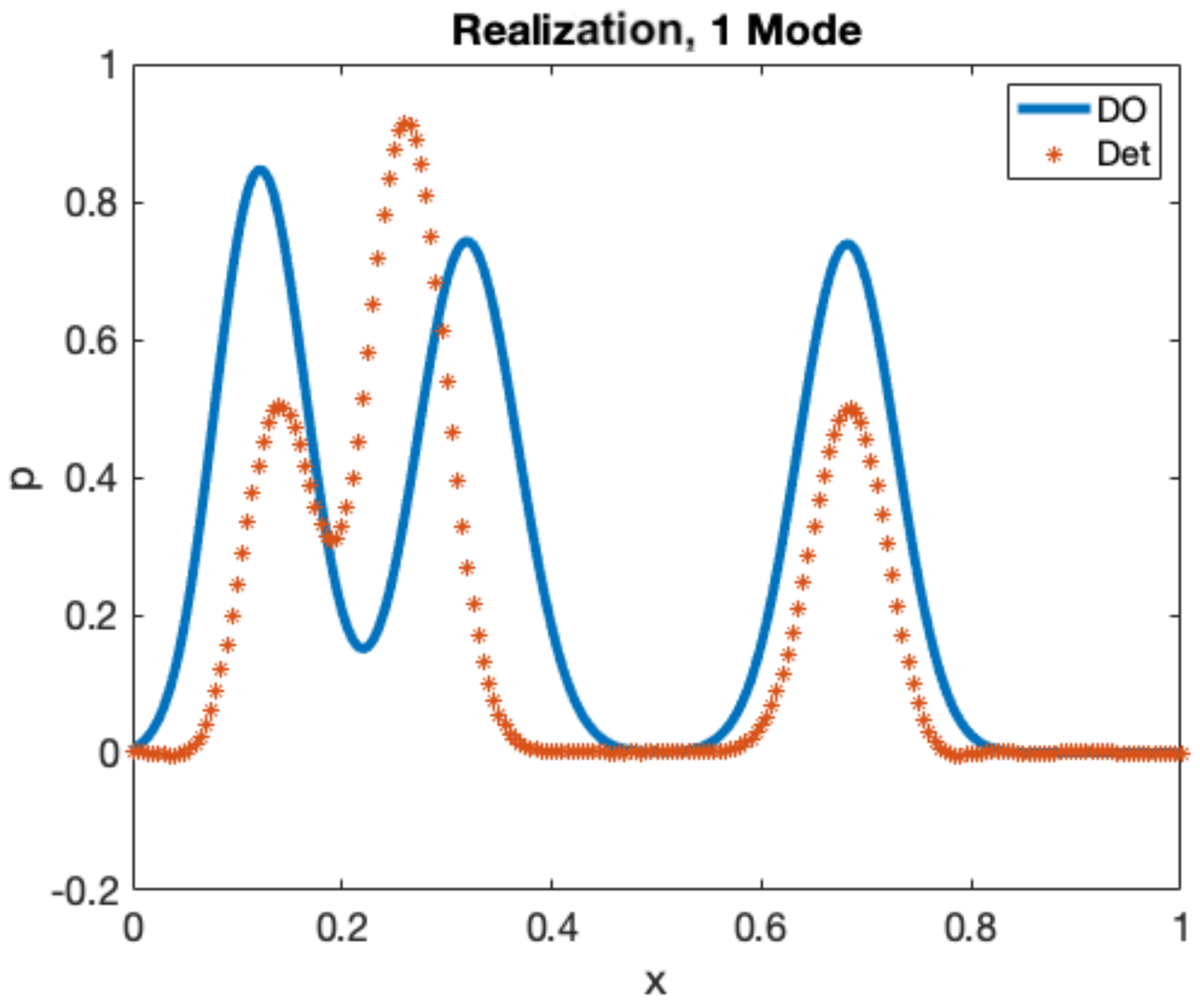

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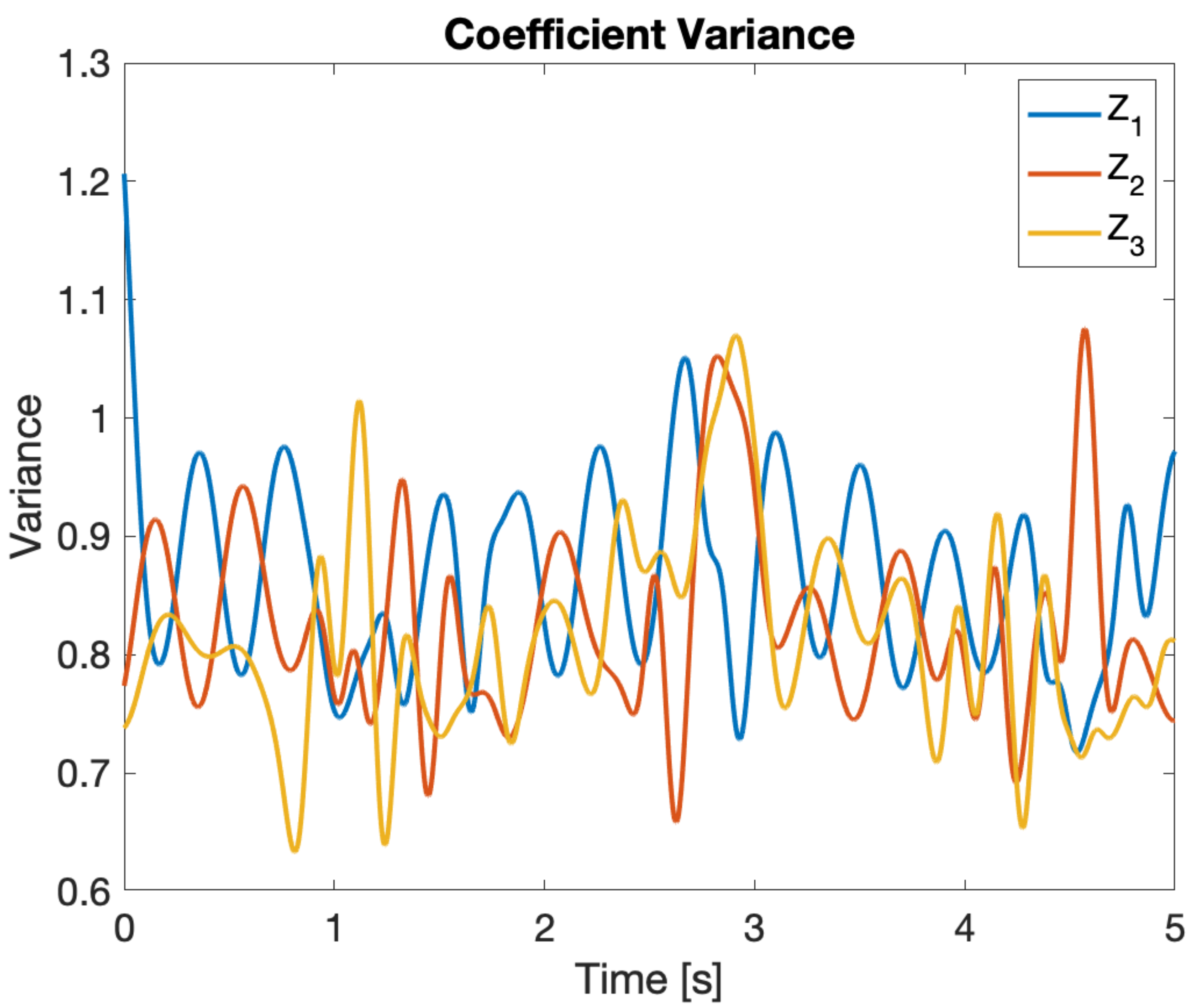

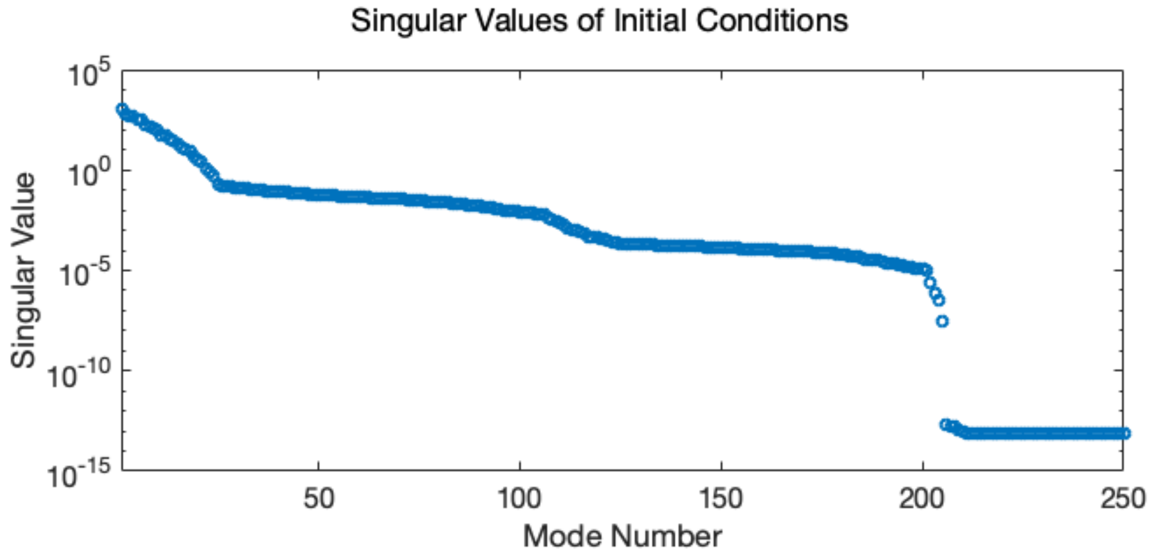

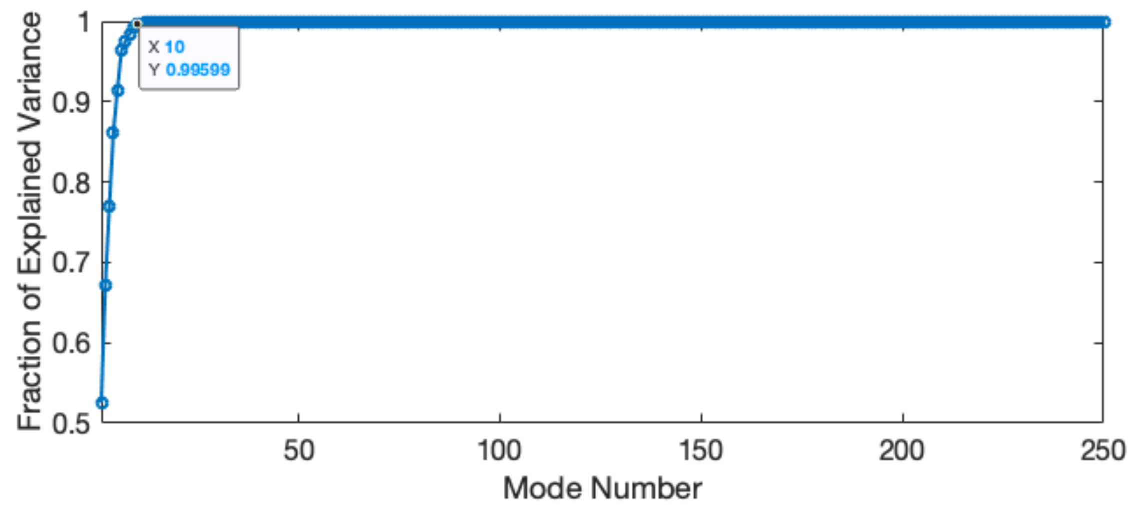

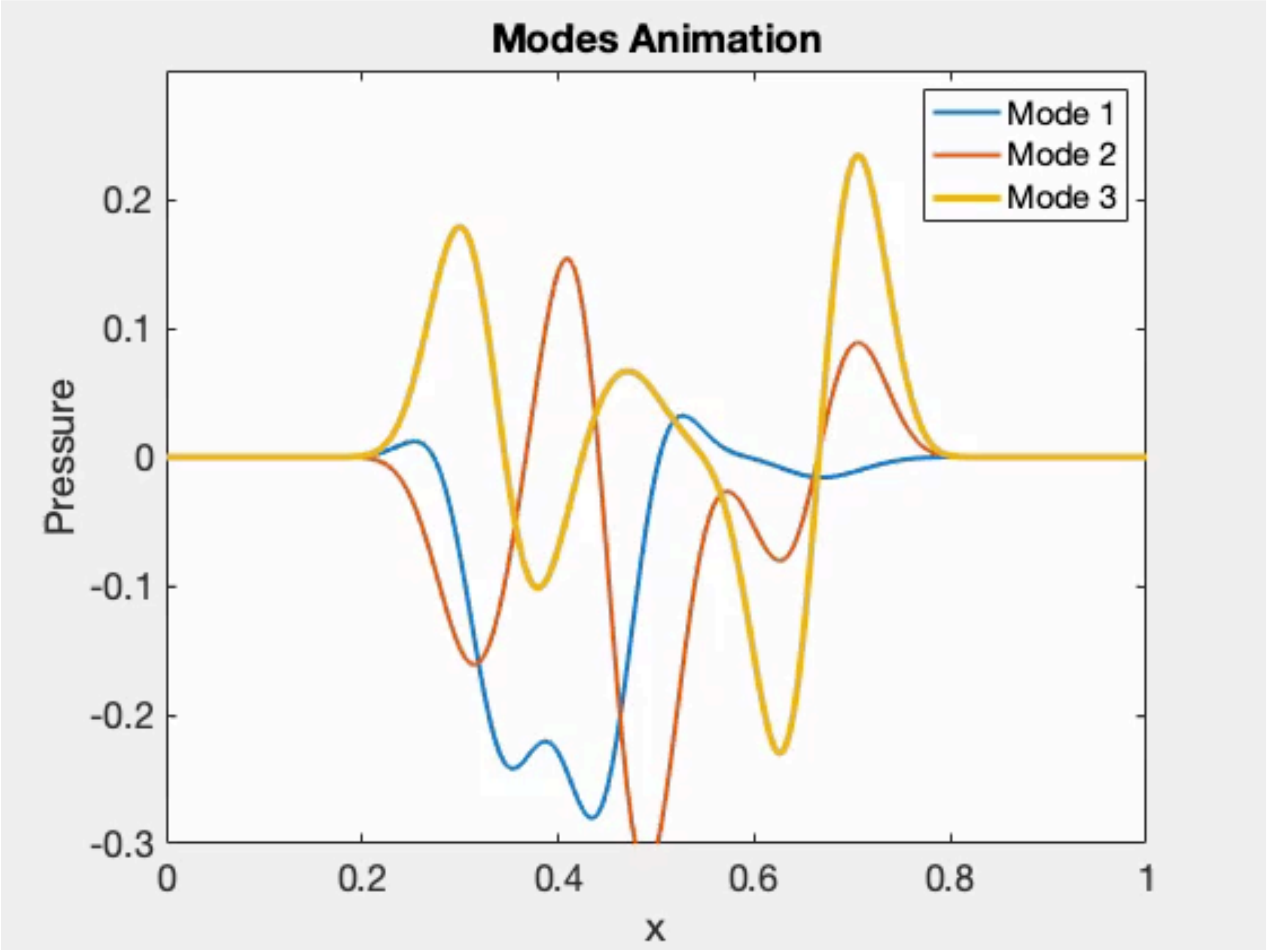

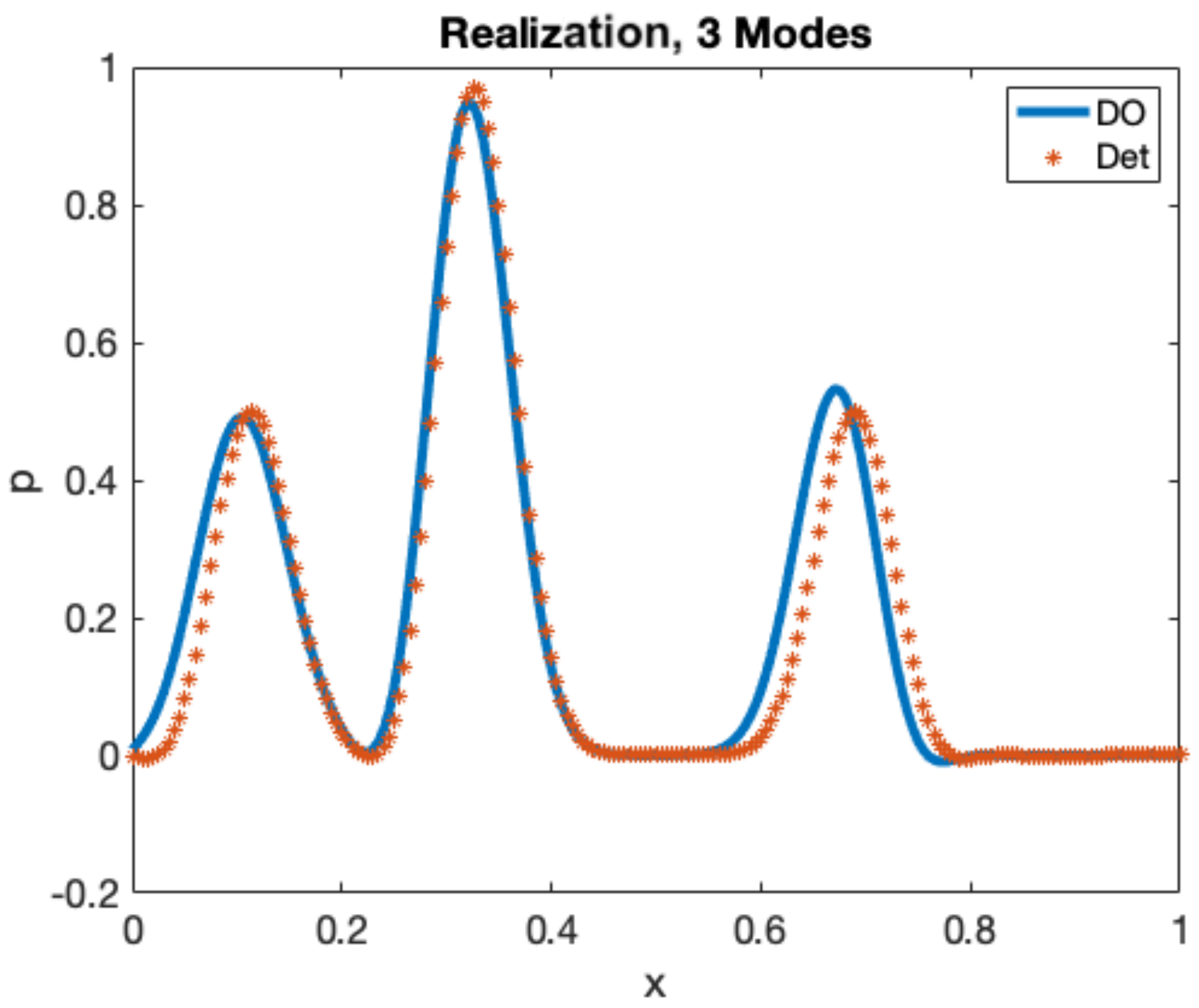

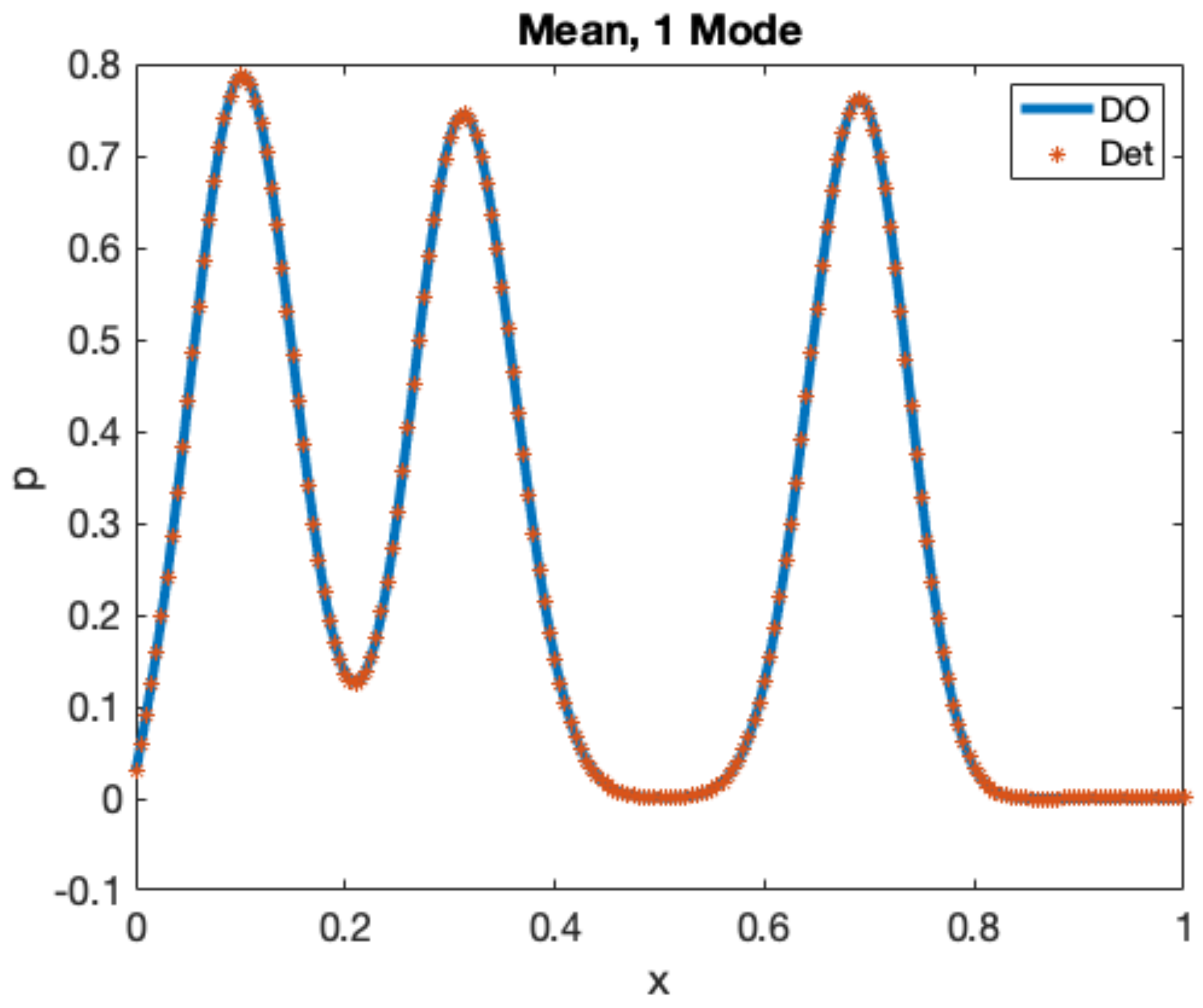

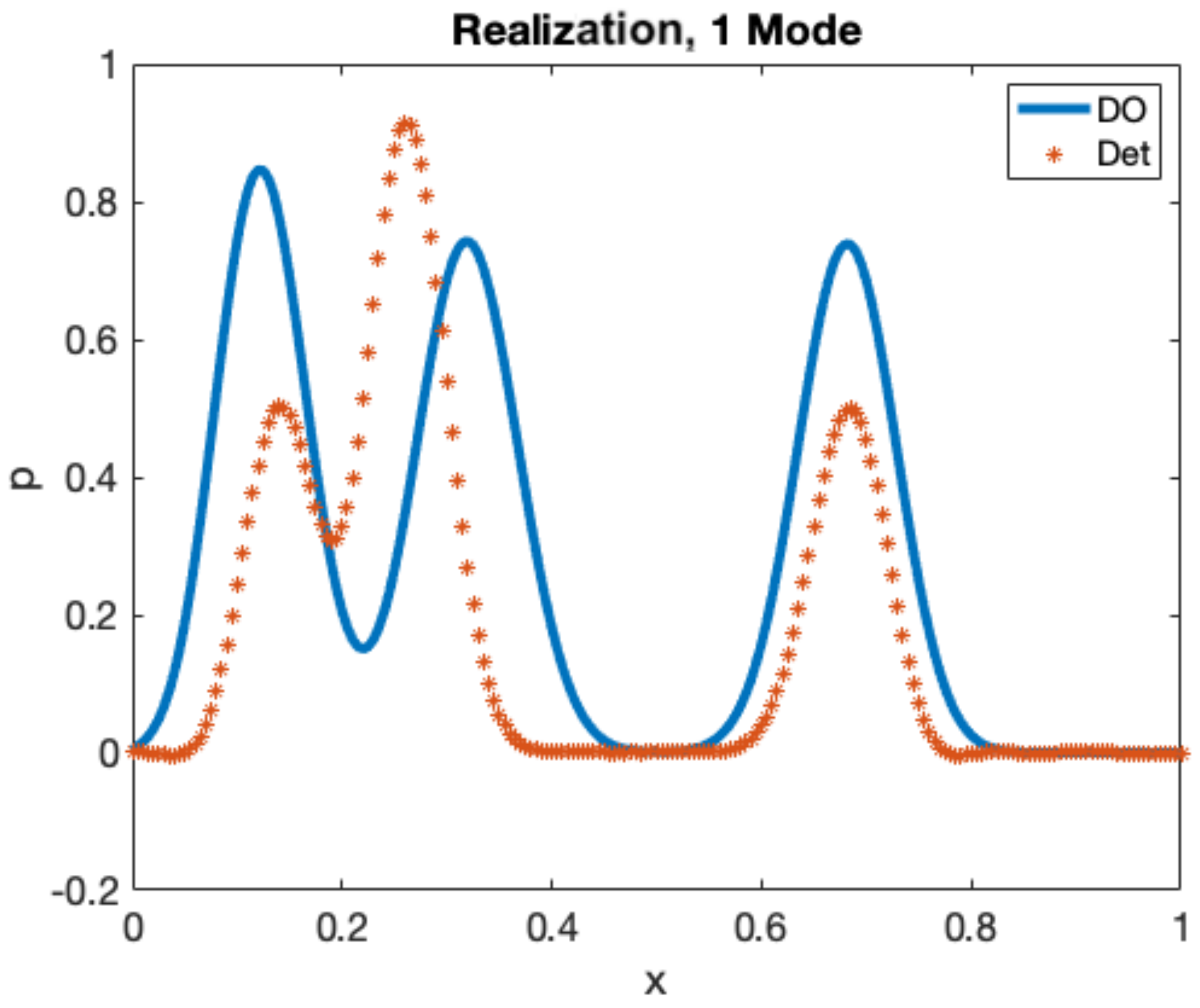

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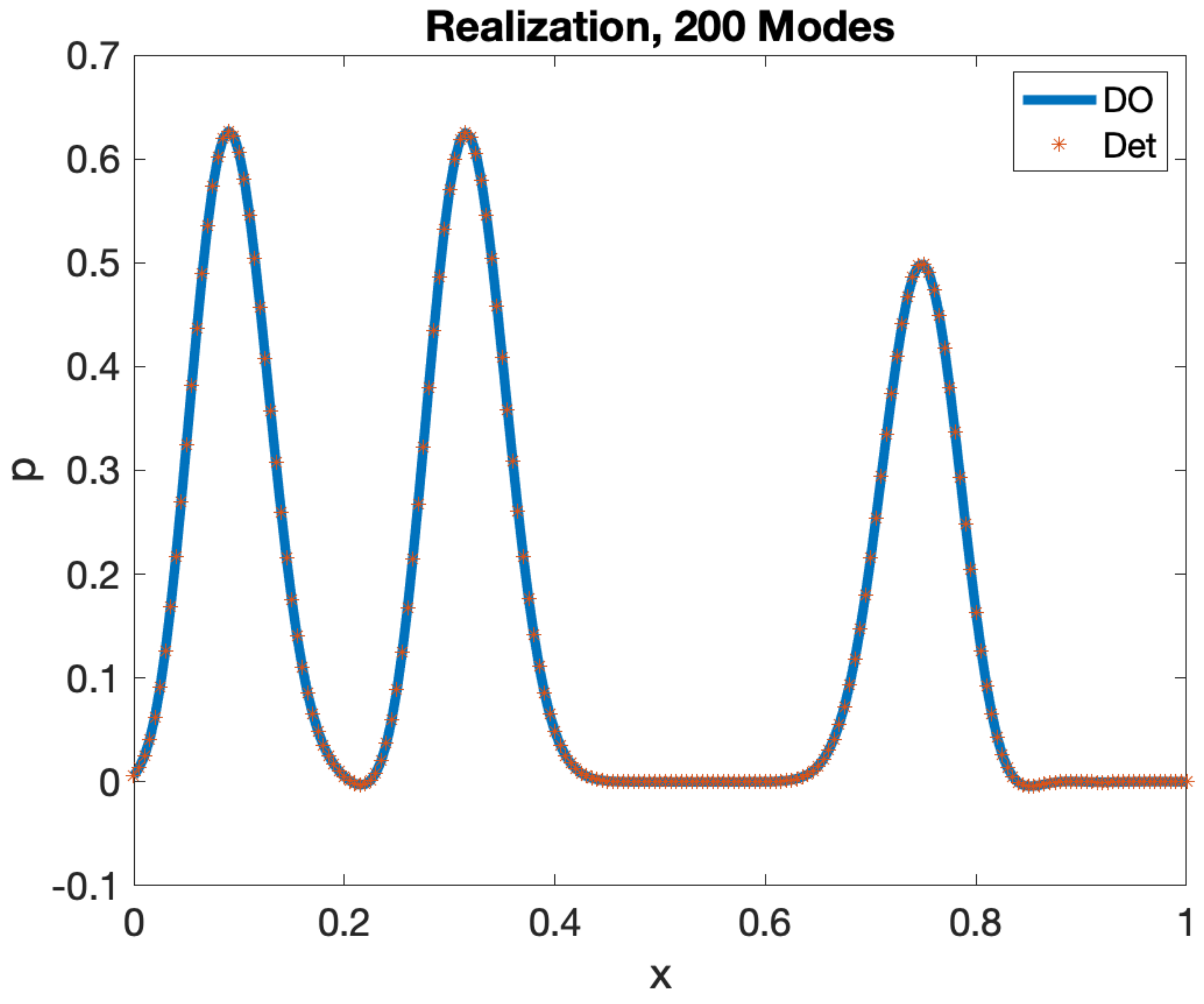

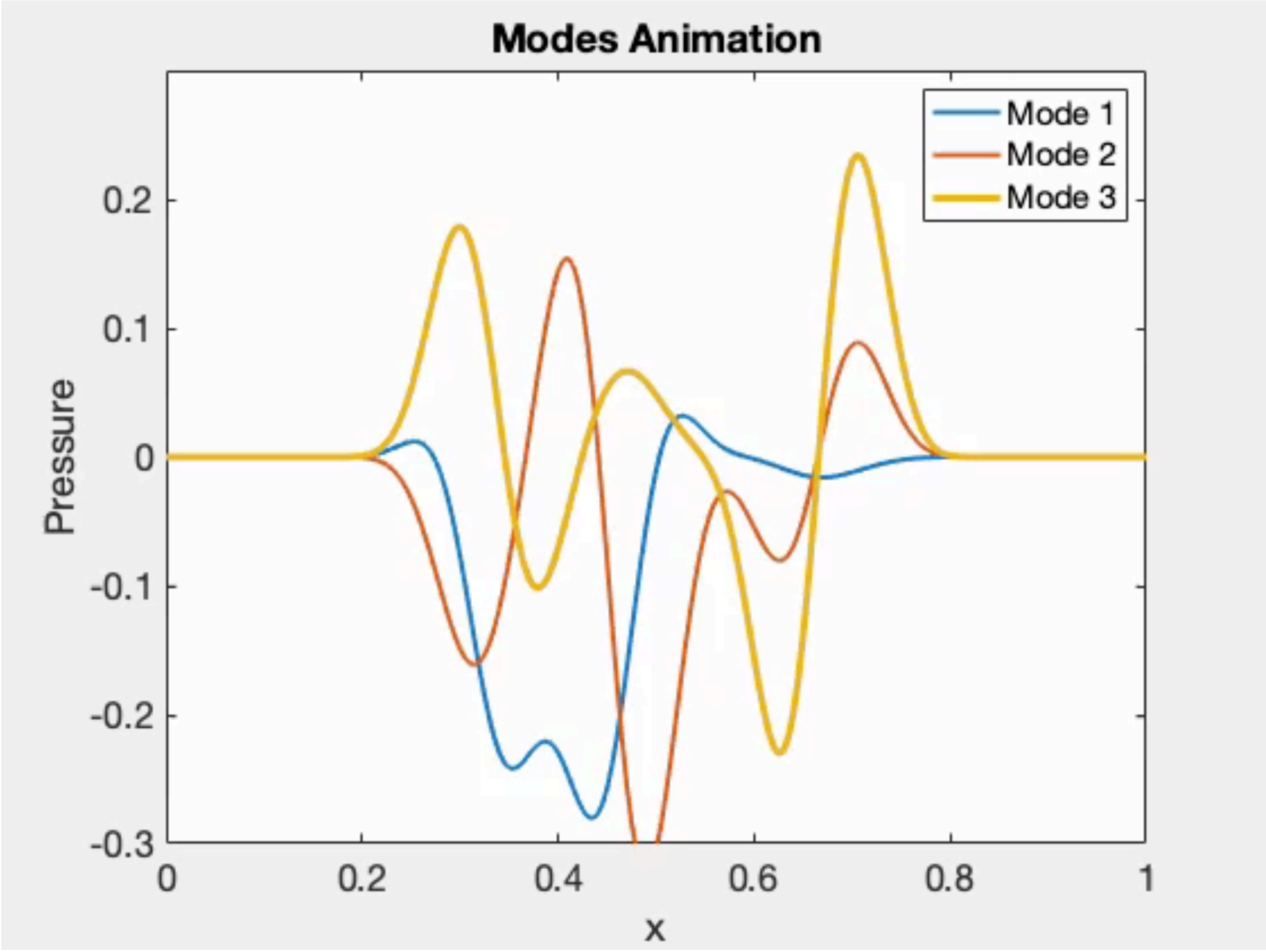

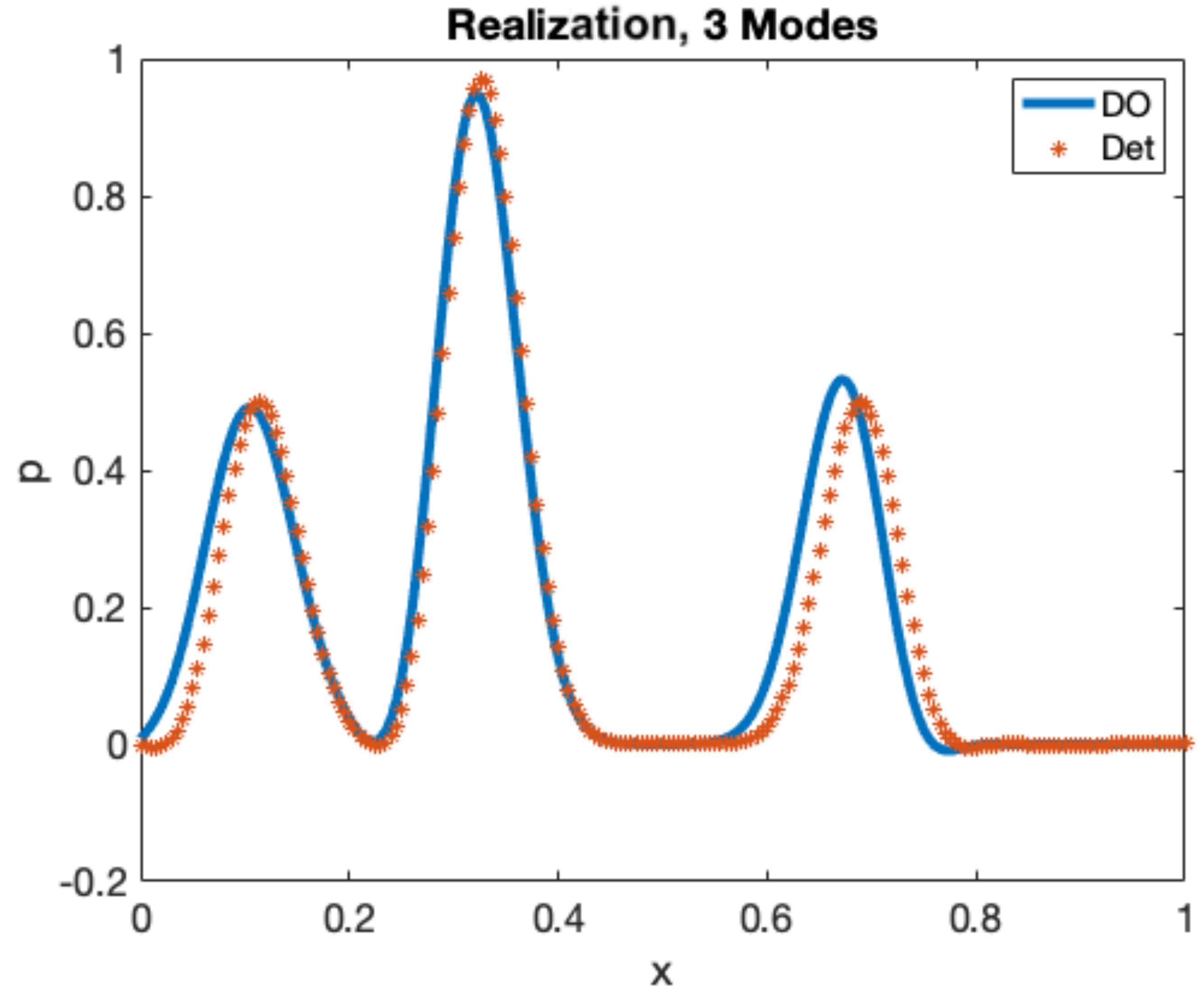

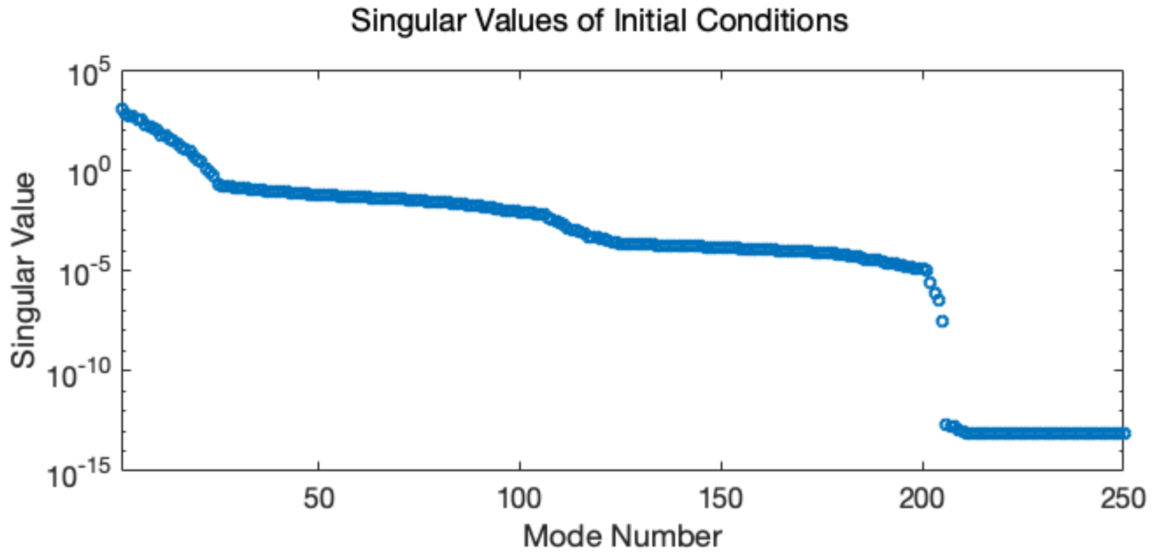

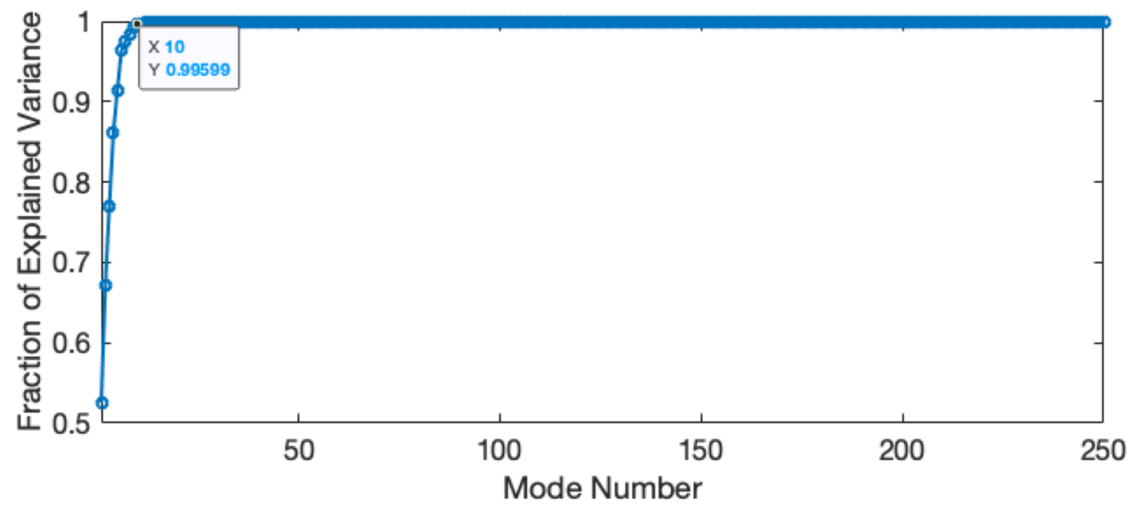

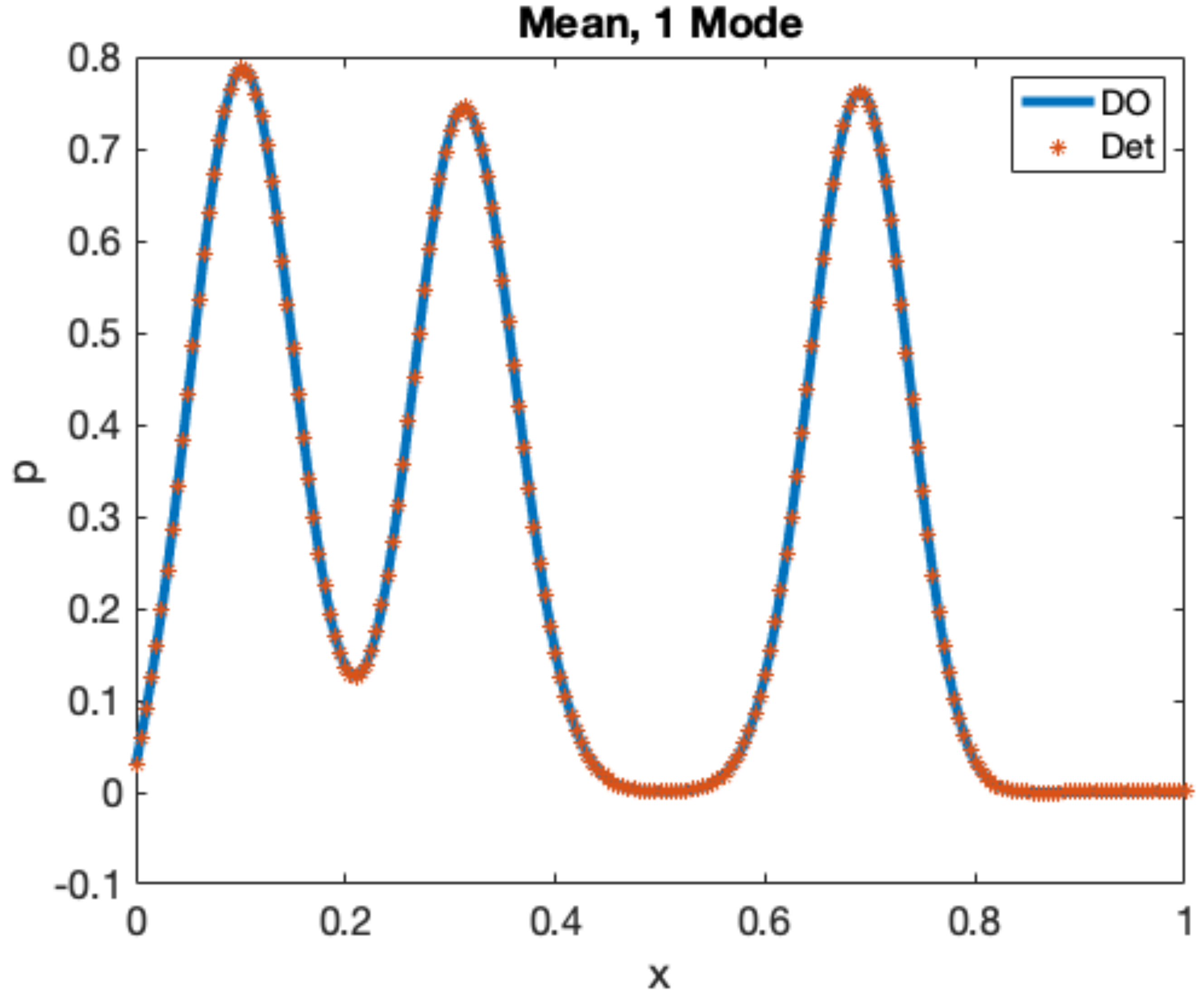

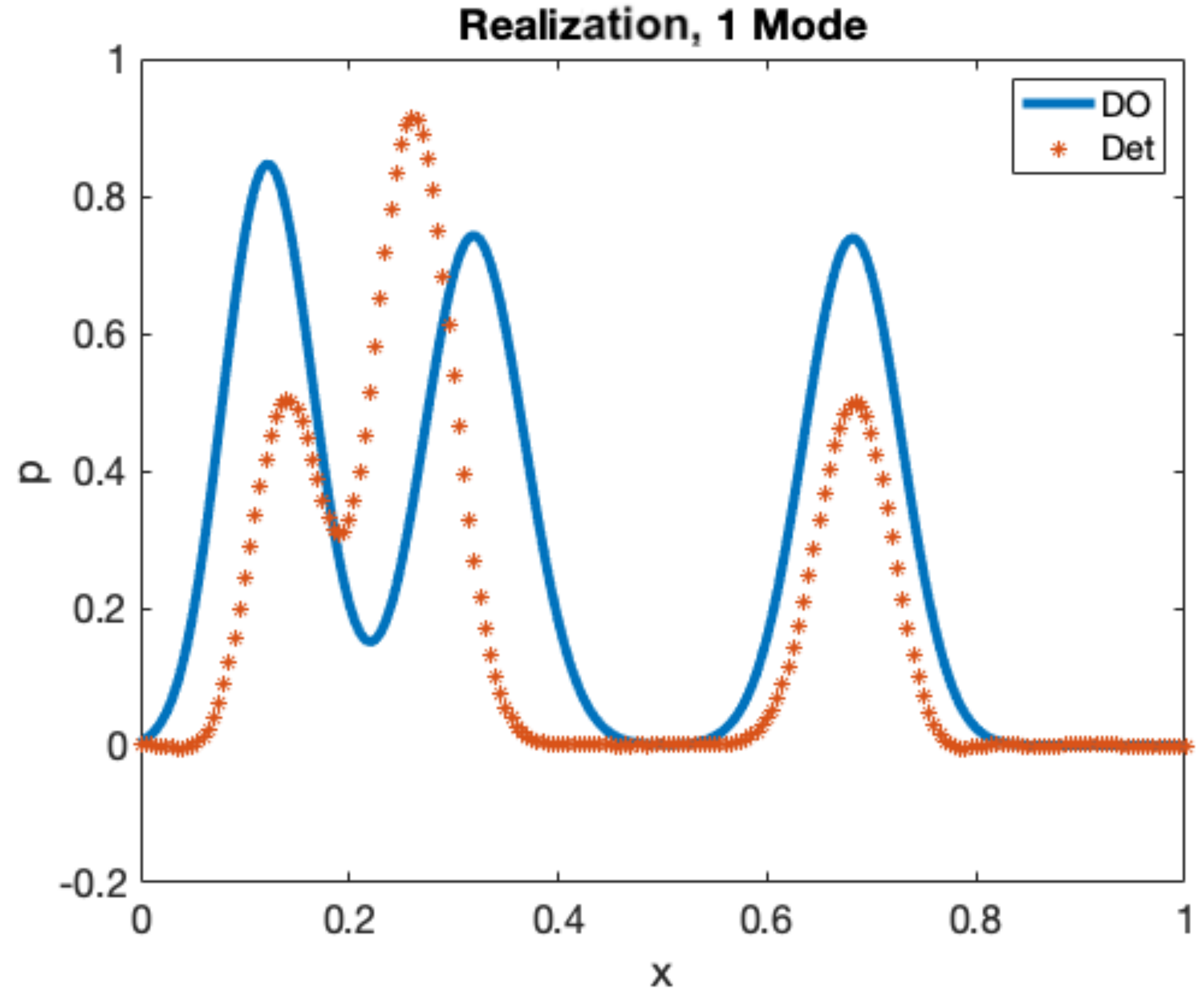

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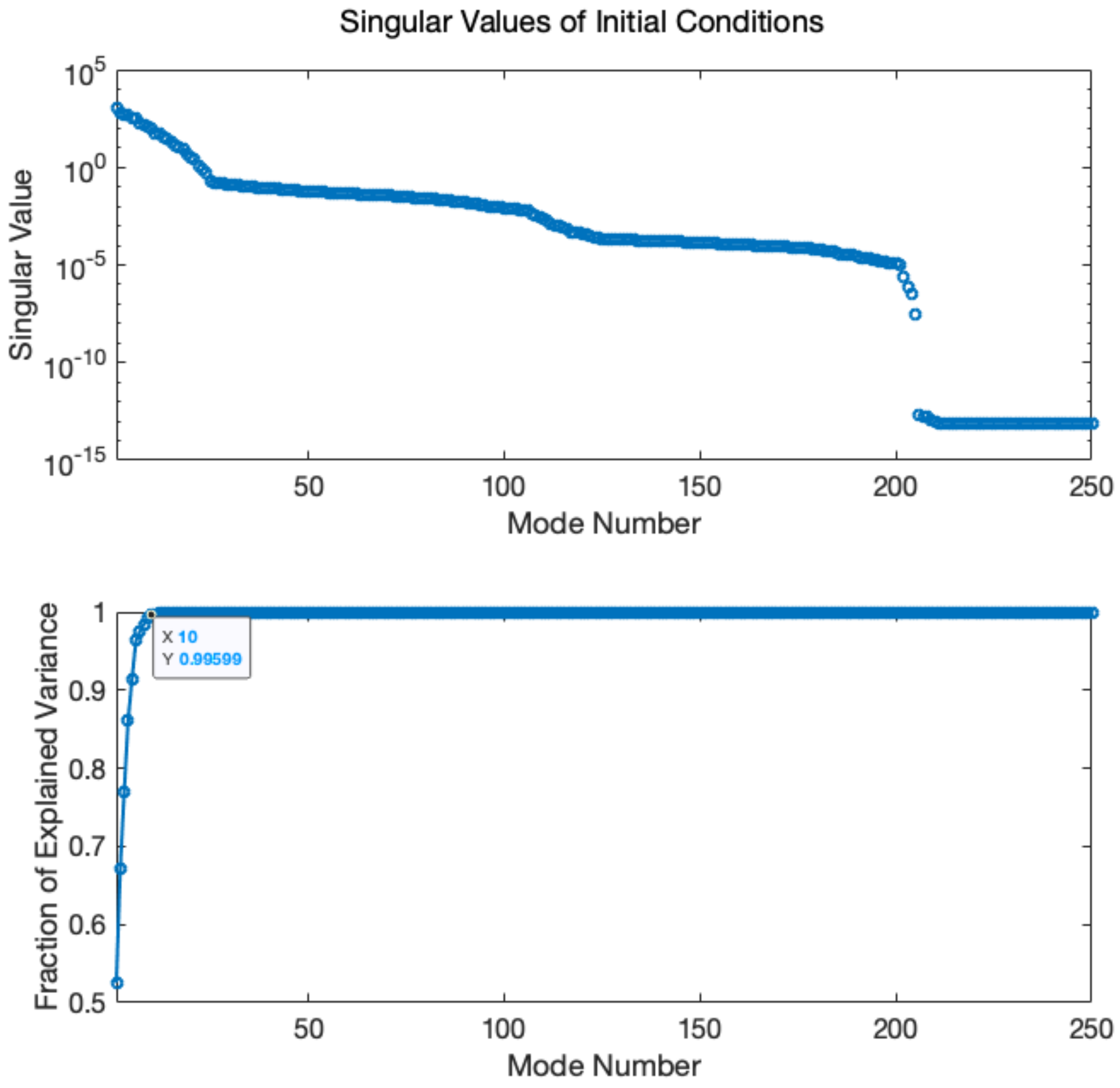

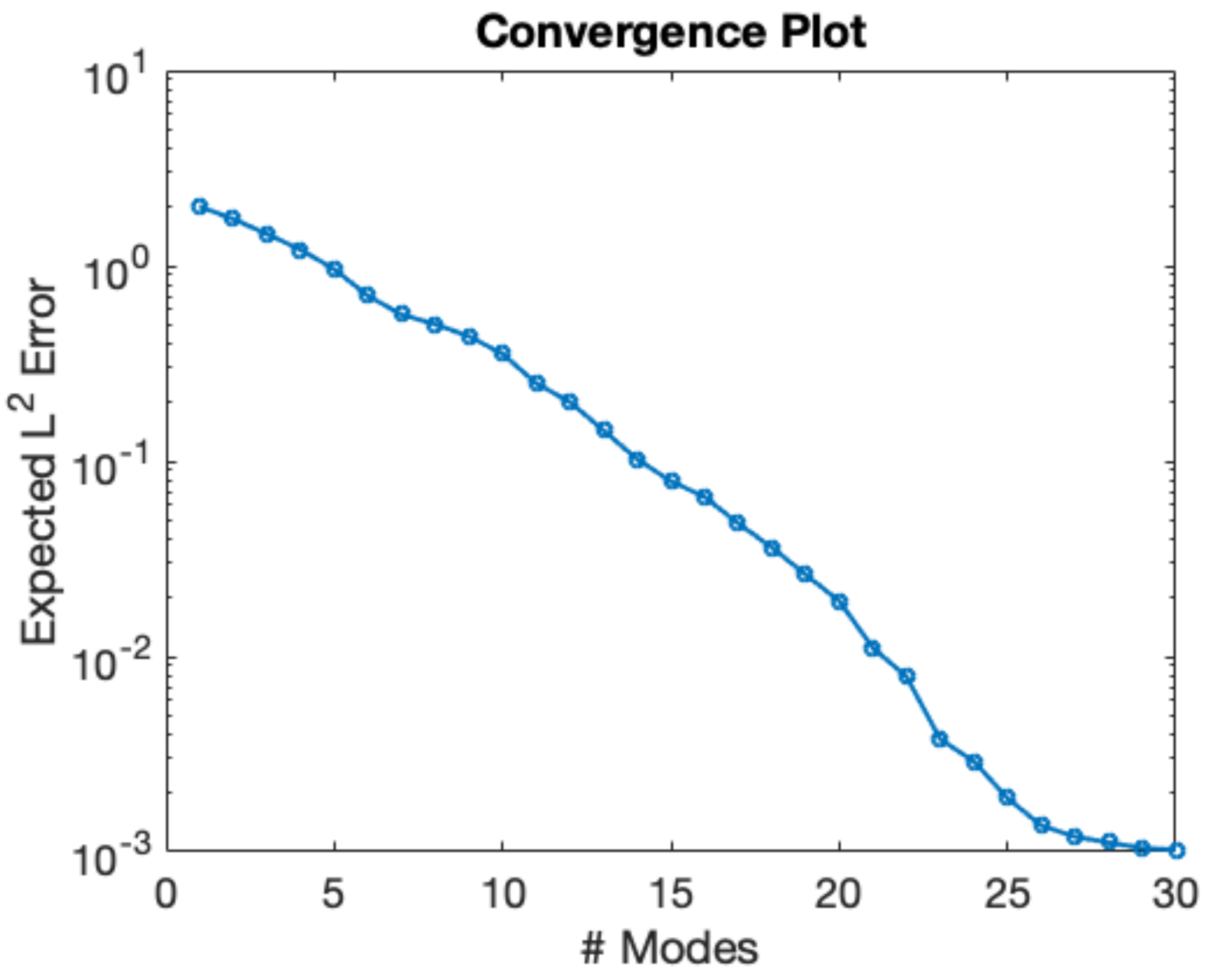

Thanks

To Wael, Manan, and Pierre!